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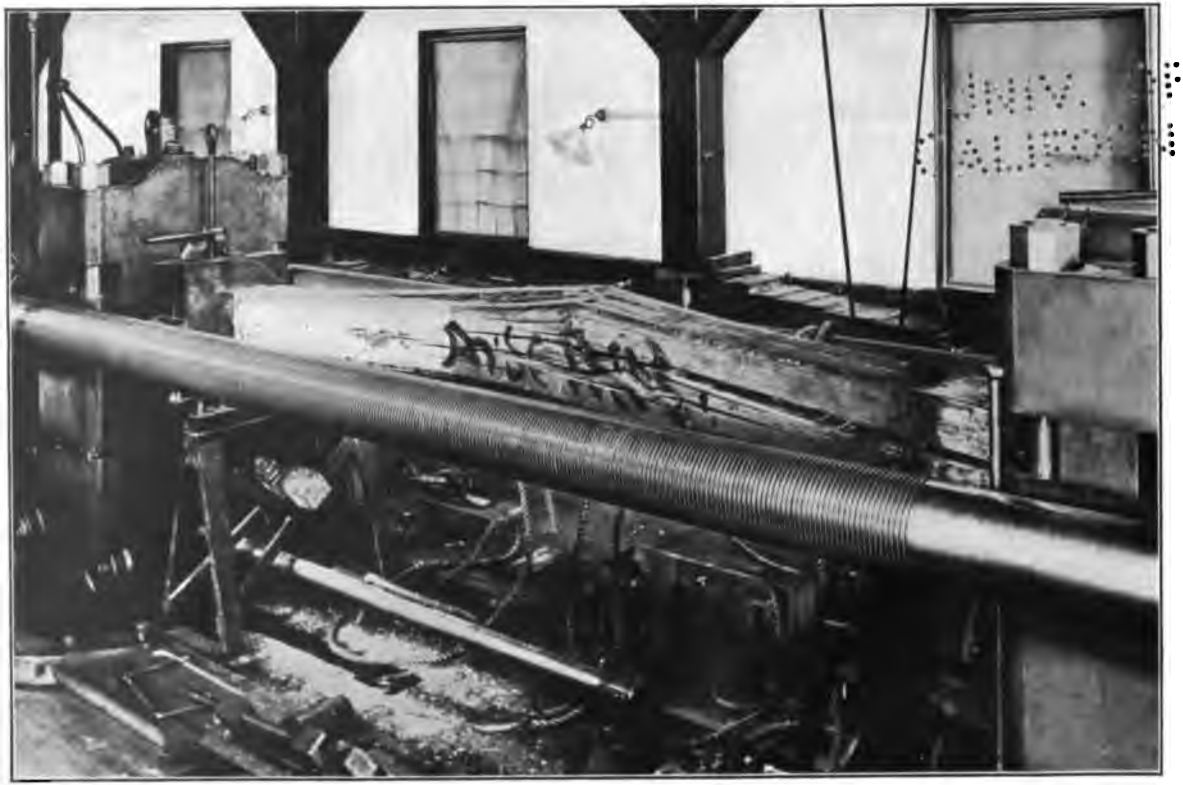

A view of the 400-ton testing-machine at the U. S. Arsenal, Watertown, Mass., immediately after compressing a timber column to failure.

(Frontispiece.)

THE

ELASTICITY AND RESISTANCE

OF THE

MATERIALS OF ENGINEERING.

UNIV. OF CALIFORNIA

BY

WM. H. BURR, C.E.,

PROFESSOR OF CIVIL ENGINEERING IN COLUMBIA UNIVERSITY IN THE CITY OF NEW YORK;
CONSULTING ENGINEER; MEMBER OF THE AMERICAN SOCIETY OF CIVIL ENGINEERS;
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PREFACE TO SIXTH EDITION.

THE present edition of this work constitutes almost a new book, one-half or more of the entire volume being new matter. Much of the latter contains original matter not hitherto published. Portions of those chapters on combined bending and direct stress and on the theory and design of concrete-steel members are of this nature. The effort has been to set forth the treatment of concrete-steel beams and other members with complete fulness to meet the requirements of that rapidly extending field of engineering construction.

It is believed that the value of the book is greatly enhanced by the tables which the author is permitted to include in this volume through the generous courtesy of the Cambria Steel Company. These practical tables, found at the end of the book, are taken directly from the Cambria Steel Company's Handbook, and they will not only enable any instructor to give with the greatest facility extended practical exercises of computations and design in his instruction work, but they will also be found of great value in structural practice.

The advanced analytic matter relating to the general theory of elasticity in amorphous solid bodies, and the exact theories of torsion and flexure, have been placed at the end of the volume in Appendix I. While this matter

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may not be of immediate use to the ordinary practitioner, every real student of the subject of resistance of materials should be familiar with the treatment of these general problems, special solutions of which only are required in the every-day practice of the engineer.

The author desires to express his indebtedness to Lieut.-Col. John W. Butler, Ord. Dept. U. S. A., commanding Watertown Arsenal, Watertown, Mass., for photographs of the 400-ton Emery machine at the Watertown Arsenal and material tested to destruction in that machine, half-tones of which will be found at a number of places in the book; also to Mr. John Sterling Deans, Chief Engineer, The Phœnix Bridge Company, for similar courtesies, and to Mr. Myron S. Falk, C.E., M.E., for suggestions and assistance in preparing the manuscript, for proof-reading and other courtesies of substantial character.

W. H. B.

COLUMBIA UNIVERSITY, April 7, 1903.

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ELASTICITY AND RESISTANCE OF MATERIALS.

PART I.—ANALYTICAL.

CHAPTER I.

ELEMENTARY THEORY OF ELASTICITY IN AMORPHOUS SOLID BODIES.

Art. 1.—General Statements.

THE molecules of all solid bodies known in nature are more or less free to move toward, or from, or among each other. Resistances are offered to such motions, which vary according to the circumstances under which they take place and the nature of the body. This property of resistance is termed the *elasticity* of the body.

The summation of the displacements of the molecules of a body, for a given point, is called the *distortion* or *strain* at the point considered. The *force* by which the molecules of a body resist a strain, at any point, is called the *stress* at that point. This distinction between *stress* and *strain* is fundamental and important.

Stresses are developed, and strains caused, by the application of force to the exterior surface of the material.

These stresses and strains vary in character according to the method of application of the external forces. Each stress, however, is accompanied by its own characteristic strain and no other. Thus there are shearing stresses and shearing strains, tensile stresses and tensile strains, compressive stresses and compressive strains. Usually a number of different stresses with their corresponding strains are coexistent at any point in a body subjected to the action of external forces.

It is a matter of experience that strains always vary continuously and in the same direction with the corresponding stresses. Consequently the stresses are continuously increasing functions of the strains, and any stress may be represented by a series composed of the ascending powers (commencing with the first) of the strains multiplied by proper coefficients. When, as is usually the case, the displacements are very small, the terms of the series whose indices are greater than unity are exceedingly small compared with the first term, whose index is unity. Those terms may consequently be omitted without essentially changing the value of the expression. Hence follows what is ordinarily termed Hooke's law:

The ratio between stresses and corresponding strains, for a given material, is constant.

This law is susceptible of very simple algebraic representation. If a piece of material, whose normal cross section is A, is subjected to either tensile or compressive stress, its length L will be changed by the amount ΔL. If P be the external force or loading which produces that deformation or change of length, the amount of force or stress, supposed to be uniformly distributed, acting on 1 square inch of normal cross-section of the piece, will be found by dividing the total force P by the area of cross-section A. This amount of uniformly distributed stress is called the "intensity of stress," and it is a most important quantity. In dealing with the effects of forces or stresses in all engineering work, the amount of such force or stress on a square unit of area, usually a square inch in American practice, and called the intensity, is often the main object sought, for it determines the question whether material is carrying too much or too little load, as well as many other related questions.

Again, the important consideration as to strain is the fractional change in length of the entire piece, and not the total change in length expressed in the unit adopted, ordinarily an inch. This fractional change of length is the same as the amount of actual change of each linear unit of the piece, as is found by dividing ΔL by L. Inasmuch as that fraction expresses the amount of change in length for each unit, it is frequently called the rate of change of length or rate of deformation. Hooke's law is to the effect that the intensity of stress is proportional to the rate of strain, and its analytic expression may readily be written.

Let p represent the intensity of any stress and l the strain per unit of length, or, in other words, the rate of strain. If E is a constant coefficient, Hooke's law will be given by the following equation:

$$p=\frac{P}{A}=\frac{\Delta L}{L}E=El. \quad . \quad . \quad . \quad . \quad . \quad ①$$

If the intensity of stress varies from point to point of a body, Hooke's law may be expressed by the following differential equation:

$$\frac{dp}{dl}=E. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If p and l are rectangular coordinates, eqs. (1) and (2) are evidently equations of a straight line passing through the origin of coordinates. It will hereafter be seen that the line under consideration is essentially straight for comparatively small strains in any case, and for some materials it has no straight portions.

Art. 2.—Coefficients of Elasticity.

In general the coefficient E in eq. (1) of the preceding article is called the *coefficient of elasticity*, or, sometimes, *modulus of elasticity*. The coefficient of elasticity varies both with the kind of material and kind of stress. It simply expresses *the ratio between stress and strain.*

The characteristic strain of a tensile stress is evidently an *increase* of the linear dimensions of the body in the direction of action of the external forces.

Let this increase per unit of length be represented by l, while p and E represent, respectively, the corresponding intensity and coefficient. Eq. (1) of the preceding article then becomes

$$p = El, \quad \text{or} \quad E = \frac{p}{l}. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

E is then the coefficient of elasticity for tension.

The characteristic strain for a compressive stress is evidently a *decrease* in the linear dimensions of the body in the direction of action of the external forces. Let l_1 represent this decrease per unit of length, p_1 the intensity of compressive stress, and E_1 the corresponding coefficient. Hence

$$p_1 = E_1 l_1, \quad \text{or} \quad E_1 = \frac{p_1}{l_1}. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

E_1 consequently is the coefficient of elasticity for compression.

The characteristic strain for a shearing stress may be determined by considering the effect which it produces on the layers of the body parallel to its plane of action.

In Fig. 1 let *ABCD* represent one face of a cube, another of whose faces is fixed along *AD*. If a shear acts in the face *BC*, whose plane is normal to the plane of the paper, all layers of the cube parallel to the plane of the shearing stress, i.e., *BC*, will slide over each other, so that the faces *AB* and *DC* will take the positions *AE* and *DF*. The amount of distortion or strain per unit of length will be represented by the angle $EAB = \phi$. If the strain is small, there may be written ϕ, sin ϕ, or tan ϕ indifferently.

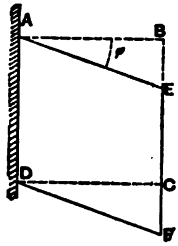


FIG. 1.

Representing, therefore, the intensity of shear, coefficient, and strain by S, G, and ϕ, respectively, eq. (1) of Art. 1 becomes

$$S = G\phi, \quad \text{or} \quad G = \frac{S}{\phi}. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

It will be seen hereafter that there are certain limits of stress within which eqs. (1), (2), and (3) are essentially true, but beyond which they do not hold; this limit is called the *limit of elasticity*, and is not in general a well-defined point.

The line *Okghn* exhibited in Fig. 2 represents the actual strains in a piece of structural steel 1 inch in length with 1 square inch of cross-section. *O* is the origin of coordinates, and the loads per square inch, i.e., intensities of stresses, are shown by the vertical ordinates drawn parallel to *OC* from *OD* to the strain curve, while the strains per unit of length, that is, per inch, are laid off as horizontal ordinates of the curve parallel to *OD*. If *Op′* is the intensity of stress, p' corresponding to the point k of the strain curve, while Ol' is the resulting strain per unit of length, then $p' = El'$. Again, if g is at the upper limit of the straight portion of the curve for which the intensity of stress and rate of strain are p and l respectively, the relation between those two quantities is shown by eq. (1). Since E, also as shown by eq. (1), is equal to the quotient of p divided by l, Fig. 2 shows that it is equal to the tangent of the angle between OD and the straight portion Og of the strain curve, it being supposed that the rates of strain are laid down at their actual or natural sizes. If the strain line is curved, the first term of eq. (2) of Art. 1, the differential ratio, will represent the tangent of the angle between the curve and the horizontal axis OD in Fig. 2. The point g, being at the upper limit of constant proportionality between intensity of stress and rate of strain, is called the elastic limit, above which it is seen that the strains increase far more rapidly than the stresses until the point n is reached, where actual rupture takes place. The nearly horizontal portion of the curve between g and h and a little

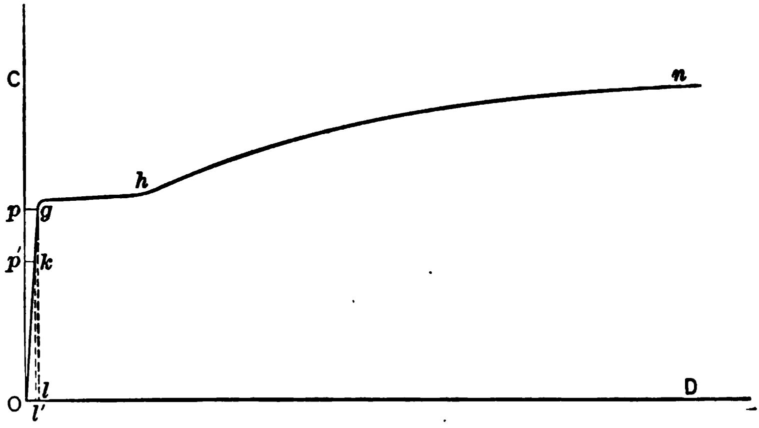


FIG. 2.

above *g* indicates the "stretch limit," an intensity of stress where the material is said first to "break down" or stretch rapidly under tensile stress without much increase of the latter.

Art. 3.—Direct Stresses of Tension and Compression.

The direct stresses of tension and compression always produce shearing stresses and strains on all planes in the interior of a body except those perpendicular and parallel to those direct stresses. If, in Fig. 1, a straight piece of material *CD* is subjected to the tensile stress induced by the forces *P* equal and opposite to each other, there will be pure tension only on all planes or sections of the piece at right angles to the direction of the forces *P*, such as *HK*. On all planes passing through the longitudinal axis of the piece there will be no stress whatever, if, as is supposed, the forces *P* are uniformly distributed over the sections of application *DF* and *BC*.

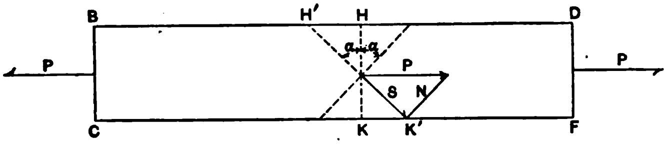


FIG. 1.

On every oblique plane or section in all parts of the piece as *H'K'*, supposed to be perpendicular to the plane of the diagram, there will be shear as well as direct stress of tension normal to it, the intensities of both the shear and the normal stress being dependent upon the angle α between *HK* and *H'K'*. The force *P* may be resolved by the triangle of forces into two components, one at right angles to *H'K'*, represented by *N*, and the other along or tangential to *H'K'*, represented by *S*. If

A represents the area of the normal section HK, the area of the oblique section $H'K'$ will be $A \sec \alpha$. The value of the normal stress N will be $N=P \cos \alpha$, but $S=P \sin \alpha$. The intensity of the normal tensile stress on $H'K'$ will be, therefore,

$$n=\frac{N}{A \sec \alpha}=\frac{P \cos \alpha}{A \sec \alpha}=p \cos^2 \alpha. \quad . \quad . \quad (1)$$

The intensity of shear on the same plane $H'K'$ will be

$$s=\frac{S}{A \sec \alpha}=\frac{P \sin \alpha}{A \sec \alpha}=p \sin \alpha \cos \alpha. \quad . \quad . \quad (2)$$

When the angle α is zero, s in eq. (2) becomes zero, while n in eq. (1) becomes equal to p, i.e., the intensity of direct tensile stress on the normal section. On the other hand, when the angle α has the value of 90°, both n and s become zero, i.e., there is no stress whatever on a longitudinal, axial plane.

Inasmuch as the angle α may have any value whatever from zero to 90° on either side of HK, it is clear that both shearing and normal tensile stresses will be found concurrently on every oblique plane in the piece. As has been observed in the preceding article, these shearing stresses induce the lateral strains under which the normal cross-sections of a piece subjected to pure tension decrease in area while they increase under the action of pure compression.

Eqs. (1) and (2) have been written on the assumption that the external forces P produce tension in the material, but precisely the same equations apply to the condition of pure compression, the only difference being that in the latter case the external forces P would be directed toward each other from the ends of the piece, instead of away from each other.

Art. 4.—Lateral Strains.

If a body, as indicated in Fig. 1, be subjected to tension, it has been shown in Art. 3 that all of its oblique cross-sections, such as FE and GH, will sustain shearing stresses in consequence of the component of the tension tangential to those oblique sections. These tangential stresses will cause the oblique sections, in both directions, to slide over each other. Consequently *the normal cross-sections of the body will be decreased;* and if the normal cross-sections of the body are made less, its capacity to resist the external forces acting on AB and CD will be correspondingly diminished.

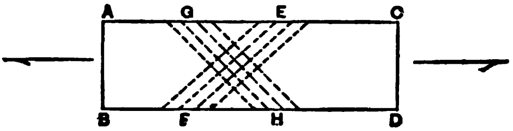


Fig. 1.

If the body is subjected to compression, oblique sections of the body will be subjected to shears, but in directions *opposite* to those existing in the previous case. The effect of such shears will be an *increase* of the lateral dimensions of the body and a corresponding increase in its capacity of resistance.

These changes in the lateral dimensions of the body are termed "lateral strains"; they always accompany direct strains of tension and compression.

It is to be observed that lateral strains *decrease* a body's resistance to tension, but *increase* its resistance to compression. Also, that if they are prevented, both kinds of resistance are *increased.*

Consider a cube, each of whose edges is a, in a body subjected to tension. Let r represent the ratio between the lateral and direct rates of strains, and let it be supposed to be the same in all directions. If l, as in Art. 2, represents the direct unit strain, the edges of the cube will become, by the tension, $a(1+l)$, $a(1-lr)$, and $a(1-rl)$. Consequently the volume of the resulting parallelopiped will be

$$a^3(1+l)(1-rl)^2 = a^3[1+l(1-2r)] \quad . \quad . \quad . \quad (1)$$

if powers of l higher than the first be omitted. *With r between 0 and $\frac{1}{2}$, there will be an increase of volume*, but not otherwise.

If the body is subjected to compression, the edges of the cube become $a(1-l_1)$, $a(1+r_1l_1)$, and $a(1+r_1l_1)$; while the volume of the parallelopiped takes the value

$$a^3(1-l_1)(1+r_1l_1)^2 = a^3[1+l_1(2r_1-1)]. \quad . \quad . \quad (2)$$

As before, the higher powers of l_1 are omitted. If the volume of the cube is decreased, r_1 must be found between 0 and $\frac{1}{2}$.

Art. 5.—Relation between the Coefficients of Elasticity for Shearing and Direct Stress in a Homogeneous Body.

A body is said to be homogeneous when its elasticity, of a given kind, is the same in all directions.

Let Fig. 1 represent a body subjected to tension parallel to CD. That oblique section on which the shear has the greatest intensity will make an angle of 45° with either of those faces whose traces are CD or BD; for if α is the angle which *any* oblique section makes with BD, P the total tension on BD, and A' the area of the latter surface, the total shear on any section

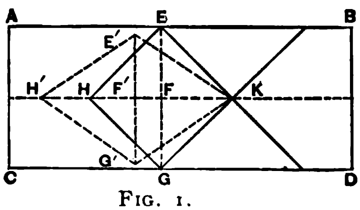


Fig. 1.

whose area is A' sec α will be P sin α. Hence the intensity of shear is

$$\frac{P \sin \alpha}{A' \sec \alpha} = \frac{P}{A'} \sin \alpha \cos \alpha. \quad . \quad . \quad . \quad . \quad (1)$$

The second member of eq. (1) evidently has its greatest value for $\alpha = 45°$. Hence if the tensile intensity on BD is represented by $\frac{P}{A'} = p$, the greatest intensity of shear will be

$$S = \frac{p}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Then by eq. (3) of Art. 2,

$$\phi = \frac{p}{2G}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

In Fig. 1 EK and KG are perpendicular to each other, while they make angles of 45° with either AB or CD. After stress, the cube $EKGH$ is distorted to the oblique parallelopiped $E'KG'H'$. Consequently $EKGH$ and $E'KG'H'$ correspond to $ABCD$ and $AEFD$, respectively, of Fig. 1, Art. 2. The angular difference $EKG - E'KG'$ is then equal to ϕ; and $EKE' = GKG' = \frac{\phi}{2}$. Also, $E'KF' = 45° - \frac{\phi}{2}$.

Using, then, the notation of the preceding articles, there will result, nearly,

$$\tan\left(45° - \frac{\phi}{2}\right) = \frac{1 - rl}{1 + l} = 1 - l(1 + r); \quad . \quad . \quad (4)$$

remembering that

$$F'K = FK(1 + l), \quad \text{and} \quad E'F' = FK(1 - rl).$$

From a trigonometrical formula there is obtained, very nearly,

$$\tan\left(45° - \frac{\phi}{2}\right) = \frac{\tan 45° - \tan\frac{\phi}{2}}{\tan 45° + \tan\frac{\phi}{2}} = \frac{1 - \frac{\phi}{2}}{1 + \frac{\phi}{2}} = 1 - \phi. \quad . \quad (5)$$

From eqs. (4) and (5),

$$\phi = l(1+r). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Substituting from eq. (3), as well as from eq. (1) of Art. 2,

$$G = \frac{E}{2(1+r)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

It has already been seen in the preceding article that r must be found between 0 and $\frac{1}{2}$, consequently *the coefficient of elasticity for shearing lies between the values of* $\frac{1}{3}$ *and* $\frac{1}{2}$ *of that of the coefficient of elasticity for tension.*

This result is approximately verified by experiment.

Since precisely the same form of result is obtained by treating compressive stress, instead of tensile, there will be found, by equating the two values of G,

$$\frac{E}{1+r} = \frac{E_1}{1+r_1}, \quad \text{or} \quad \frac{E_1}{E} = \frac{1+r_1}{1+r}. \quad . \quad . \quad . \quad . \quad (8)$$

It is clear, from the conditions assumed and operations involved, that the relations shown by eqs. (7) and (8) can only be approximate.

PROBLEMS FOR CHAPTER I.

Problem 1.—A wrought-iron bar 3″ × $\frac{3}{4}$″ in section is subjected to a tensile force of 27,000 pounds. The stretch for a gauged length of 21′ 6″ was .1122″. Find the intensity of tensile stress in the material, the coefficient of elasticity E, and the rate of strain, i.e., the strain per linear inch.

Partial Ans. Intensity of stress = 12,000 pounds per square inch; $E = 27{,}600{,}000$ pounds per square inch.

Problem 2.—A steel eye-bar 8″ × 2″ in section carries a total load of 128,000 pounds, under which there is a stretch of .016″ in a gauged length of 5 ft. Find the intensity of stress, rate of strain, and coefficient of elasticity E. $E' = 33,200,000$.

Problem 3.—A 2″ round steel bar, for which the coefficient of elasticity E is 29,000,000 pounds, is subjected to a tensile force of 46,400 pounds. Find the intensity of stress, the rate of strain, and the total tensile strain for a length of 18 ft.

Problem 4.—A concrete pillar 24″×24″ in section and 8 ft. high carries a total (compressive) load of 115,200 pounds. If the coefficient of elasticity for the concrete is 3,000,000 pounds per square inch, what will be the rate of compressive strain and the shortening, first, for the total height 8 ft. of pillar, and, second, for 12″, under the preceding load?

Problem 5.—In Problems 1 and 2, if the ratio r of lateral to direct strains is .42, what will be the change in volume of a portion of each bar 1 ft. long? Also, find changes in cross dimensions of bars.

Problem 6.—In Problem 3 the diameter of the round bar decreased .00045″. Find the ratio r between direct and lateral unit strains, and also the increase of volume of 3 ft. length of the bar.

Problem 7.—In Problems 5 and 6 find the coefficient of elasticity, G, for shearing in terms of the direct coefficient of elasticity E.

Problem 8.—In Problem 2 find the total normal and tangential stresses and their intensities on plane sections making angles of 18°, 35°, and 53° with the axis of the piece.

Problem 9.—In Problem 3 find the total normal and tangential stresses and their intensities on plane sections making angles of 31°, 45°, and 72° with the axis of the piece.

CHAPTER II.

HOLLOW CYLINDERS AND SPHERES AND TORSION.

Art. 6.—Thin Hollow Cylinders and Spheres.

IF a straight, closed, hollow cylinder be subjected to an interior pressure with the intensity q', sufficiently greater than the exterior pressure q_1, the tendency will be to split the cylinder longitudinally. Fig. 1 represents such a cylinder, the sides of which are supposed to be so thin that the stress to which they are subjected may be considered uniformly distributed throughout the material.

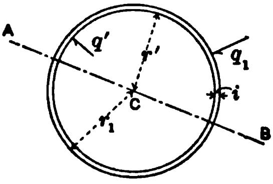


FIG. 1.

AB represents the diametral plane through the axis of the cylinder, the thickness, i, of the shell being supposed in this case to be so small that the cylinder may be considered a "thin" cylinder.

The interior radius is r' and the exterior radius r_1. If a unit length of cylinder be considered, the total circumferential tension developed in the material on one side of the cylinder will be $q'r' - q_1r_1$, it being supposed that the interior pressure is so much greater than the exterior that tension only will be induced in the material. Obviously, if the exterior pressure were much larger than the interior, compression would exist instead of tension. The intensity of tensile stress t in the sides of the cylinder will therefore be

$$t=\frac{q'r'-q_1r_1}{i}. \qquad (1)$$

This value of t expresses the tendency of the cylinder to split along a diametral plane under the action of the interior pressure q'.

If the ends of the cylinder are closed, the internal pressure against them will tend to force them off or to pull the cylinder apart around a section normal to the axis. The force F tending to produce this result will be

$$F=\pi(q'r'^2-q_1r_1^2). \qquad (2)$$

The area of normal section of the cylinder will be $\pi(r_1^2-r'^2)$. Hence the intensity of stress developed by this force will be

$$f=\frac{F}{\pi(r_1^2-r'^2)}=\frac{q'r'^2-q_1r_1^2}{r_1^2-r'^2}. \qquad (3)$$

If the exterior pressure is so small that it may be considered zero, eqs. (1) and (3) give

$$t=\frac{q'r'}{i}, \qquad (4)$$

$$f=\frac{q'r'^2}{r_1^2-r'^2}. \qquad (5)$$

When the thickness of the shell is small r' may be placed equal to $\frac{r'+r_1}{2}$, and this value introduced in eq. (5) will give

$$f=\frac{q'r'}{2(r_1-r')}=\frac{q'r'}{2i}. \qquad (6)$$

f in eq. (6) is seen to be but half as much as t in eq. (4). In this case, therefore, if the material has the same ultimate resistance in both directions, the cylinder will fail longitudinally when the interior intensity is only half great enough to produce transverse rupture.

In designing thin cylinders it will usually be necessary to determine the thickness i, so that the tensile stress t in the metal shall not exceed the prescribed value h. After writing h for t in eq. (1), also $r_1 - r'$ for i, then dividing both sides of the equation by r', there will result

$$h\frac{r_1}{r'} = h + q' - q_1\frac{r_1}{r'}.$$

This equation readily gives

$$r_1 - r' = i = r'\left(\frac{h+q'}{h+q_1}\right) - r'. \quad . \quad . \quad . \quad . \quad (7)$$

If the exterior pressure q_1 is so small that it may be considered zero, the thickness given by eq. (7) takes the following form:

$$i = \frac{q'r'}{h}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

This is the same value that will be found by solving eq. (4) for i.

The expression for the thickness of the material of the cylinder to resist the longitudinal tension having the intensity f can be found with equal ease. If f_1 be written for f in eq. (3), as the greatest permissible longitudinal tension, then if both numerator and denominator of the second member of that equation be divided by r'^2, there will result

$$\frac{r_1^2}{r'^2} = \frac{f_1 + q'}{f_1 + q_1}.$$

The solution of this equation at once gives the desired thickness:

$$r_1 - r' = i = r'\left(\frac{f_1 + q'}{f_1 + q_1}\right)^{\frac{1}{2}} - r'. \quad . \quad . \quad . \quad . \quad (9)$$

If q_1 is so small that it may be neglected, it is simply to be made zero in eq. (9).

If the exterior pressure q_1 were considerably larger than q', the resulting stresses in the sides of the cylinder would be compression, but the formulæ for the resulting intensities would be precisely the same as the preceding, as long as the cylinder retained its circular shape.

The case of stresses in a thin hollow sphere or thin spherical shell may be treated in the same general manner. The hemispherical ends of a metallic cylindrical tank or reservoir may be illustrated by the skeleton section in Fig. 2.

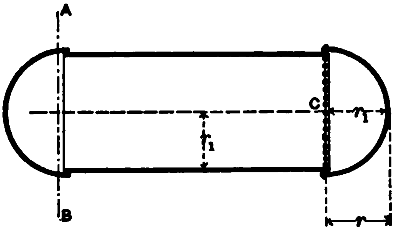


Fig. 2.

As indicated in the figure the internal radius of each end is r_1, while r is the external radius. The internal and external intensities of pressure are as shown in Fig. 1. The force tending to tear off the hemispherical ends of the tank along the line AB, Fig. 2, is $\pi(q'r_1^2 - q_1r^2)$. The section of metal resisting this force with the intensity t is $\pi(r^2 - r_1^2)$. The intensity of stress developed in the metal will therefore be

$$t = \frac{q'r_1^2 - q_1r^2}{r^2 - r_1^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

If the external pressure is so small that there may be taken $q_1 = 0$, eq. (10) will take the form

$$t = \frac{q' r_1^2}{r^2 - r_1^2} = \frac{q' r_1}{2i}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

In this last equation $i = r - r_1$, and the interior radius is placed equal to one half the sum of the interior and exterior radii, as may be done without sensible error. The interior radius being given, the thickness of metal required to withstand a given internal pressure q' without stressing the metal above a given working value t may be written as follows from eq. (11):

$$i = \frac{q' r_1}{2t}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

If the value of the thickness i should be desired in terms of both the interior and exterior pressures, it can easily be written by the aid of eq. (10); if both numerator and denominator of the second member of that equation be divided by r_1^2, there may at once be found

$$\frac{r}{r_1} = \left(\frac{t + q'}{t + q_1}\right)^{\frac{1}{2}}.$$

After multiplying this equation through by r_1, then subtracting that quantity from each side of the resulting equation, the desired value of the thickness will be

$$i = r - r_1 = r_1 \left(\frac{t + q'}{t + q_1}\right)^{\frac{1}{2}} - r_1. \quad . \quad . \quad . \quad . \quad (13)$$

By giving a proper working value to the tensile intensity t and inserting the values of the pressures, the thickness i will at once result.

In this last set of equations it has been assumed that no metal is taken out of the section AB by rivets. The proper allowance for metal so deducted will be plainly evident after the subject of riveted joints has been treated.

Art. 7.—Thick Hollow Cylinders and Spheres.

If the sides or walls of hollow cylinders and spheres subjected to high internal pressures are great in comparison with the internal radius, the tensile stress in the metal may not be assumed to be uniformly distributed, and it is necessary to determine entirely different formulæ from those established in the preceding article.

The normal section of a thick hollow cylinder is shown in Fig. 1, r' being the internal radius and r_1 the external, with the intensities of internal and external pressures q' and q_1 respectively. It is supposed that the internal pressure so greatly exceeds the external that the metal sustains tensile stress only. If the cylinder be supposed to be divided into a great number of thin concentric portions, the elastic stretching of the metal will cause a much higher tension to exist in the interior portions than in the exterior. The determination of the law of variation of the intensity of the tensile stress in the sides of the cylinder is the chie purpose of the investigation. If any diametral section, such as AB, Fig. 1, be assumed, it is clear that the sum of all the tensile stresses developed in that section must be equal to the excess of the internal pressure over the external. A unit length of cylinder will be considered in the following formulæ.

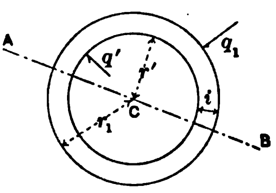


Fig. 1.

The tensile stress in the sides of the cylinder, whose intensity will be represented by t, and which is developed in any diametral section, as AB, has a circumferential direction, and for that reason it is sometimes called "hoop tension."

The variation of this tensile intensity t carries with it a corresponding variation in intensity of the radial pressure whose intensity is q, having the values q' in the interior of the cylinder and q_1 at the external surface.

The amount of tension on a radial section of thickness dr will be tdr, and if that differential expression be integrated so as to extend over the entire thickness of one wall or side of the cylinder, it must be equal to the effort of the internal pressure in excess of the external to split the cylinder along one of its sides. The following equation is the analytical expression of this condition:

$$q'r' - q_1r_1 = \int_{r'}^{r_1} tdr. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

It is only necessary to determine a proper expression for t in terms of r in order to make the integration indicated in eq. (1), and thus completely solve the problem. If the following values of t and q be assumed, it will be found that eq. (1) is completely satisfied:

$$t = a + \frac{b}{r^2} \quad \text{and} \quad q = -a + \frac{b}{r^2}. \quad . \quad . \quad . \quad (2)$$

The quantities a and b of eq. (2) are arbitrary constants which are to be found by the condition imposed in eq. (1) and by the further condition that q must be equal to q' when r is equal to r', and to q_1 when r is equal to r_1. The substitution of the value of t from eq. (2) in eq. (1) will give

$$\int_{r'}^{r_1} tdr = a(r_1 - r') - b\left(\frac{1}{r_1} - \frac{1}{r'}\right)$$
$$= q'r' - q_1r_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Again, by first making r equal to r' and then r equal to r_1 in the second of eqs. (2), there will result

$$q' = -a + \frac{b}{r'^2} \quad \text{and} \quad q_1 = -a + \frac{b}{r_1^2}. \quad . \quad . \quad (4)$$

By subtracting the second from the first of eqs. (4),

$$b = \frac{(q' - q_1)r'^2 r_1^2}{r_1^2 - r'^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The substitution of b from eq. (5) in the second of eqs. (3) will give

$$a = \frac{q' r'^2 - q_1 r_1^2}{r_1^2 - r'^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

If the value of a and b as determined in eqs. (5) and (6) be introduced into the two eqs. (2), the intensities of the compression and tension in the metal of the cylinder will be

$$q = -\frac{q' r'^2 - q_1 r_1^2}{r_1^2 - r'^2} + \frac{(q' - q_1)r'^2 r_1^2}{r_1^2 - r'^2} \frac{1}{r^2}, \quad . \quad . \quad . \quad (7)$$

$$t = \frac{q' r'^2 - q_1 r_1^2}{r_1^2 - r'^2} + \frac{(q' - q_1)r'^2 r_1^2}{r_1^2 - r'^2} \frac{1}{r^2}. \quad . \quad . \quad . \quad (8)$$

Eqs. (7) and (8) are the general values of the intensities of the internal stresses in the shell of the cylinder, q acting in a radial direction and t in a circumferential direction. The maximum tensile stress t' will exist at the interior surface of the cylinder and will be found by making $r = r'$ in eq. (8):

$$t = t' = \frac{q'(r_1^2 + r'^2) - 2q_1 r_1^2}{r_1^2 - r'^2} = h. \quad . \quad . \quad . \quad . \quad (9)$$

The least intensity of tension will exist at the exterior surface of the cylinder and will be indicated by t_1. It has the following value found by making $r = r_1$ in eq. (8):

$$t = t_1 = \frac{2q'r'^2 - q_1(r_1^2 + r'^2)}{r_1^2 - r'^2} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (10)$$

It is essential to find an expression for the thickness i of the cylinder wall required for given pressures so that the maximum tensile stress shall not exceed a given value h, and it is readily found by first dividing both numerator and denominator of the third of eqs. (9) by r' and then solving the resulting equation for $\frac{r_1}{r'}$,

$$\frac{r_1}{r'} = \left(\frac{q' + h}{2q_1 - q' + h}\right)^{\frac{1}{2}}.$$

If both members of this last equation be multiplied by r', and then if r' be subtracted from both sides of the resulting equation, the desired expression for the thickness will be

$$i = r_1 - r' = r'\left(\frac{q' + h}{2q_1 - q' + h}\right)^{\frac{1}{2}} - r'. \quad . \quad . \quad . \qquad (11)$$

The internal radius r_1 being known, eq. (11) will give the proper thickness of metal when the working tensile stress or hoop tension h is prescribed together with the actual pressures q' and q_1. It will be observed that the denominator of the fraction in the third member of eq. (11) becomes zero when $2q_1 + h = q'$; hence the resulting value of i would be indefinitely great. This shows that when the intensity of the internal pressure is equal to or greater than twice the intensity of the external pressure added to the greatest allowed tensile stress in the metal, it is impossible to make the wall of the cylinder thick enough to meet the requirements of the case, whatever may be the internal radius.

If the external pressure is so small that it may be neglected, it is necessary only to place $q_1 = 0$ in the preceding equations.

It is seen by eq. (8) that the intensity of the hoop tension t decreases quite rapidly as the radius increases.

In all ordinary engineering operations it is not necessary to use these formulæ for thick hollow cylinders, but they are required for such special purposes as the design of modern high-powered ordnance and some special hydraulic machinery. A more elegant and complete if not conclusive demonstration of eqs. (7) and (8) will be found in App. I. The preceding method, however, is probably more simple, and does not require the general equations of equilibrium used in the theory of elasticity in solid bodies.

Equations for thick hollow spheres may readily be established in precisely the same general manner followed in the preceding paragraphs for thick hollow cylinders, but such spheres are so seldom used that any other demonstration will not be given than the general one found in App. I.

Art. 8.—Torsion in Equilibrium.

The state of stress called torsion is produced when a straight bar of material, like a piece of round shafting, is twisted. Such a bar is represented in Fig. 1, the axis of the piece being AB, and its normal cross-section having any shape whatever. In engineering practice the outline of that normal section is usually circular, although it is occasionally square.

The twisting of the bar is done by the action of two equal and opposite couples acting in two planes, each normal to the axis, but at any desired distance apart. The two couples are represented in Fig. 1 at each end of the piece in the two normal sections *A* and *B*. The forces and lever-arm of one couple are respectively *P* and *e*, and *P'* and *e'* of the other. The moment of the first couple will be *Pe* and that of the second couple *P'e'*, and if pure torsion is to be produced these two moments must be equal, but opposite to each other. Inasmuch as the moment of a couple is the product of the force by the lever-arm, the forces and lever-arms of the two twisting couples may vary to any extent as long as the moments remain unchanged.

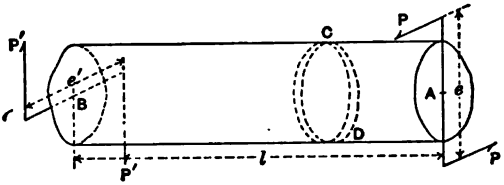
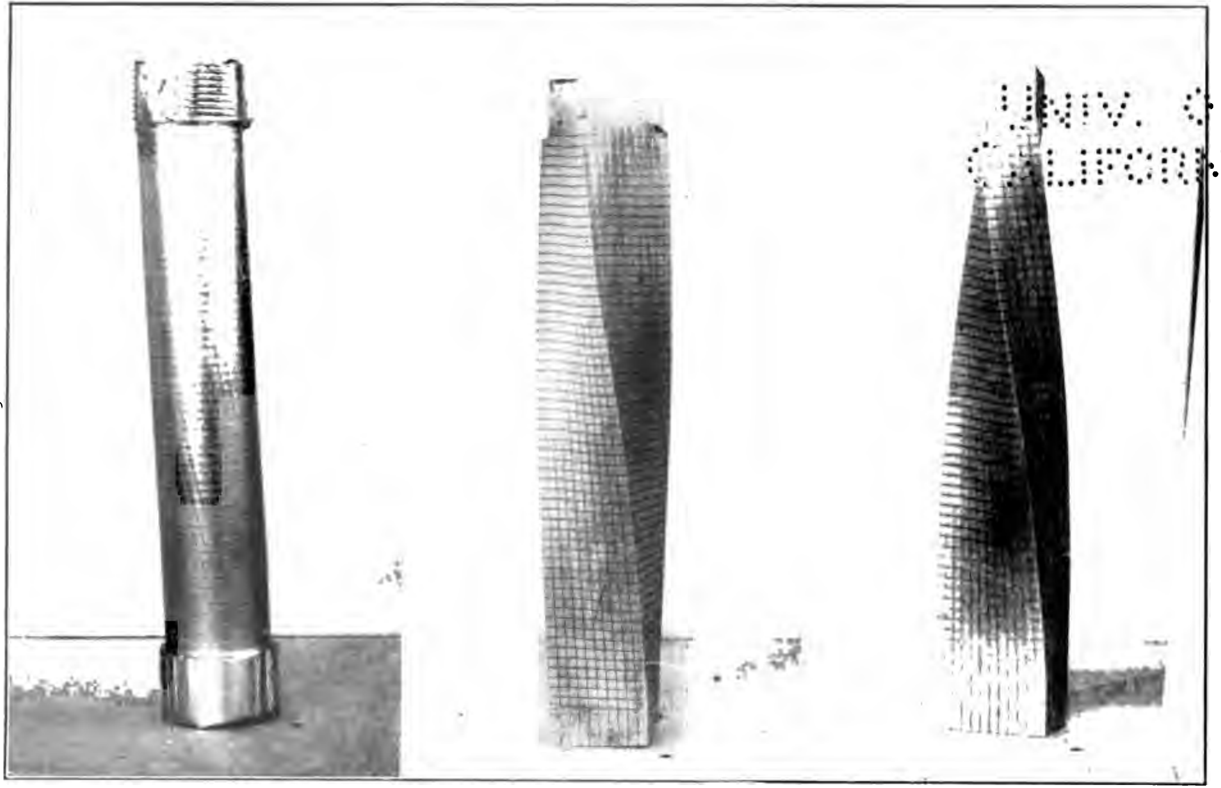


FIG. 1.

Although the system of forces to which a bar in torsion is subjected is such as to be in equilibrium, any portion of the piece will tend to have its normal sections like those at *CD* rotated over each other, the result being a small sliding motion around the axis of the piece. Hence a torsive stress is wholly a shearing stress on normal sections of the piece subjected to torsion. It is further important to observe that inasmuch as a couple produces the same effect wherever it may act in its own plane, the actual twisting moment need not be applied with its forces symmetrically disposed in reference to the axis of the piece; indeed, both of those forces may be anywhere on one side of the piece without varying the conditions of torsion or torsive stress to any extent whatever.



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View of three cylinders subjected to torsion beyond the elastic limit. The lines shown were drawn at right angles to each other on the original test pieces before being subjected to torsion. Those lines show the characteristic distortion of sections normal to the axes of the pieces.

(To face page 25.)

It is known from the general theory of stress in a solid body that although there can be no stresses of tension and compression parallel to the axis of a bar under torsion, or at right angles to it, there will be such stresses of varying intensities on oblique planes. Inasmuch as the result of torsion is to slide normal sections each past its neighbor, the elastic torsive shear like any other shear will not change the volume of the body. The principal shearing strains will produce deformation without changing the dimensions whose product gives the volume.

The exact and complete mathematical theory of torsion deduced from the general equations of equilibrium of stresses in an elastic solid, without extraneous assumptions, will be found in App. I. Those formulæ show accurately the state of torsive stress in bars of any elastic material and of various shapes of cross-section. For the general purposes of engineering practice that general demonstration is rather complicated. Hence it is often avoided by making certain approximate assumptions based to some extent on experimental observations which lead to an approximate and simpler theory, yielding formulæ accurate only for the circular normal section, but which are not materially in error for the square section. These formulæ are, however, far from accurate for certain other sections. In this article only the formulæ of the simpler theory, called the common theory of torsion, will be given.

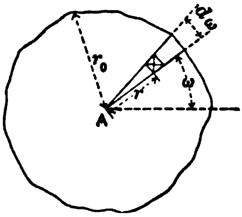

FIG. 2.

Fig. 2 is supposed to represent the normal section of a bar of material of any shape, subjected to torsion by the application of couples as shown in Fig. 1. The fundamental assumptions of the common theory of torsion are that the intensity of shearing stress varies directly as the distance from a central point at which that intensity is zero, and that that central point is located at the centre of gravity or the centroid of the section. It is also implicitly assumed that the normal sections which are plane before torsion remain plane during torsion. In Fig. 2, A is supposed to be the centre of gravity of the section at which the intensity of shear, i.e., the shear per square unit of section, is zero. The distance from the centre A to any point of the section is represented by r, and to the most remote point in the perimeter of the section by r_0. In accordance with the assumed law, the greatest intensity of shear T_m in the section will be found at the distance r_0 from its centre. While this is accurately true for the circular section, it is quite erroneous for a number of other sections. Hence the intensity at the distance unity from the centre A will be $\frac{T_m}{r_0}$, and at the distance r from the centre it will have the value

$$s = \frac{r}{r_0} T_m. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The element of the section at the distance r from A will be

$$r d\omega . dr. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Hence the shear on that element is

$$dS = \frac{r}{r_0} T_m . r d\omega . dr = \frac{T_m}{r_0} r^2 dr . d\omega. \quad . \quad . \quad . \quad (3)$$

The direction of action of this torsive shear is around the circumference of a circle whose radius is r; hence if moments of all these small shears, dS, be taken about the centre or point of no shear, A, the lever-arm of each small force, dS, will be r, and the differential moment will be

$$dM = rdS = \frac{T_m}{r_0} r^3 dr . d\omega. \quad . \quad . \quad . \quad . \quad (4)$$

The total moment of torsion therefore will be

$$M = \int_0^{2\pi}\int_0^{r_0} \frac{T_m}{r_0} r^3 dr . d\omega = \frac{T_m}{r_0}\int_0^{2\pi}\int_0^{r_0} r^3 dr\, d\omega = \frac{T_m}{r_0} I_p. \quad (5)$$

The quantity I_p is the polar moment of inertia of the section.

For a circular section

$$I_p = \frac{\pi r_0^4}{2} = \frac{\pi d^4}{32}\ (d = \text{diameter}) = \frac{Ad^2}{8}. \quad . \quad . \quad (6)$$

For a square section (b = side of square)

$$I_p = \frac{b^4}{6} = \frac{Ab^2}{6}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

For a rectangular section (b = one side and c = the other side)

$$I_p = \frac{bc^3 + b^3 c}{12} = \frac{A(b^2 + c^2)}{12}. \quad . \quad . \quad . \quad . \quad (8)$$

For an elliptical section (a_1 and b_1 being semi-axes)

$$I_p = \frac{\pi(a_1^3 b_1 + a_1 b_1^3)}{4} = \frac{\pi a_1 b_1 (a_1^2 + b_1^2)}{4}. \quad . \quad . \quad (9)$$

Using the notation of Fig. 1, the following equation of moments may be written, Pe being the moment of the external twisting couple and M the moment of the internal torsive shearing stresses in any normal section:

$$Pe = M = \frac{T_m}{r_0} I_p. \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

It is clear from Art. 2, if ϕ_0 is the shearing strain at the distance r_0 from the centre, that $T_m = G\phi_0$, G being the coefficient of elasticity for shearing. Also, since the intensity of shearing varies directly as the distance from the centre A, it is equally clear that the shearing strain ϕ varies directly as the distance from the centre, so that if α represents the shearing strain at unit's distance from A

$$\phi = r\alpha \quad \text{and} \quad \phi_0 = r_0\alpha. \quad . \quad . \quad . \quad . \quad (11)$$

Hence in general

$$T = Gr\alpha, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

and as a maximum

$$T_m = Gr_0\alpha. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

α is evidently *the angle through which one end of a fibre of unit's length and at unit's distance from the centre or axis is turned.* It is called the *angle of torsion.*

If l is the length of the piece twisted, the total angle through which the end of the fibre at unit's distance from the axis will be turned is

$$\textit{Total angle of torsion} = \alpha l. \quad . \quad . \quad . \quad (14)$$

If the fibre is at the distance r_0 from the axis one end will be twisted around beyond the other by an amount equal to

$$\textit{Total strain of torsion} = r_0\alpha l. \quad . \quad . \quad . \quad (15)$$

By the aid of eq. (13) eq. (5) may be written

$$Pe = M = G\alpha I_p = \frac{G\phi_0}{r_0} I_p. \quad . \quad . \quad . \quad . \quad (16)$$

If ϕ_0 is observed experimentally

$$G = \frac{Pe \, . \, r_0}{\phi_0 I_p}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

The angle through which a shaft will be twisted by the moment Pe, if the length is l, is

$$\alpha l = \frac{Pel}{GI_p} = \frac{lT_m}{Gr_0}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

If G is in pounds per square inch, as is usual, the preceding formulæ require all dimensions to be in inches, while α will be arc distance at radius of one inch.

If $12l$ is written for l the unit for the latter quantity must be the foot.

By inserting the value of I_p from eq. (6) in eq. (5),

$$Pe = M = \frac{2T_m}{d} \cdot \frac{\pi d^4}{32} = \frac{\pi T_m d^3}{16};$$

$$\therefore\ d = 1.72\sqrt[3]{\frac{Pe}{T_m}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Eq. (19) will give the diameter of a shaft capable of resisting the twisting moment represented by Pe with the maximum torsive shear in the extreme fibres of T_m.

The main cross dimensions of other sections may be found similarly by the use of eqs. (7), (8), and (9).

It is frequently convenient to compute the greatest intensity T_m from the twisting moment M. For this purpose the equation preceding eq. (19) gives

$$T_m = 5.1\frac{M}{d^3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

These equations complete all that are required for the practical use of the common theory of torsion. In some cases it may be necessary to use accurate formulæ for other shapes of section than the circular. In those cases the exact formulæ of App. I should be employed. The practical applications of the preceding formulæ to such

HP + . $d = 68.5\sqrt[3]{\frac{HP}{nT_m}}$

problems as arise in engineering practice will be made in a subsequent article.

PROBLEMS FOR CHAPTER II.

Problem 1.*—A steel boiler shell 58″ in diameter carries an internal pressure of 110 pounds per square inch, the external atmospheric pressure being taken at 15 pounds per square inch. If the thickness of metal is $\frac{3}{8}$″, what is the intensity of stress in the steel plates on plane sections taken through the axis of the boiler? The ends of the boiler being supposed closed, what will be the intensity of stress in the same steel plates, on the annular section of the metal made by a plane at right angles to the axis of the shell?

Partial Ans. Intensity of stress on the longitudinal section of the plates is 7347 pounds per square inch.

Problem 2.*—Design a steel boiler shell 68″ in diameter for an internal pressure of 225 pounds per square inch above the atmosphere, the greatest intensity of allowed stress being 8000 pounds per square inch. Then find the intensity of stress in the annular section of the shell made by a plane at right angles to the axis of that shell.

Partial Ans. Thickness required is $\frac{15}{16}$″.

Problem 3.*—In Fig. 2 of Art. 6 a diameter of cylinder of 2 ft. and a pressure of 1800 pounds per square inch above the atmosphere are taken. What is the total force tending to tear the semispherical head from the cylinder and what must be the thickness of steel plate to resist it if the greatest working intensity of stress in the steel is 10,000 pounds per square inch? Also find the thickness of the steel shell required to prevent tearing apart along a longitudinal section if the greatest working intensity is 9000 pounds per square inch.

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* The steel shell is treated as if there were no riveted joints, although obviously such joints must be used in actual boilers. This assumption is made as these problems are to be solved prior to the study of riveted joints.

Problem 4.—Each foot of head of water corresponds to .4335 lb. per square inch of pressure. Design cast-iron water-pipes 6″, 20″, and 48″ in diameter for heads of 75 and 180 ft., the greatest working intensity of tensile stress in the cast iron being 2000 pounds per square inch.

Problem 5.—Let the pressure in the bore of a 12″ steel rifle be taken at 28,015 pounds per square inch, the external atmospheric pressure being taken at 15 pounds per square inch. Let it further be assumed that the greatest intensity of tensile stress in the metal shall not exceed 40,000 pounds per square inch, this intensity existing at the interior surface of the gun at the instant of discharge. It is first required to find the thickness of metal. Eq. (11) of Art. 7 will give that thickness. Then find the intensity of tensile stress in the metal at the exterior surface and at the distances of 2″, 4″, and 6″ from the interior surface. The solution of eqs. (11), (7), and (8) of Art. 7 will give the required answers. The thickness of metal will be found to be 8.28″ and the intensity of stress at the exterior surface 11,980 pounds.

Problem 6.—A round bar of steel $2\frac{7}{8}$ inches in diameter is twisted by a force of 2100 pounds acting with a lever-arm of 17 inches. Two sections 25 ft. apart are turned .185 inch in reference to each other, i.e., the total strain of torsion for a length of bar of 25 feet has that value. Find the *total angle of torsion*, the *angle of torsion* and the coefficient of elasticity, G, for shearing (i.e., for torsion).

Ans. $\alpha = .00043$; $\alpha l = .129$; and $C = 13{,}000{,}000$ lbs.

Problem 7.—The greatest permitted working intensity of torsive shearing is 8000 pounds per sq. in. Design a steel shaft to carry a twisting moment produced by a force of 1900 pounds, acting with a lever-arm of 84 inches. If the coefficient of elasticity for shearing is 12,000,000 pounds, what will be the angle of torsion? Also what will be the total angle of torsion and total strain of torsion for a length of shaft of 13 feet?

Problem 8.—A shearing force of 44,000 pounds acting on the normal section of a two-inch square steel bar produced a shearing strain of .001834 inch for a length of two inches. What is the coefficient of elasticity, G, for shearing?

Prob 7. Ans.

$d = 4.65''$ $\alpha = .000144$ rad $r_0 \alpha l = .104''$

$\alpha l = .0224$

CHAPTER III.

FLEXURE.

Art. 9.—The Common Theory of Flexure.

A STRAIGHT piece or bar of material is subjected to flexure or bending when it is acted upon by loads or forces at right angles to its axis, the loads and supporting forces taken as a whole constituting a system in equilibrium.

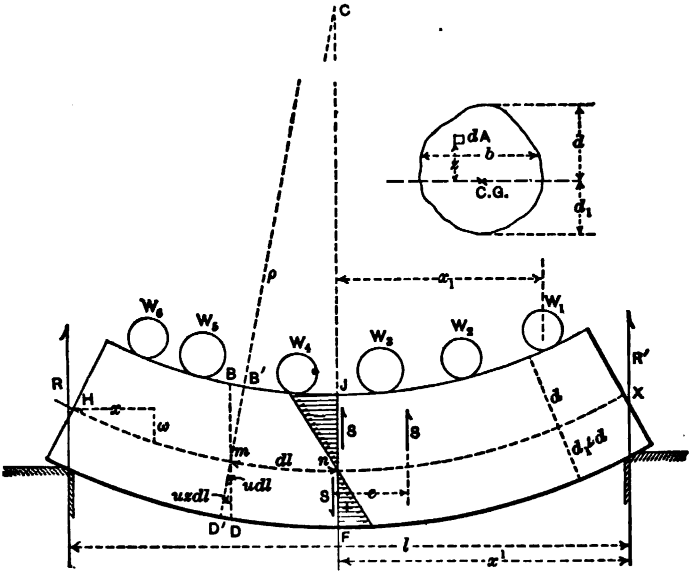
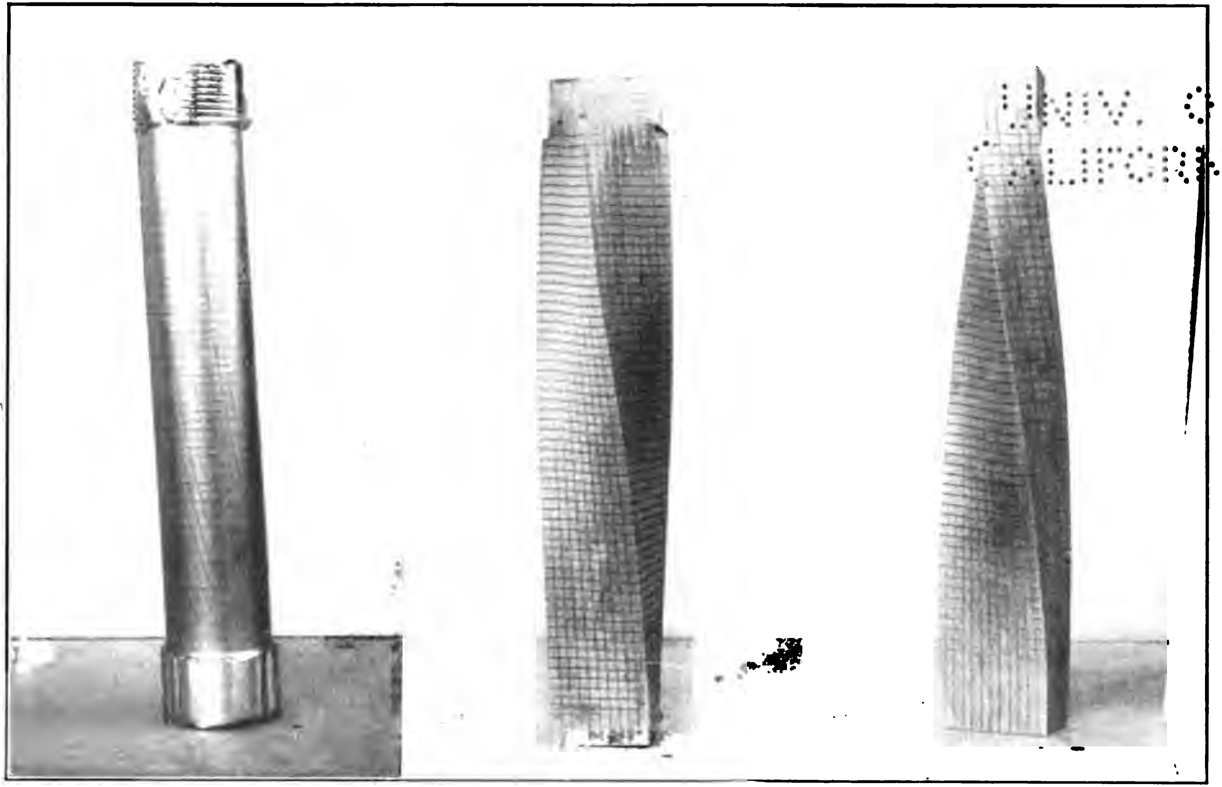


Fig. 1.



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View of three cylinders subjected to torsion beyond the elastic limit. The lines shown were drawn at right angles to each other on the original test pieces before being subjected to torsion. Those lines show the characteristic distortion of sections normal to the axes of the pieces.

(To face page 25.)

It is known from the general theory of stress in a solid body that although there can be no stresses of tension and compression parallel to the axis of a bar under torsion, or at right angles to it, there will be such stresses of varying intensities on oblique planes. Inasmuch as the result of torsion is to slide normal sections each past its neighbor, the elastic torsive shear like any other shear will not change the volume of the body. The principal shearing strains will produce deformation without changing the dimensions whose product gives the volume.

The exact and complete mathematical theory of torsion deduced from the general equations of equilibrium of stresses in an elastic solid, without extraneous assumptions, will be found in App. I. Those formulæ show accurately the state of torsive stress in bars of any elastic material and of various shapes of cross-section. For the general purposes of engineering practice that general demonstration is rather complicated. Hence it is often avoided by making certain approximate assumptions based to some extent on experimental observations which lead to an approximate and simpler theory, yielding formulæ accurate only for the circular normal section, but which are not materially in error for the square section. These formulæ are, however, far from accurate for certain other sections. In this article only the formulæ of the simpler theory, called the common theory of torsion, will be given.

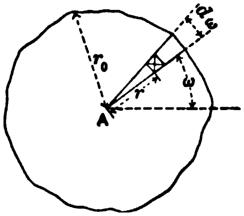

FIG. 2.

Fig. 2 is supposed to represent the normal section of a bar of material of any shape, subjected to torsion by the application of couples as shown in Fig. 1. The fundamental assumptions of the common theory of torsion are that the intensity of shearing stress varies directly as ternal stresses of tension and compression shown by the two shaded triangles. Hence in order that there may be equilibrium the sum of those stresses of tension and compression must be equal to zero. This latter condition will determine in a simple manner the position of the neutral axis. If a is the intensity of either the tensile or compressive stress at the distance unity from the neutral axis, then by the second of the preceding fundamental assumptions the intensity N, at any other distance z from the same axis or line of no stress, will be $N=az$. Again, if A is the area of the normal section of the beam, dA will be the area of an indefinitely small portion of that section, so that the amount of internal stress acting on it will be $az.dA$. If this differential amount of stress be integrated for the entire section, the preceding condition of equilibrium for either portion of the beam requires that the sum represented by that total integration shall be equal to zero; or if d_1 and d represent the distances of the most remote fibres on either side of the neutral axis, the following equations may be written:

$$\int_{-d}^{d_1} azdA = a\int_{-d}^{d_1} zdA = 0,$$

or

$$\int_{-d}^{d_1} zdA = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Eq. (1) shows that the static moment of the entire section about the neutral axis is equal to zero, and therefore that the neutral axis passes through the centre of gravity or the centroid of the normal section.

It is next necessary to determine the expression for the bending moment of the internal stresses of any section, such as JF of Fig. 1, which is induced by and must be equal to the moment of the external forces acting upon either one of the two portions into which the beam is divided by that section.

In Fig. 1, let *mn* represent a differential length, dl of the neutral surface, and let ρ represent the radius of curvature of dl after flexure, also as shown in Fig. 1, C being the centre of curvature. If u is the direct or longitudinal strain of a unit length of fibre at the distance unity from the neutral axis, when stressed with the intensity a, the strain in dl under that intensity will be udl. BD is drawn parallel to JF, and represents the position of BD before flexure. The triangle $D'mD$ is, therefore, similar to Cmn. Consequently there may be written

$$\frac{udl}{1}=\frac{dl}{\rho}; \therefore u=\frac{1}{\rho}. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

Or *the rate of strain*, i.e., the strain of a unit length of fibre at distance unity from the neutral axis, *is equal to the reciprocal of the radius of curvature.*

By the fundamental law or assumption of the common theory of flexure already given

$$\textit{Rate of strain at distance } z=\frac{z}{\rho}.$$

Then, by the fundamental law between stress and strain, the intensity N of the direct stress at any distance z is

$$N=E\frac{z}{\rho}=Euz. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

If b is the variable breadth of section, the differential of the total stress is

$$Nbdz=\frac{E}{\rho}.(bdz).z. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

The differential moment of the internal stresses about the neutral axis will be

$$dM = N . bdz . z = \frac{E}{\rho} . (bdz) . z^2 ; \quad . \quad . \quad . \quad . \quad (5)$$

$$\therefore M = \frac{E}{\rho} \int_{-d}^{d_1} (bdz) . z^2 = \frac{EI}{\rho} ; \quad . \quad . \quad . \quad . \quad (6)$$

in which I is the moment of inertia of the section of the beam about the neutral axis.

If x is the horizontal coordinate of the neutral surface, and w the deflection or sag of the beam at any point, as indicated in Fig. 1, when the curvature is small

$$\frac{1}{\rho} = \frac{d^2w}{dx^2}$$

and

$$M = EI \frac{d^2w}{dx^2} . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Eq. (7) is the fundamental equation by which the deflection of a bent beam is found, whatever may be the character or amount of the loading. As indicated, it is strictly true only when the deflections are small; in other words, when they are produced by strains within the elastic limit of such beams as are ordinarily used in engineering practice. That equation is easily integrated in all ordinary cases, if the value of the external bending moment M is expressed in terms of x, as will be abundantly illustrated in succeeding articles.

Another equation of great practical value remains to be established. Let it first be observed that the intensity of stress a, at the distance of unity from the neutral surface of a bent beam is $a = Eu$, by Hooke's law, and further by eq. (2)

$$a = Eu = \frac{E}{\rho}. \qquad . \ . \ . \ . \ . \ . \ . \qquad (8)$$

If the value of $\frac{E}{\rho}$ from eq. (8) be substituted in eq. (6) there will result

$$M = aI. \qquad . \ . \ . \ . \ . \ . \ . \ . \qquad (9)$$

If the greatest intensity of stress in a normal section of a bent beam at the distance d_1 from the neutral axis be represented by k, then $a = \frac{k}{d_1}$, and eq. (9) will take the form

$$M = \frac{kI}{d_1}. \qquad . \ . \ . \ . \ . \ . \ . \qquad \textcircled{10}$$

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Eq. (10) is one of the most important equations in the whole subject of the resistance of materials in consequence of its frequent use in the practical operation of designing beams or girders. Its employment is rendered exceedingly simple and convenient by tables in which may be found computed the moments of inertia I for all the rolled sections, as well as values of the quantity $\frac{I}{d_1}$, called the "section modulus." These tables are found in the various "Hand-books" published by steel-producing companies, and they obviate essentially all numerical computations for the determination of either moment of inertia or section modulus. Other tables may also be found which give the moments of inertia of a great variety of built sections, i.e., composite sections formed of various commercial rolled shapes such as plates, angles, channels, and I beams.

In all the preceding expressions where the quantity

M appears it is to be taken to represent the bending moment of the external forces, including the reactions, applied to a beam, the moment being taken about the neutral axis of the section under consideration. This external moment must necessarily be equal to the moment of the internal stresses represented by the last members of the preceding moment equations involving the greatest intensity of stress k of the section and the moment of inertia I of the latter.

There are one or two approximate features involved in the preceding analysis, the character of which is not discoverable when the fundamental laws of the theory of flexure are assumed rather than demonstrated, but which appear plainly evident in the true demonstration of the theory of flexure in App. I. It is obvious that the compression produced at the exterior surface of a bent beam at the points of loading is neglected or ignored in the preceding demonstrations; but this does not sensibly affect the accuracy of the formulæ which have been reached. There is, however, one result of the assumptions made which materially affects the accuracy of the formulæ of the common theory of flexure for comparatively short beams. If the accurate analysis be followed it will be found that the formulæ of that theory involve in reality the further assumption that the depth of the beam, i.e., in the direction of the loading, is small in comparison with the length of span. The limit of ratio of length of span to depth above which the formulæ may be applied with strict accuracy cannot be definitely assigned, but there are many beams, especially of timber, employed in engineering practice which are much too short in comparison with their depths to permit an accurate application of the formulæ of the common theory of flexure. This observation bears with special emphasis on computations for pins in pin-connected bridges which are treated as short beams. As a matter of fact, the common theory of flexure cannot be applied to such short thick beams with any degree of accuracy whatever. It is, however, entirely permissible to use these formulæ as general expressions, even under such loosely approximate conditions, into which empirical quantities established under the actual conditions of use are introduced, but they are not to be used in any other way. By such a procedure the formulæ of the common theory of flexure have become of inestimable value to the civil engineer, but it is imperative to realize under what conditions they may be employed with strict accuracy and under what conditions the introduction of quantities established by practical tests is required.

Art. 10.—The Distribution of Shearing Stress in the Normal Section of a Bent Beam.

The longitudinal fibres of a beam under loading take their stresses of tension and compression from the shearing stresses which are induced on vertical and horizontal planes in the interior of the beam. In order to realize what takes place in the interior of a beam let it be supposed to be divided into an indefinitely large number of small rectangular portions like those shown in the upper part of Fig. 1, and on a somewhat larger scale in the lower part. The vertical loading and reactions induce transverse shears, i.e., shearing stresses on vertical transverse planes, which, as known from the general theory of stresses in solid bodies, induce shears of equal intensity on horizontal planes. The result is that which is shown in the lower portion of Fig. 1. On the faces of the indefinitely small rectangular portions of the beam there are induced shears in pairs having the same intensity and acting either toward or from a given edge. Each horizontal layer of the beam is, therefore, made to slide a little over the adjoining layers above and below it, as shown at *A* and *A'* in the lower part of Fig. 1.

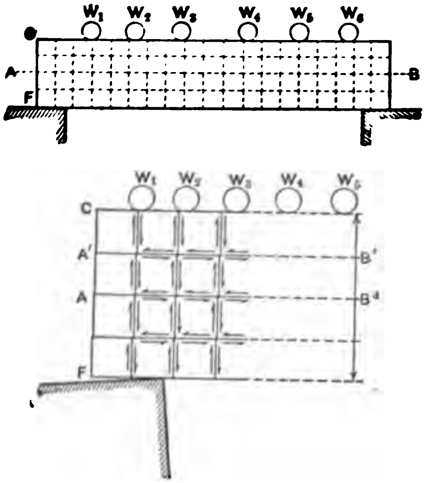


FIG. 1.

Carefully remembering these general conditions, let the bending moment in the section *ad* of the beam in Fig. 2 be represented by M and let the total transverse shear at

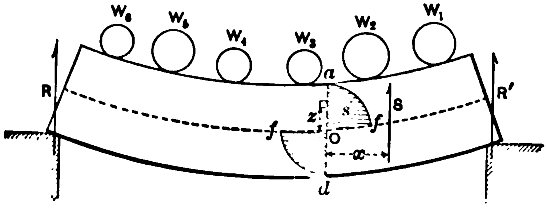


FIG. 2.

the same section be represented by S. Then if x measured horizontally from the section *ad* be so taken that $x = \frac{M}{S}$, and if the intensity of the direct stress of tension or compression at the distance z from the neutral axis be represented by k, there may at once be written

$$M = Sx = \frac{kI}{z}; \quad \therefore\ k = \frac{Sz}{I}x. \quad . \quad . \quad . \quad . \quad (1)$$

k is thus seen to be a function of both z and x. If z be unchanged while x varies, the small variation of k for an indefinitely small variation of x will be

$$\frac{\partial k}{\partial x}dx = \frac{Sz}{I}dx. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If s is the intensity of the transverse shear at the distance z from the neutral axis, the variation of that intensity for the indefinitely short distance dz (x remaining unchanged) will be $\frac{ds}{dz}dz$, and if the breadth or width of the beam is b, the variation of longitudinal shear on the small horizontal area bdx for the small distance dz will be

$$\frac{\partial s}{\partial z}dz(bdx). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The small shear given by expression (3) is equal to the variation of k found by multiplying the members of eq. (2) by bdz, hence

$$\frac{\partial s}{\partial z}dz \,.\, bdx = \frac{Sz}{I}dx \,.\, bdz; \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$\therefore\ \frac{ds}{dz} = \frac{Sz}{I}, \quad \text{or} \quad ds = \frac{S}{I}zdz. \quad . \quad . \quad . \quad . \quad (5)$$

It is obvious that the intensity of the shear at the exterior surface of the beam is zero; in other words, $s = 0$, when $z = d$ the distance of the extreme fibre of the section from the neutral axis. Hence eq. (5) must be integrated between the limits of z and d, and that integration will give

$$\therefore\ s=\frac{S}{I}\int_{z}^{d} z dz=\frac{S}{2I}(d^2-z^2). \quad . \quad . \quad . \quad (6)$$

The intensity s has its maximum value where $z=0$, i.e., at the neutral axis; hence

$$(\text{max.})\ s=\frac{Sd^2}{2I}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad ⑦$$

If the section is rectangular, $I=\frac{8bd^3}{12}=\frac{2bd^3}{3}$

and

$$(\text{max.})\ s=\frac{3}{2}\cdot\frac{S}{2bd}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

In other words, the maximum intensity of shear found at the neutral axis is $\frac{3}{2}$, the average shear of the entire section.

It is to be remembered that this intensity of shear s, at all points in the entire beam, acts on both the vertical and horizontal planes, i.e., this shear acts on longitudinal or horizontal planes parallel to the neutral surface as well as upon the vertical section of the beam.

Eq. (6) is the equation of a parabola with its vertex in the neutral surface. Hence if Of be laid off, as shown in Fig. 2, at any convenient scale to represent the maximum value of s, as given in eq. (7), and if from f as vertices the two branches of parabolic curves fa and fd be described as shown, any horizontal abscissa of the curves drawn from the line ad will represent the intensity of shear at that point. The origin of coordinates for eq. (6) is at O in Fig. 2.

With such material as timber, in the case of beams, the longitudinal shear represented by s in either eq. (7) or eq. (8) may be the governing quantity in design. The capacity of timber to resist shear along its fibres is comparatively so small that where the spans are relatively short failure will take place by shearing along the neutral surface before the extreme fibres yield either in tension or compression. In the design of timber beams, therefore, and in other similar cases, it is necessary to test by computation the maximum value of s as well as to determine the greatest intensity of tensile or compressive stress in the extreme fibres, as will be completely shown in a later article.

Art. 11.—External Bending Moments and Shears in General.

Beams subjected to pure bending only will be treated here.

A beam is said to be *non-continuous* if its extremities simply *rest* at each end of the span and *suffer no constraint whatever.*

A beam is said to be *continuous* if its length is equal to two or more spans, or if its ends, in case of one span (or more) suffer constraint.

A *cantilever* is a beam which overhangs its span, one end of which is in no manner supported. Each of the overhanging portions of an open swing bridge is a cantilever truss.

Fig. 1 represents a beam simply supported at each end, carrying the loads W_1, W_2, W_3, etc. Let bending moments be taken for any section, as JF, at the distance x' from the right-hand abutment, at which location the reaction R' acts. The load W_1 is at the distance x_1 from the section, W_2 at the distance x_2, and W_3 at the distance x_3 from the same section, the last distance not being shown in the figure. The bending moment desired will be the following:

$$M = R'x' - W_1x_1 - W_2x_2 - W_3x_3. \quad . \quad . \quad . \quad (1)$$

This equation is typical of all external bending moments for a beam simply supported at each end, whatever may be the system of loading or its amount, or whatever may be the location of the section. This equation is frequently written in the following form:

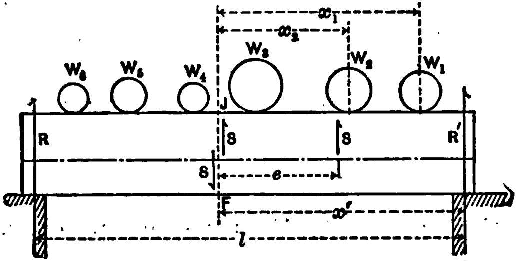


Fig. 1.

$$M = \Sigma Wx. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The summation sign indicates that the sum is to be taken of the products formed by multiplying each external force, whether loading or reaction, by its lever-arm or normal distance from the section in question. It is a common and convenient mode of expressing the general value of the bending moment in any case whatever.

In eq. (1) the differentials of x', x_1, x_2, and x_3 are all equal, so that if that equation be differentiated, the first derivative of M will have the following form:

$$\frac{dM}{dx} = R' - W_1 - W_2 - W_3 = \Sigma W = S. \quad . \quad . \quad (3)$$

It will be at once evident that S in eq. (3) is the total transverse shear in the section for which the bending moment M is written, since the algebraic sum of R' and the loads between the end of the beam and the section constitutes that shear. Indeed, the usual manner of determining the total transverse shear is the simple operation of summing up all the external forces acting on one of the portions of the beam formed by the section in question; the external forces, such as the reaction, acting in one direction being given one sign, and those, like the loading, acting in the other direction being given the opposite sign. The shear, therefore, becomes the numerical difference of the two sets of forces having opposite directions.

Eq. (3) thus establishes the following important principle: *The total transverse shear at any section is equal to the first differential coefficient of the bending moment considered a function of x.*

In Fig. 1 the force S is supposed to be the resultant of the three loads W_1, W_2, and W_3, and the reaction R', i.e., the force S is supposed to represent that resultant both in line of action and magnitude. The bending moment M is, therefore, equal to Se, e being the normal distance of the line of action of S from the section, so that the actual bending moment upon any section of a bent beam may always be represented by the transverse shear, located as the resultant of all the external forces producing the bending moment, multiplied by its lever-arm. This is a simple but important matter of observation.

In the section JF let the two equal and opposite forces S and $-S$, numerically equal, act in opposite directions; they will not, therefore, affect the equilibrium of the beam or any portion of it in any way whatever. As far as the equilibrium of the portion of the beam JF is concerned, the loads and the reactions may be supposed to be displaced by the couple S, $-S$, with the lever-arm e, and the shear S acting upward in the section JF. The importance of this particular feature of the analysis consists in showing that in every bent beam carrying loads the action of the external forces (including the reaction) producing the bending is equivalent to a couple whose moment is Se acting about the neutral axis of the section and the total transverse shear S acting in the section. The shear S evidently tends to move or slide one portion of the beam past the other, and an essential part of the operation of designing beams and trusses is its determination at various sections with correspondingly various positions of loading.

As is well known, the analytical condition for a maximum or minimum bending moment in a beam is

$$\frac{dM}{dx}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

From eqs. (3) and (4) is to be deduced the following principle: *The greatest or least bending moment in any beam is to be found in that section for which the shear is zero.*

The greatest bending moment obviously is the only one of importance in the design of beams and trusses, and eq. (4) shows that the section in which it will be found can be located by simple inspection of the loading. It is only necessary to sum up the reaction at one end and the loads adjacent to it, until the point is reached where the summation is zero. This point will usually be found where a load is supported. In that case the single load may arbitrarily be divided into two parts, supposed to act indefinitely near to each other, so that one of the parts may be just sufficient to make the zero summation desired. A single practical operation will make this feature perfectly clear and simple.

If the loading is uniformly continuous and of the intensity p, in each of the equations (1), (2), and (3) pdx is to be used for each of the separate loads W_1, W_2, W_3, etc. The bending moment thus becomes

$$M = R'x' - \Sigma Wx = R'x' - \int_0^x x.pdx = R'x' - \tfrac{1}{2}px^2. \quad (5)$$

The expression for the shear then becomes

$$\frac{dM}{dx} = S = R' - px. \quad . \quad . \quad . \quad . \quad . \quad (6)$$

A second differentiation gives

$$\frac{d^2M}{dx^2} = -p.. \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Or, the second differential coefficient of the moment considered a function of x is equal to the intensity of the continuous load.

This method of passing from formulæ for concentrated loads to those for continuous loads is perfectly simple and frequently employed.

Art. 12.—Intermediate and End Shears.

The beam shown in Fig. 1 is supposed to carry any loading whatever, and the figure is consequently intended to exhibit a uniform load in addition to a load of concentrations. Inasmuch as all beams and other similar pieces have considerable weight, and sometimes great weight, ordinarily considered uniformly distributed over the span, this condition of loading is that which exists in all actual cases. The amount of uniform loading per linear unit, usually a foot, is represented by p, while the concentrations, as heretofore, are represented by W_1, W_2, etc.

The determination of the transverse shear at any section of a beam or truss is one of the most frequent as well as one of the most important computations required in the design of structures. As has already been indicated, it is an extremely simple computation. It is first necessary, after knowing the position of the loading, to find the reactions at both ends of the span. In Fig. 1 the various weights or loads are separated by the distances shown, a' being the distance from W_1 to the reaction R or end of the span. W_6 is supposed to rest at the right end of the span for a purpose that will presently appear. The reaction R'' at the left end of the span (not shown) resulting from the concentrated loads only will have the following value:

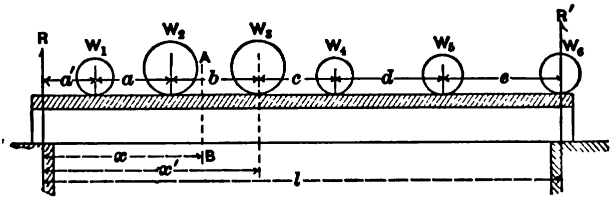


FIG. 1.

$$R'' = W_1\left(\frac{a+b+\ \dots\ +e}{l}\right) + W_2\left(\frac{b+c+\ \dots\ +e}{l}\right) + W_3\left(\frac{c+d+e}{l}\right) + W_4\frac{d+e}{l} + W_5\frac{e}{l}. \quad (1)$$

The reaction R''' at the other end of the span (not shown) can be expressed by a similar equation, but it is simpler and more direct to write it as follows:

$$R''' = W_1 + W_2 + W_3 + W_4 + W_5 - R''. \quad . \ . \quad (2)$$

Obviously the sum of the two reactions R'' and R''' must be equal to the total concentrated loading.

That part of the reaction due to the uniform load extending over the span l will clearly be one half of that load or

$$R_1 = \tfrac{1}{2}pl = R_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The reaction R_1 is supposed to be found at the left end of the span and R_2 at the right end. The total reactions then will be as follows. At left end of the span:

$$R = R'' + \tfrac{1}{2}pl. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

At right end of the span:

$$R' = R''' + \tfrac{1}{2}pl. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The transverse shear at any intermediate section of the beam whatever may now readily be written. Let the section AB at the distance x from the left end of the span first be considered. The total loading between that section and the end of the span is $W_1 + W_2 + px$, and it acts downward. As the reaction R acts upward the expression for the shear will be

$$S = R - W_1 - W_2 - px. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

In this case the section considered has been taken between two weights; let the section at the weight W_3 be considered, that weight being at the distance x' from the end of the span. The amount of uniform load over the length x' is simply px', but inasmuch as the weight W_3 is located at the section under consideration, the portion of that weight which may be taken as resting on the left of the section considered is indeterminate. In such cases it is proper and customary to take any portion or all of the weight as resting on either side of the section, but indefinitely near to it. If it is a case where the maximum shear is desired, the single weight should be taken in such a position as to make the transverse shear as great as possible. If the case is one where it is desired to find the section at which the total load from that section to the end of the span is equal to the reaction, any portion may be taken which is found necessary to make the equality. If, for instance, $px' + W_1 + W_2$ is less than R while $px' + W_1 + W_2 + W_3$ is greater than R, then that portion of W_3 which would be considered on the left of the section but indefinitely near to it would be $R - px' - W_1 - W_2$. The remaining portion of W_3 would be considered as resting at the right of the section but indefinitely near to it. In such a case the transverse shear is zero at the weight W_3.

Again, let it be desired to find the greatest upward shear at W_3, it being supposed that R is greater than the total load between W_3 and the left end of the span. In this case no portion of W_3 would be considered as acting to the left of the section, but the expression for the shear would be

$$S = R - px' - W_1 - W_2. \quad \ldots \ldots \quad (7)$$

It can be seen from the preceding statements that the maximum transverse shear in the beam will occur at the ends of the span where the value of the shear is the end reaction. Inasmuch as the end reaction R or R' is thus the greatest shear in the entire span, it is a most important quantity to determine in the design of beams and trusses; it is the most important single factor in the determination of the amount of material required at the end sections of both beams and trusses. The value of this end shear is given by the values for R and R' in eqs. (4) and (5).

Since the total transverse shear in any section of a beam is simply the summation of all the external loads, including the reactions from one end of the span up to the section considered, it is evident, first, that that summation may be made from either end of the span, and second, that the amounts so found will be equal numerically but affected by opposite signs. In determining the shear, therefore, in any given case, it is usual to make the summation from that end of the span which can be used with the greatest convenience in computation.

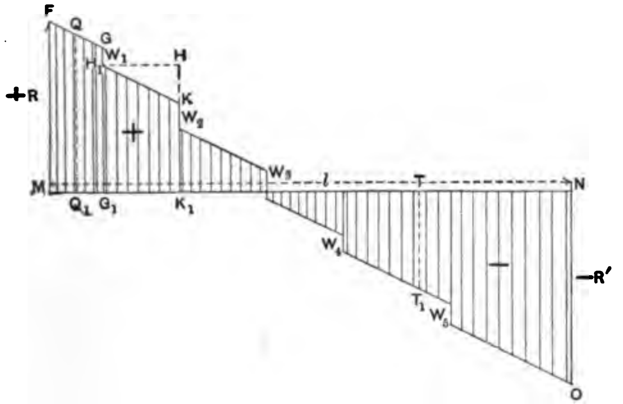


FIG. 2.

Fig. 2 exhibits a graphical representation of the preceding treatment of intermediate and end shears, MN being the length of span shown in Fig. 1. MF is the reaction R laid off at a convenient scale. The weights or loads W_1, W_2, W_3, etc., are laid off vertically downward in their proper locations at the same scale, as shown. The vertical distance of G below F is the amount of uniform load pa' between R and W_1 in Fig. 1, also laid down by the same scale. GG_1 is, therefore, the shear in the beam of

Fig. 1 immediately to the left of W_1, and H_1G_1 is the shear immediately to the right of the same load. Similarly, H_1H being drawn horizontally, HK is the amount of uniform loading pa between W_1 and W_2. The remainder of the diagram is drawn in the same manner.

Any vertical ordinate drawn from MN either up or down to the broken line $FGH_1K \dots O$ represents the shear at the corresponding point in the span at the same scale used in laying off the reactions and loads. QQ_1 is the shear at the point or section of beam at Q_1, while TT_1 is the shear at the section T. The shear is zero at W_3 where it changes its sign. At that point also will be found the greatest bending moment in the beam.

As the diagram is drawn the shears on the left of W_3 and above MN are positive, those on the right of W_3 and below MN being negative; but the diagram might have been drawn with equal propriety so as to have made R' and the shears between it and W_3 positive and those between that load and R negative.

A glance at the diagram shows that the end shears, equal to the reactions, are the greatest in the span.

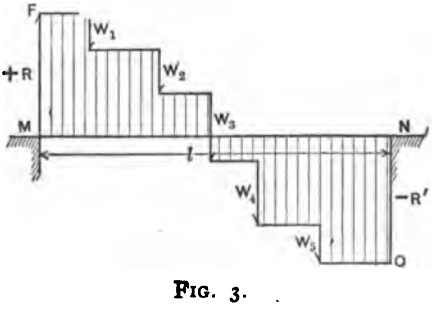


FIG. 3.

If a beam carries a load of concentrations only its shear diagram will be illustrated by Fig. 3, in which there are five loads, the diagram being composed of rectangles only. If, again, the load is wholly uniform Fig. 4 will represent the shear diagram composed of two triangles with their apices at C, the centre of the span and point of no shear. Any vertical ordinate drawn from MN in either figure

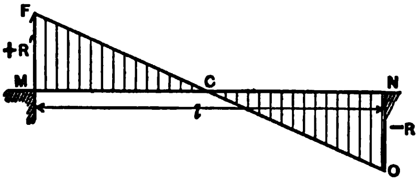


FIG. 4.

to the stepped line in the one case and to the straight line in the other will represent the shear at the section of beam from which the ordinate is drawn. Those diagrams represent completely the graphical treatment of shears in all cases.

Art. 13.—Maximum Reactions for Bridge Floor Beams.

Three transverse floor beams of a railroad bridge are represented in Fig. 1 separated by the two spans l_1 and l which, in a bridge, represent the panel lengths. The members AB and BC supporting the weights W_1, W_2, etc., indicate the stringers which carry the railroad track and the train. The two beams or stringers AB and BC are considered simple non-continuous beams resting on the floor beams, but not necessarily nor usually on their tops. The problem is to determine the position of the locomotive or other train loads on the adjacent two short spans l_1 and l, so that the reaction R on the floor beam between shall have its greatest value.

In Fig. 1 let a section of the beam be shown at R, and let the x's be measured from the right and left ends of l and l_1, while W_1, W_2, etc., W, W', etc., represent the

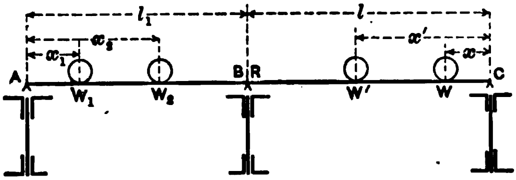


FIG. 1.

weights or wheel concentrations resting on the two spans The reaction R will then have the value

$$R=\frac{W_1x_1+W_2x_2+\text{etc.}}{l_1}+\frac{Wx+W'x'+\text{etc.}}{l}. \quad . \quad (1)$$

If the whole system of loading move to the left by the distance Δx, the new reaction will be

$$R'=R-\frac{(W_1+W_2+\text{etc.})\,\Delta x}{l_1}+\frac{(W+W'+\text{etc.})\,\Delta x}{l}.$$

In that position, which gives a maximum or minimum, $R'-R=0$; hence

$$(W_1+W_2+W_3+\text{etc.})\frac{l}{l_1}=(W+W'+W''+\text{etc.}). \quad . \quad (2)$$

It will seldom happen that eq. (2) will be satisfied unless a concentration rest on the point R, so that the proper portion of it may be taken for one span or the other, precisely as in the problems of maximum shear and maximum moments.

Ordinarily the two adjacent spans are equal, or $l=l_1$;

$$\therefore \; W_1+W_2+\text{etc.}=W+W'+\text{etc.} \quad . \quad . \quad . \quad . \quad (3)$$

Eq. (3) shows that when the two spans are equal, the amounts of load each side of R must also be equal.

After the proper position of loading has been determined, eq. (1) will give the maximum reaction desired.

In computing this reaction it is to be remembered that there will usually be a whole weight resting at B which is to be added to the amounts given by eq. (1) in order to find the total reaction on the floor beam at B.

Art. 14.—Greatest Bending Moment Produced by Two Equal Weights.

Fig. 1 represents a non-continuous beam with the span l

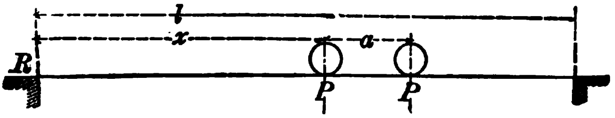


FIG. 1.

supporting two equal weights P, P. These two weights or loads are to be kept at a constant distance apart denoted by a.

It is required to find that position of the two loads which will cause the greatest bending moment to exist in the beam, and the value of that moment. The reaction R is to be found by the simple principle of the lever. Its value will therefore be

$$R = \frac{l - \left(x + \frac{a}{2}\right)}{l} . 2P. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Since the reaction R can never be equal to $2P$, ΣP, or the shear, must be equal to zero at the point of application of one of the loads P. In searching for the greatest moment, then, it will only be necessary to find the moment about the point of application of one of the forces P. It will be most convenient to take that one nearest R.

The moment desired will be

$$M = Rx = 2P\left(x - \frac{x^2}{l} - \frac{ax}{2l}\right). \quad . \quad . \quad . \quad . \quad (2)$$

$$\therefore \frac{dM}{dx} = 0 = 2P\left(1 - \frac{2x}{l} - \frac{a}{2l}\right);$$

$$\therefore x = \frac{l}{2} - \frac{a}{4}.$$

This value in eq. (2) gives

$$M_1 = \frac{P}{2l}\left(l - \frac{a}{2}\right)^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Since

$$\frac{d^2M}{dx^2} = -\frac{4P}{l},$$

it appears that M_1 is a maximum.

The shear S in the section RP of the span will be the reaction R as given by eq. (1):

$$S = 2P - \frac{2P}{l}\left(x + \frac{a}{2}\right). \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Throughout the section a the shear S' will be

$$S' = S - P = P - \frac{2P}{l}\left(x + \frac{a}{2}\right). \quad . \quad . \quad . \quad . \quad (5)$$

Finally, between the right abutment and the nearest weight the shear S_1 will be

$$S_1 = S - 2P = -\frac{2P}{l}\left(x + \frac{a}{2}\right). \quad . \quad . \quad . \quad . \quad (6)$$

If the separating distance, a, between the two weights be increased a value may be reached so great as to make the bending moment of the pair of weights less than that produced by placing one of them at the centre of the span. This limiting value of a may easily be found. The moment at the centre of span produced by placing a single weight P there is

$$M' = \frac{P}{2} \cdot \frac{l}{2} = \frac{Pl}{4}.$$

By using eq. (3)

$$M' = M_1; \quad \therefore \frac{Pl}{4} = \frac{P}{2l}\left(l - \frac{a}{2}\right)^2. \quad . \quad . \quad . \quad (7)$$

By solving this equation

$$a = l(2 - \sqrt{2}) = .586l. \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Whenever, therefore, the separating distance a is equal to or greater than .586 span length, the moment should be found by placing a single weight P at the centre of the span.

Art. 15.—Position of Uniform Load for Greatest Shear and Greatest Bending Moment at any Section of a Non-Continuous Beam — Bending Moments of Concentrated Loads.

A continuous load of uniform density is frequently employed in structural operations both for beams and trusses, and it is essential to place such a load so as to produce the greatest effect both for shears and moments. The position of loading for the greatest shear will first be found.

A continuous train of any given uniform density advances along a simple beam of span l. It is required to determine what position of loading will give the greatest shear at any specified section.

In Fig. 1, CD is the span l, and A is any section for which it is required to find the position of the load for the greatest transverse shear. The load is supposed to advance continuously from C to any point B. Let R be the reaction at D, and ΣP the load between A and B. The shear S' at A will be

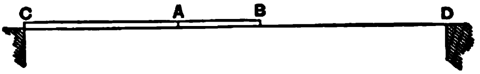


FIG. 1.

$$R-\Sigma P=S'. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Let R' be that part of R which is due to ΣP, and R'' that part due to the load on CA; evidently R' is less than ΣP. Then

$$R'+R''-\Sigma P=S'.$$

If AB carries no load, R' and ΣP disappear in the value of S. Hence

$$R''=S$$

is the shear for the head of the train at A. S is greater than S' because ΣP is greater than R'. But no load can be taken from AC without decreasing R''. Hence *the greatest shear at any section will exist when the load extends from the end of the span to that section, whatever be the density of the load.*

In general, the section will divide the span into two unequal segments. The load also may approach from either direction. The greater or smaller segment, then, may be covered, and, according to the principle just established, either one of these conditions will give a maximum shear. A consideration of these conditions of loading in connection with Fig. 1, however, will show that *these greatest shears will act in opposite directions.*

When the load covers the greater segment the shear is called a *main* shear; when it covers the smaller, it is called a *counter* shear.

The determination of the greatest bending moment at any section A of a beam or truss, exemplified by Fig. 1, traversed by a continuous train of uniform density is a very simple matter. It is clear that every part of the uniform load resting on the beam will produce bending at any section considered; and it is further obvious that every part of that uniform loading will create a bending moment at A of the same sign. It follows, therefore, that the entire span should be covered by the uniform train in order to produce a maximum bending moment at any section of the beam or truss, and that this single position of the train will give the maximum bending moment throughout the entire span.

The preceding position of moving load is taken only for a train of uniform density or for a series of uniform concentrations, each pair of which is separated by the same distance as every other pair, i.e., for a uniformly distributed system of uniform concentrations.

The general case of a simple beam loaded with any system of weights may be represented by Fig. 2, in which the beam carries three loads W_1, W_2, and W_3, spaced as shown. The reactions or supporting forces R and R' are determined in the usual manner by the law of the lever. Hence

$$R = W_3\frac{d}{l} + W_2\frac{d+c}{l} + W_1\frac{d+c+b}{l}. \quad . \quad . \quad . \quad (2)$$

A similar value may be written for R', but it is simpler after having found one reaction to write

$$R' = W_1 + W_2 + W_3 - R. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The beam itself being supposed to have no weight, the bending moments at the points of application of the loads will be

$$\left.\begin{aligned} M_1 &= Ra, \\ M_2 &= R(a+b) - W_1 b, \\ M_3 &= R(a+b+c) - W_1(b+c) - W_2 c. \end{aligned}\right\} \quad . \quad . \quad (4)$$

After substituting the value of R from eq. (2) in eqs. (4) the moments in the latter equations will be completely known.

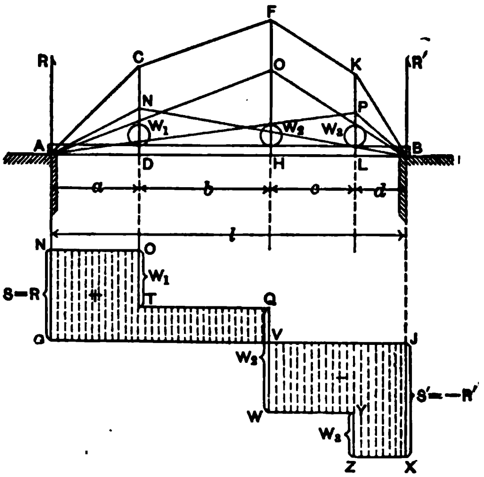


FIG. 2.

The bending moment produced by each weight will be represented by the ordinates of the triangles shown in Fig. 2, the resultant moments at the points of application of the weights being given by eqs. (4). The ordinate *CD* represents M_1 in eqs. (4) by any convenient scale. Similarly *FH* represents M_2 in eqs. (4), and *KL*, M_3. The lines *AC*, *CF*, *FK*, and *KB* are then drawn. Any vertical intercept between *AB* and the polygon *ACFKB*, found in the manner explained, will represent the bending moment at the point where the intercept is drawn, and to the scale at which M_1, M_2, and M_3 are laid down. This intercept is simply the sum of the intercepts of the triangles, each representing the partial bending moment due to a single weight.

Obviously the bending moments of any number of loads of any magnitude or of a uniform load, even, may be treated or represented in the same manner.

The lower portion of Fig. 2 is the shear diagram drawn precisely as explained for Fig. 3 of Art. 12.

Art. 16.—Greatest Bending Moment in a Non-Continuous Beam Produced by Concentrated Loads.

The position of the moving load for the greatest bending moment at any section of a non-continuous beam may be very simply determined. In Fig. 1, let *FG* represent any such beam of the span l, and let any moving load whatever, as $W_1 \ldots W_{n'} \ldots W_n$ advance from *F* toward *G*. Let *C* be the section at which it is desired to determine the maximum bending moment, and let n' loads rest to the left of *C*, while n is the total number of loads on the span. Finally, let x' represent the distance of $W_{n'}$ from *C* and to the left of that point, while x is the distance of W_n to the left of *F*. If a is the distance between W_1 and W_2, b the distance between W_2 and W_3, c the distance between W_3 and W_4, etc., the reaction R at *G* will be

$$R=\begin{cases} W_1\dfrac{a+b+c+\ \ldots\ +x}{l} \\ +W_2\dfrac{b+c+\ \ldots\ +x}{l} \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ +W_n\dfrac{x}{l} \end{cases} \quad .\ .\ .\ .\ \quad (1)$$

The bending moment M about C will then take the value

$$M=Rl'-\begin{cases} W_1(a+b+c+\ \ldots\ +x') \\ +W_2(\quad b+c+\ \ldots\ +x') \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ +W_{n'}x'. \end{cases}$$

Or, after inserting the value of R from above,

$$\left.\begin{aligned} M&=\frac{l'}{l}[W_1a+(W_1+W_2)b+(W_1+W_2+W_3)c \\ &+\ \ldots\ +(W_1+W_2+W_3+\ \ldots\ +W_n)x] \\ &-W_1a-(W_1+W_2)b-(W_1+W_2+W_3)c \\ &-\ \ldots\ -(W_1+W_2+W_3+\ \ldots +W_{n'})x'. \end{aligned}\right\}\quad .\quad (2)$$

If the moving load advances by the amount Δx, the moment becomes, since $\Delta x=\Delta x'$,

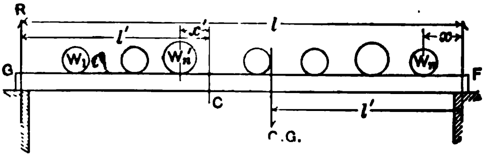


Fig. 1.

$$M'=M+\frac{l'}{l}(W_1+W_2+W_3+\ \ldots\ +W_n)\Delta x \\ -(W_1+W_2+\ \ldots\ +W_{n'})\Delta x. \quad (3)$$

Hence, for a maximum, the following value must ~~never~~ be ~~negative~~: zero.

$$M' - M = \Delta x \left\{ \frac{l'}{l}(W_1 + W_2 + W_3 + \ldots + W_n) - (W_1 + W_2 + \ldots + W_{n'}) \right\} = 0. \tag{4}$$

Or the desired condition for a maximum takes the form

$$\frac{l'}{l} = \frac{W_1 + W_2 + \ldots + W_{n'}}{W_1 + W_2 + W_3 + \ldots + W_n}. \quad . \quad . \quad . \tag{⑤}$$

impr

It will seldom or never occur that this ratio will exactly exist if $W_{n'}$ is supposed to be a *whole* weight; hence $W_{n'}$ will usually be that part of a whole weight at C which is necessary to be taken in order that the equality (5) may hold.

It is to be observed that if the moving load is very irregular, so that there is a great and arbitrary diversity among the weights W, there may be a number of positions of the moving load which will fulfil eq. (5); some one of which will give a value greater than any other; this is the absolute maximum desired.

From what has preceded, it follows that $W_{n'}$ may always be taken at the point C in question; hence x' in eq. (2) may always be taken equal to zero when that equation expresses the greatest value of the moment. The latter may then take either of the two following forms:

$$\left. \begin{aligned} M = \frac{l'}{l}[W_1 a + (W_1 + W_2)b + \ldots + (W_1 + W_2 \\ + \ldots + W_n)x] - W_1 a - (W_1 + W_2)b \\ - \ldots - (W_1 + W_2 + \ldots + W_{n'-1})(?) \end{aligned} \right\}. \tag{⑥}$$

$$\left. \begin{aligned} M = \frac{l'}{l}[W_1(a + b + \ldots + x) + W_2(b + c + \ldots + x) \\ + W_3(c + d + \ldots + x) + \ldots + W_n x] \\ - W_1(a + b + \ldots + ?) - W_2(b + \ldots + ?) \\ - \ldots - W_{n'-1}(?) \end{aligned} \right\}. \tag{6a}$$

In these equations x corresponds to the position of maximum bending, while the sign (?) represents the distance between the concentrations $W_{n'-1}$ and $W_{n'}$.

It has already been shown that for any given condition of loading the greatest bending moment in the beam will occur at that section for which the shear is zero. But if the shear is zero, the reaction R must be equal to the sum of the weights $(W_1+W_2+\ldots+W_{n'})$ between G and C; the latter now being the section at which the greatest moment in the span exists.

Hence for that section eq. (5) will take the form

$$\frac{l'}{l}=\frac{R}{W_1+W_2+W_3+\ldots+W_n}. \quad . \quad . \quad . \quad (7)$$

But if $c.g.$ is the centre of gravity of the load in Fig. 1, and if its distance from F is l',

$$R=\frac{l'}{l}(W_1+W_2+\ldots+W_n). \quad . \quad . \quad . \quad . \quad (8)$$

This equation in connection with eq. (7) shows that the centre of gravity of the load is at the same distance from one end of the beam as the section or point of greatest bending is from the other. In other words, *the distance between the point of greatest bending for any given system of loading and the centre of gravity of the latter is bisected by the centre of span.*

If the load is uniform, therefore, it must cover the whole span.

It is to be observed that eq. (6) is composed of the sums W_1, W_1+W_2, etc., multiplied by the distances a, b, c, etc. Again, as in the equation immediately preceding eq. (2), the expression for the moment, M, may be taken as composed of the positive products of each of the single weights

W_1, W_2, etc., multiplied by its distance from any point distant x to the right of W_n and of the negative products similarly taken in reference to the section located by x', as shown by eq. (6*a*).

The practical application of the preceding formulæ can therefore best be effected by means of a tabulation of moments like that shown in Table I, taken from the standard specifications of the N. Y. C. & H. R. R. R. Co. for 1902. The wheel weights and train loads shown in the table are for one rail only, i.e., they are half those for one track. By comparing the weights and spacings with those in Fig. 1 and eq. (6) it will be seen that $W_1 = 10{,}000$ lbs.; $W_2 = 20{,}000$ lbs.; $W_3 = 20{,}000$ lbs., etc., and that $a = 8$ ft.; $b = 5$ ft.; $c = 5$ ft., etc.

In using tables similar to Table I for shears as well as for moments it is convenient to have the summations of weights, distances between weights or spacings of loads, and the moments already described from each extremity of the system or series. An examination of the table with the explanatory notes at the bottom will make its composition perfectly clear. The figures on the right of the heavy stepped line are found by taking moments of each load or weight proceeding from load 10 to the right. For example: $20{,}000 \times 8 = 160{,}000$; $20{,}000 \times (8+5) + 20{,}000 \times 8 = 420{,}000$; $20{,}000 \times (5+5+8) + 20{,}000 \times (5+8) + 20{,}000 \times 8 = 780{,}000$. Also in line 10, $13{,}000 \times (5+9) + 13{,}000 \times 9 = 299{,}000$, etc. Again, beginning at the right end of line 5, $13{,}000 \times 5 = 65{,}000$; $13{,}000 \times (5+5) + 13{,}000 \times 5 = 195{,}000$, etc. All the lines from 5 to 14 are made up in a similar manner.

The method of finding shears set forth in Art. 12 and of finding the greatest bending moments by eqs. (5), (6), and (6*a*) of this article, by the aid of Table I, leads to the numerical results shown in Table II. The series of concentrated loadings exhibited in Tables I and II are identical, and they are given in the specifications of Mr. Theodore Cooper, C.E., as the locomotive loading "E 40." It will be observed that the entire train for Table II is composed of two coupled locomotives followed by a uniform train of 2000 lbs. per linear foot for each rail, while in Table I the train load is 2250 lbs. per linear foot.

Table II is taken from a paper by Mr. O. E. Selby, Jun. Am. Soc. C. E., in the "Trans. Am. Soc. C. E.," Vol. XLII, 1899. In general, the computation of maximum shears at arbitrary points like quarter-points and the centre is largely tentative by the use of the general methods of Art. 12, but Mr. Selby obtained his shears (of Table II) by means of equivalent loads, which he explains in his paper.

When a uniform train load is a part of the system of loading it is only necessary to consider any desired portion of it as acting through its centre of gravity, i.e., through its mid-point. Taking that centre as its point of application the separating space is the distance from that point to the nearest concentration. If in Table II 20 feet of train load be taken, that train weight will be 40,000 pounds applied at the distance $10+5=15$ feet from load 18. This simple operation is all that is needed for any uniform load or for a series of sections of uniform load.

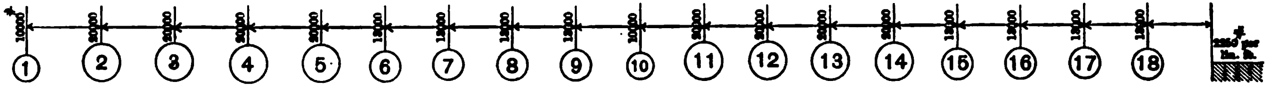
PROBLEM.

Let a railroad plate girder with an effective span of 80 feet be traversed from right to left by the moving load shown in Table II, except that there will be taken but one locomotive followed by the uniform train load of 2000 pounds per linear foot. It is required to find the greatest bending moments and shears at the centre and quarter-points of the span, the dead load or own weight of the span and track being taken at 1400 pounds per linear foot.

TABLE I.

CONSOLIDATION ENGINE, N. Y. C. & H. R. R. R.—1902.

10	30	50	70	90	103	116	129	142	152	172	192	212	232	245	258	271	284	1
284	274	254	234	214	194	181	168	155	142	132	112	92	72	52	39	26	13	2
8	13	18	23	32	37	43	48	56	64	69	74	79	88	93	99	104	109	3
109	101	96	91	86	77	72	66	61	53	45	40	35	30	21	16	10	5	4
8′	5′	5′	5′	9′	5′	6′	5′	8′	8′	5′	5′	5′	9′	5′	6′	5′	5′	



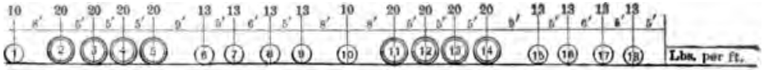
16364	15274	13254	11334	9514	7794	6793	5857	4999	4206	3676	2776	1976	1276	676	403	195	65	5
14944	13904	11984	10164	8444	6824	5888	5017	4224	3496	3016	2216	1516	916	416	208	65		6
13589	12599	10779	9059	7439	5919	5048	4242	3514	2851	2421	1721	1121	621	221	78		65	7
12041	11111	9411	7811	6311	4911	4118	3390	2740	2155	1785	1205	725	345	65		78	221	8
10816	9936	8336	6836	5436	4136	3408	2745	2160	1640	1320	840	460	180		65	208	416	9
8728	7938	6518	5198	3978	2858	2247	1701	1233	830	600	300	100		117	209	559	884	10
7668	6928	5608	4388	3268	2248	1702	1221	818	480	300	100		100	282	529	854	1244	11
6708	6018	4798	3678	2658	1738	1257	841	503	230	100		100	300	547	859	1249	1704	12
5848	5208	4088	3058	2148	1328	912	561	288	80		100	300	600	912	1289	1744	2264	13
4632	4072	3112	2252	1492	832	520	273	104		160	420	780	1240	1656	2137	2696	3320	14

Line 1. Summation of loads from left end. 2. Summation of loads from right end. 3. Summation of distances from left end. 4. Summation of distances from right end. 5. Moments about end of uniform load. 6 to 14. Moments about wheels 18 to 10 respectively. Figures to left of heavy line are moments of loads to left of wheel indicated. Figures to right of heavy line are moments of loads to right of wheel indicated. Moments in thousands of Ft.-lbs. Summations of loads in thousands of pounds.

TABLE II.

MAXIMUM MOMENTS AND SHEARS IN PLATE GIRDERS.

Moments in Thousands of Ft.-lbs. Shears in Thousands of Lbs. Weights on One Rail.



Span, Feet.	Moment.	Shears.			Span, Feet.	Moment.	Shears.			Span, Feet.	Moment.	Shears.			Span, Feet.	Moment.	Shears.		
		End.	¼ P't.	Center.			End.	¼ P't.	Center.			End.	¼ P't.	Center.			End.	¼ P't.	Center.
....	......	......	......	......	36	548.6	70.6	44.1	20.6	66	1539.5	105.0	63.6	29.7	96	2993.9	144.8	83.3	38.3
....	......	......	......	......	37	574.3	71.9	44.8	21.0	67	1581.5	106.4	64.2	30.0	97	3049.0	146.2	84.2	38.5
8	40.0	27.5	17.5	10.0	38	600.0	73.1	45.4	21.3	68	1623.5	107.8	64.8	30.2	98	3106.5	147.5	85.0	38.8
9	46.9	28.0	18.9	10.0	39	626.6	74.3	46.0	21.7	69	1665.5	109.2	65.4	30.5	99	3162.3	148.8	85.8	39.1
10	56.3	30.0	20.0	10.0	40	655.6	75.4	46.8	22.0	70	1706.9	110.5	65.9	30.7	100	3219.9	150.0	86.6	39.4
11	65.7	32.7	20.9	10.9	41	684.6	76.8	47.5	22.3	71	1749.3	111.8	66.5	31.1	101	3277.6	151.2	87.3	39.6
12	80.0	35.0	21.7	11.7	42	713.6	78.1	48.2	22.6	72	1792.2	113.3	67.0	31.4	102	3335.9	152.4	88.1	39.9
13	95.0	36.9	22.3	12.3	43	742.6	79.4	48.9	22.9	73	1833.9	114.8	67.5	31.7	103	3410.6	153.7	88.8	40.1
14	110.0	38.6	23.6	12.9	44	771.6	80.6	49.5	23.2	74	1879.2	116.3	68.0	32.0	104	3475.2	154.9	89.5	40.4
15	125.0	40.0	25.0	13.3	45	800.6	81.7	50.1	23.4	75	1925.8	117.7	68.6	32.3	105	3537.6	156.1	90.3	40.6
16	140.0	42.5	26.3	13.8	46	829.6	82.8	50.7	23.7	76	1972.0	119.1	69.2	32.6	106	3600.3	157.3	90.9	40.9
17	155.0	44.7	27.4	13.8	47	858.6	83.8	51.4	23.9	77	2019.1	120.4	69.9	32.9	107	3666.6	158.5	91.7	41.1
18	170.0	46.7	28.3	13.9	48	887.6	84.8	52.1	24.2	78	2064.4	121.7	70.5	33.2	108	3745.3	159.6	92.4	41.3
19	186.6	48.4	29.2	14.0	49	918.8	86.0	52.8	24.5	79	2112.3	123.0	71.1	33.4	109	3818.4	160.8	93.2	41.6
20	206.3	50.0	30.0	14.0	50	950.9	87.1	53.5	24.9	80	2160.5	124.2	71.7	33.7	110	3886.8	162.0	93.9	41.8
21	226.0	51.4	31.4	14.5	51	983.1	88.2	54.1	25.2	81	2207.7	125.6	72.3	34.0	111	3958.2	163.2	94.6	42.0
22	245.7	52.7	32.7	15.0	52	1015.2	89.2	54.8	25.5	82	2256.7	126.9	73.0	34.4	112	4026.9	164.4	95.3	42.2
23	265.4	53.9	33.9	15.4	53	1047.4	90.2	55.4	25.8	83	2306.5	128.2	73.7	34.7	113	4099.0	165.5	96.0	42.5
24	285.2	55.0	35.0	15.8	54	1081.4	91.2	56.1	26.1	84	2356.3	129.5	74.4	35.0	114	4172.0	166.7	96.8	42.8
25	305.0	56.5	36.0	16.2	55	1116.9	92.1	56.8	26.4	85	2406.9	130.7	75.1	35.3	115	4245.0	167.9	97.5	43.1
26	324.8	57.9	36.9	16.5	56	1152.4	93.0	57.5	26.6	86	2459.6	132.1	75.8	35.6	116	4318.8	169.0	98.3	43.4
27	344.6	59.2	37.8	16.9	57	1187.9	94.2	58.2	26.9	87	2510.6	133.4	76.5	35.9	117	4389.5	170.2	99.0	43.7
28	365.5	60.4	38.6	17.1	58	1223.4	95.4	58.8	27.2	88	2564.2	134.7	77.1	36.2	118	4463.8	171.4	99.7	43.9
29	388.0	61.6	39.3	17.4	59	1261.0	96.5	59.5	27.5	89	2615.9	136.0	77.9	36.5	119	4538.8	172.5	100.4	44.2
30	410.5	63.0	40.0	17.7	60	1299.6	97.6	60.1	27.9	90	2670.5	137.2	78.7	36.7	120	4614.1	173.7	101.1	44.5
31	432.9	64.4	40.7	18.2	61	1338.3	98.7	60.7	28.2	91	2723.0	138.5	79.5	37.0	121	4686.5	174.8	101.8	44.7
32	455.4	65.7	41.3	18.8	62	1377.0	100.0	61.3	28.5	92	2775.9	139.8	80.3	37.3	122	4762.7	176.0	102.5	45.0
33	477.9	66.9	42.0	19.2	63	1415.6	101.3	61.8	28.8	93	2831.5	141.1	81.0	37.5	123	4836.2	177.1	103.2	45.3
34	500.4	68.1	42.8	19.7	64	1455.5	102.6	62.4	29.1	94	2885.3	142.4	81.7	37.8	124	4917.4	178.3	104.0	45.7
35	523.0	69.2	43.5	20.1	65	1497.5	103.8	63.0	29.4	95	2939.5	143.6	82.5	38.0	125	4996.4	179.4	104.7	46.0

Dead Load.

bending moments at the
the reaction, R, is

Centre,
$x = \frac{1}{2}l = 40$ ft.
560,000 ft.-lbs

end, quarter-point,

rter-point,	Centre,
$= 20$ ft.	$x = 40$ ft.
,000 lbs.	zero

r-point there will be
ain load on the span,
tender wheel, which
s $\frac{l'}{l} = \frac{1}{4}$, the criterion,

$\frac{,000}{,000}$, the first being
lence wheel 3 at the
for the maximum
), $W_n = 40,000$ lbs.
lbs. Hence the

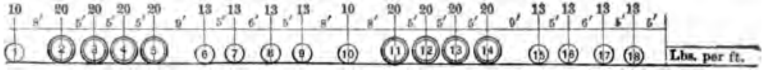
$\times 10) - 230,000$
$= 1,631,500$ ft.-lbs.

of the span ten feet
e girder, making a

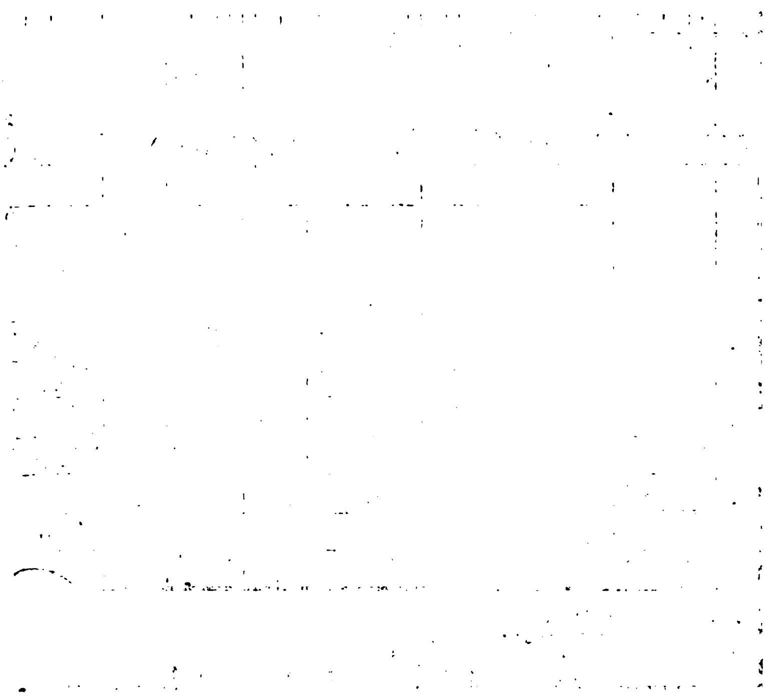
TABLE II.

MAXIMUM MOMENTS AND SHEARS IN PLATE GIRDERS.

Moments in Thousands of Ft.-lbs. Shears in Thousands of Lbs. Weights on One Rail.



Span, Feet.	Moment.	Shears.			Span, Feet.	Moment.	Shears.			Span, Feet.	Moment.	Shears.			Span, Feet.	Moment.	Shears.		
		End.	¼ P't.	Center.			End.	¼ P't.	Center.			End.	¼ P't.	Center.			End.	¼ P't.	Center.
....	......	......	......	......	36	548.6	70.6	44.1	20.6	66	1539.5	105.0	63.6	29.7	96	2993.9	144.8	83.3	38.3
....	......	......	......	......	37	574.3	71.9	44.8	21.0	67	1581.5	106.4	64.2	30.0	97	3049.0	146.2	84.2	38.5
8	40.0	27.5	17.5	10.0	38	600.0	73.1	45.4	21.3	68	1623.5	107.8	64.8	30.2	98	3106.5	147.5	85.0	38.8
9	46.9	28.9	18.9	10.0	39	626.6	74.3	46.0	21.7	69	1665.5	109.2	65.4	30.5	99	3162.3	148.8	85.8	39.1
10	56.3	30.0	20.0	10.0	40	655.6	75.4	46.8	22.0	70	1706.9	110.5	65.9	30.7	100	3219.9	150.0	86.6	39.4
11	65.7	32.7	20.9	10.9	41	684.6	76.8	47.5	22.3	71	1749.3	111.8	66.5	31.1	101	3277.6	151.2	87.3	39.6
12	80.0	35.0	21.7	11.7	42	713.6	78.1	48.2	22.6	72	1792.2	113.3	67.0	31.4	102	3335.9	152.4	88.1	39.9
13	95.0	36.9	22.3	12.3	43	742.6	79.4	48.9	22.9	73	1833.9	114.8	67.5	31.7	103	3410.6	153.7	88.8	40.1
14	110.0	38.6	23.6	12.9	44	771.6	80.6	49.5	23.2	74	1879.2	116.3	68.0	32.0	104	3475.2	154.9	89.5	40.4
15	125.0	40.0	25.0	13.3	45	800.6	81.7	50.1	23.4	75	1925.8	117.7	68.6	32.3	105	3537.6	156.1	90.3	40.6
16	140.0	42.5	26.3	13.8	46	829.6	82.8	50.7	23.7	76	1972.0	119.1	69.2	32.6	106	3600.3	157.3	90.9	40.9
17	155.0	44.7	27.4	13.8	47	858.6	83.8	51.4	23.9	77	2019.1	120.4	69.9	32.9	107	3666.6	158.5	91.7	41.1
18	170.0	46.7	28.3	13.9	48	887.6	84.8	52.1	24.2	78	2064.4	121.7	70.5	33.2	108	3745.3	159.6	92.4	41.3
19	186.6	48.4	29.2	14.0	49	918.8	86.0	52.8	24.5	79	2112.3	123.0	71.1	33.4	109	3818.4	160.8	93.2	41.6
20	206.3	50.0	30.0	14.0	50	950.9	87.1	53.5	24.9	80	2160.5	124.2	71.7	33.7	110	3886.8	162.0	93.9	41.8
21	226.0	51.4	31.4	14.5	51	983.1	88.2	54.1	25.2	81	2207.7	125.6	72.3	34.0	111	3958.2	163.2	94.6	42.0
22	245.7	52.7	32.7	15.0	52	1015.2	89.2	54.8	25.5	82	2256.7	126.9	73.0	34.4	112	4026.9	164.4	95.3	42.2
23	265.4	53.9	33.9	15.4	53	1047.4	90.2	55.4	25.8	83	2306.5	128.2	73.7	34.7	113	4099.0	165.5	96.0	42.5
24	285.2	55.0	35.0	15.8	54	1081.4	91.2	56.1	26.1	84	2356.3	129.5	74.4	35.0	114	4172.0	166.7	96.8	42.8
25	305.0	56.5	36.0	16.2	55	1116.9	92.1	56.8	26.4	85	2406.9	130.7	75.1	35.3	115	4245.0	167.9	97.5	43.1
26	324.8	57.9	36.9	16.5	56	1152.4	93.0	57.5	26.6	86	2459.6	132.1	75.8	35.6	116	4318.8	169.0	98.3	43.4
27	344.6	59.2	37.8	16.9	57	1187.9	94.2	58.2	26.9	87	2510.6	133.4	76.5	35.9	117	4389.5	170.2	99.0	43.7
28	365.5	60.4	38.6	17.1	58	1223.4	95.4	58.8	27.2	88	2564.2	134.7	77.1	36.2	118	4463.8	171.4	99.7	43.9
29	388.0	61.6	39.3	17.4	59	1261.0	96.5	59.5	27.5	89	2615.9	136.0	77.9	36.5	119	4538.8	172.5	100.4	44.2
30	410.5	63.0	40.0	17.7	60	1299.6	97.6	60.1	27.9	90	2670.5	137.2	78.7	36.7	120	4614.1	173.7	101.1	44.5
31	432 9	64.4	40.7	18.2	61	1338.3	98.7	60.7	28.2	91	2723.0	138.5	79.5	37.0	121	4686.5	174.8	101.8	44.7
32	455.4	65.7	41.3	18.8	62	1377.0	100.0	61.3	28.5	92	2775.9	139.8	80.3	37.3	122	4762.7	176.0	102.5	45.0
33	477.9	66.9	42.0	19.2	63	1415.6	101.3	61.8	28.8	93	2831.5	141.1	81.0	37.5	123	4836.2	177.1	103.2	45.3
34	500.4	68.1	42.8	19.7	64	1455.5	102.6	62.4	29.1	94	2885.3	142.4	81.7	37.8	124	4917.4	178.3	104.0	45.7
35	523.0	69.2	43.5	20.1	65	1497.5	103.8	63.0	29.4	95	2939.5	143.6	82.5	38.0	125	4996.4	179.4	104.7	46.0



Dead Load.

By eq. (6) of Art. 17 the bending moments at the quarter-point and centre are, since the reaction, R, is $40 \times 700 = 28,000$:

	Quarter-point, $x = 20$ ft.	Centre, $x = \frac{1}{2}l = 40$ ft.
$M = \frac{p}{2}(lx - x^2)$	420,000 ft.-lbs.	560,000 ft.-lbs

By eq. (7) of Art. 17, the shears at end, quarter-point, and centre are

	End, $x = 0$.	Quarter-point, $x = 20$ ft.	Centre, $x = 40$ ft.
Shear	28,000 lbs.	14,000 lbs.	zero

Moving Load.

If wheel 3 be placed at the quarter-point there will be one locomotive and 20 feet of uniform train load on the span, the latter being 5 feet from the last tender wheel, which may be represented by wheel No. 9. As $\frac{l'}{l} = \frac{1}{4}$, the criterion, eq. (5), gives either $\frac{l'}{l} = \frac{30,000}{182,000}$ or $\frac{50,000}{182,000}$, the first being too small and the second too large. Hence wheel 3 at the quarter-point is the proper position for the maximum bending moment. In eq. (6) or (6a), $W_n = 40,000$ lbs. and $x = 10$; also $W_{n'-1} = W_2 = 20,000$ lbs. Hence the bending moment has the value

$$M = \tfrac{1}{4}(4,206,000 + 142,000 \times 20 + 40,000 \times 10) - 230,000$$
$$= 1,631,500 \text{ ft.-lbs.}$$

If wheel 5 be placed at the centre of the span ten feet of uniform train load will rest on the girder, making a total of 162,000 pounds. The criterion, eq. 5, then gives $\frac{l'}{l} = \frac{70,000}{162,000}$ or $\frac{90,000}{162,000}$, the first value being less than $\frac{1}{2}$ and the second greater. Hence the trial position is correct, and the corresponding greatest moment will be

$$M = \tfrac{1}{2}(4,206,000 + 142,000 \times 10 + 20,000 \times 5) - 830,000$$
$$= 2,033,000 \text{ ft.-lbs.}$$

As the span is 80 ft., the uniform load giving the same centre moment as the actual load, i.e., the equivalent uniform load, is

$$p = \frac{8}{80 \times 80} \cdot 2,033,000 = 2541 \text{ lbs. per linear foot,}$$

the total bending moment at the quarter-point will then be

$$1,631,500 + 420,000 = 2,051,500 \text{ ft.-lbs.}$$

And at the centre

$$2,033,000 + 560,000 = 2,593,000 \text{ ft.-lbs.}$$

The end shear will be found by placing W_2 at the left end of the span, requiring 35 feet of train load to be on the girder. The reaction, i.e., the end shear, at the left end of the span will then be written by the aid of Table I:

$$\text{Reaction} = \text{end shear} = \frac{4,206,000 + (142,000 - 10,000) \times 35 + 70,000 \times 17.5 - 10,000 \times 53}{80}$$
$$= 119,013 \text{ lbs.}$$

The equivalent uniform load producing the same shear is

$$p = \frac{119,013}{40} = 2975 \text{ lbs. per linear foot.}$$

The shear at the quarter-point will be found by placing W_2 at that point, requiring 10 feet of train load to be on the girder. The reaction at the left end of the span will then be

$$R=\frac{4{,}206{,}000+142{,}000\times 10+20{,}000\times 5}{80}=71{,}575 \text{ lbs.}$$

The shear at the quarter-point will therefore be

$$71{,}575-10{,}000=61{,}575 \text{ lbs.}$$

The centre shear will be found by placing W_2 at the centre of the span, requiring W_9 to be at the right-hand extremity of the span. The reaction at the left-hand end of the span will then be

$$R=\frac{4{,}206{,}000-142{,}000\times 5}{80}=43{,}700 \text{ lbs.}$$

The centre shear will then be

$$43{,}700-10{,}000=33{,}700 \text{ lbs.}$$

The total shears at the end, quarter-point, and centre will have the values

At end.	$119{,}013+28{,}000=$	$147{,}013$ lbs.
At quarter-point. . . .	$61{,}575+14{,}000=$	$75{,}575$ lbs.
At centre.	$=$	$33{,}700$ lbs.

There may be more than one position of concentrated loadings satisfying the criterion, eq. (5), as may be seen by trial. Such maxima must frequently be sought, and they can be found, if existing, with little trouble.

Art. 17.—Moments and Shears in Special Cases.

Certain special cases of beams are of such common occurrence, and consequently of such importance, that a somewhat more detailed treatment than that already given may be deemed desirable. The following cases are of this character:

Case I.

Let a non-continuous beam supporting a single weight P at any point be considered, and let such a beam be represented in Fig. 1. If the span RR' is represented by

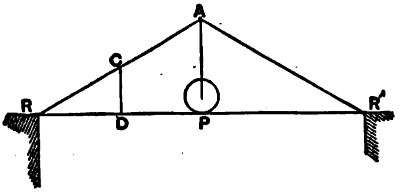


FIG. 1.

$$l = a + b = RP + R'P,$$

the reactions R and R' will be

$$R = \frac{b}{l}P, \quad \text{and} \quad R' = \frac{a}{l}P. \quad . \quad . \quad . \quad . \quad (1)$$

Consequently, if x represents the distance of any section in RP from R, while x' represents the distance of any section of $R'P$ from R', the general values of the bending moments for the two segments a and b of the beam will be

$$M = Rx, \quad \text{and} \quad M' = R'x'. \quad . \quad . \quad . \quad . \quad (2)$$

These two moments become equal to each other and represent *the greatest bending moment in the beam* when

$$x = a \quad \text{and} \quad x' = b,$$

or *when the section is taken at the point of application of the load P.*

Eq. (2) shows that the moments vary directly as the distances from the ends of the beam. Hence if AP (normal to RR') is taken by any convenient scale to represent the greatest moment, $\frac{ab}{l}P$, and if RAR' is drawn, any intercept parallel to AP and lying between RAR' and RR' will represent the bending moment for the section at its foot by the same scale. In this manner CD is the bending moment at D.

The shear is uniform for each single segment; it is evidently equal to R for RP and R' for $R'P$. It becomes zero at P, where is found the greatest bending moment.

Case II.

Again, let Fig. 2 represent the same beam shown in Fig. 1, but let the load be one of uniform intensity, p, extending from end to end of the beam. Let C be placed at the centre of the span, and let R and R', as before, represent the two reactions. Since the load is symmetrical in reference to C,

$$R = R'.$$

For the same reason the moments and shears in one half of the beam will be exactly like those in the other; consequently reference will be made to one half of the beam only. Let x and x_1 then be measured from R toward C. The forces acting upon the beam are R and p, the latter being uniformly continuous. Applying the formulæ for the bending moment at any section x, remembering that x_1 has all values less than x,

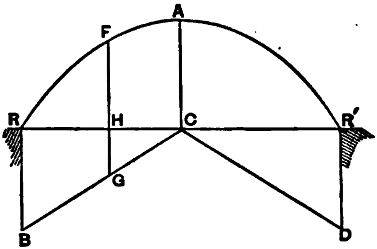


Fig. 2.

$$M = Rx - p\int_0^x (x - x_1)dx_1;$$

$$\therefore\ M = Rx - \frac{px^2}{2}. \quad \dots\dots\dots \quad (3)$$

If l is the span, at C, M becomes

$$M_1 = \frac{Rl}{2} - \frac{pl^2}{8}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

But because the load is uniform

$$R = \frac{pl}{2}.$$

Hence

$$M_1 = \frac{pl^2}{8} = \frac{Wl}{8} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

if W is put for the total load. Placing

$$R = \frac{pl}{2},$$

in eq. (3),

$$M = \frac{p}{2}(lx - x^2). \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The moments M, therefore, are proportional to the abscissæ of a parabola whose vertex is over C, and which passes through the origin of coordinates R. Let AC, then, normal to RR', be taken equal to M_1, and let the parabola RAR' be drawn. Intercepts, as FH, parallel to AC, will represent bending moments in the sections, as H, at their feet.

The shear at any section is

$$S = \frac{dM}{dx} = R - px = p\left(\frac{l}{2} - x\right), \quad . \quad . \quad . \quad . \quad (7)$$

or *it is equal to the load covering that portion of the beam between the section in question and the centre.*

Eq. (7) shows that the shear at the centre is zero; it also shows that $S = R$ at the ends of the beam. It further demonstrates that *the shear varies directly as the distance from the centre.* Hence, take RB to represent R and draw BC. The shear at any section, as H, will then be represented by the vertical intercept, as *HG*, included between *BC* and *RC*.

The shear being zero at the centre, the greatest bending moment will also be found at that point. This is also evident from inspection of the loading.

Eq. (2) of *Case I* shows that if a beam of span l carries a weight $\frac{W}{2}$ at its centre, the moment M at the same point will be

$$M_1 = \frac{W}{4} \cdot \frac{l}{2} = \frac{Wl}{8}. \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The third member of eq. (8) is identical with the third member of eq. (5). It is shown, therefore, *that a load concentrated at the centre of a non-continuous beam will cause the same moment, at that centre, as double the same load uniformly distributed over the span.*

Eqs. (5) and (8) are much used in connection with the bending of ordinary non-continuous beams, whether solid or flanged; and such beams are frequently found.

Case III.

The third case to be taken is a cantilever uniformly loaded; it is shown in Fig. 3. Let x be measured from the free end A, and let the uniform intensity of the load be represented by p. The load px acts with its centre at the distance $\frac{1}{2}x$ from the section x. Hence the desired moment will be

$$M = -px \cdot \frac{x}{2} = -\frac{px^2}{2}. \quad . \quad . \quad (9)$$

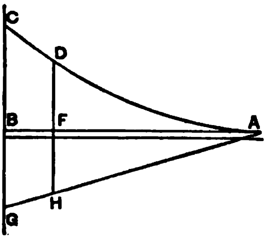


Fig. 3.

If $AB = l$, the moment at B is

$$M_1 = -\frac{pl^2}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

The negative sign is used to indicate that the *lower* side of the beam is subjected to compression. In the two preceding cases, evidently the *upper* side is in compression.

The shear at any section is

$$S=\frac{dM}{dx}=-px. \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Hence the shear at any section is the load between the free end and that section.

Eq. (9) shows that the moments vary as the *square* of the distance from the free end; consequently the moment curve is a parabola with the vertex at A, and with a vertical axis. Let BC, then, represent M_1 by any convenient scale and draw the parabola CDA. Any vertical intercept, as DF, will represent the moment at the section, as F, at its foot.

Again, let BG represent the shear pl at B, then draw the straight line AG. Any vertical intercept, as HF, will then represent the shear at the corresponding section F.

Art. 18.—Recapitulation of the General Formulæ of the Common Theory of Flexure.

It is convenient for many purposes to arrange the formulæ of the Common Theory of Flexure in the most general and concise form. In this article the preceding general formulæ for shear, strains, resisting moments, and deflections will be recapitulated and so arranged. In order to complete the generalization, the summation sign Σ will be used instead of the sign of integration.

In Fig. 1, let ABC represent the centre line of any bent beam; AF, a vertical line through A; CF, a horizontal line through C, while A is the section of the beam at which the deflection (vertical or horizontal) in reference to C, the bending moment, the shearing stress, etc., are to be determined. As shown in figure, let x be the horizontal coordinate measured from A, and y the vertical one measured from the same point; then let z be the horizontal distance from the same point to the point of application of any external vertical force P. To complete the notation, let D be the deflection desired; M_1, the moment of the external forces about A; S, the shear at A; P', the strain (extension or compression) per unit of length of a fibre parallel to the neutral surface and situated at a normal distance of unity from it; I, the general expression of the moment of inertia of a normal cross-section of the beam, taken in reference to the neutral axis of that section; E, the coefficient of elasticity for the material of the beam; and M the moment of the external forces for any section, as B.

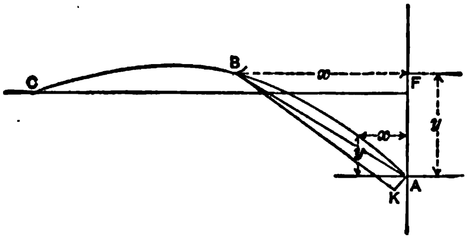


Fig. 1.

Again, let Δ be an indefinitely small portion of any normal cross-section of the beam, and let y' be an ordinate normal to the neutral axis of the same section. By the "common theory" of flexure, the intensity of stress at the distance y' from the neutral surface is $(y'P'E)$. Consequently the stress developed in the portion Δ of the section is $EP'y'\Delta$, and the resisting moment of that stress is $EP'y'^2\Delta$.

The resisting moment of the whole section will therefore be found by taking the sum of all such moments for its whole area.

Hence

$$M = EP'\Sigma y'^2 \Delta = EP'I.$$

Hence, also,

$$P' = \frac{M}{EI}.$$

If n represents an indefinitely short portion of the neutral surface, the strain for such a length of fibre at unit's distance from that surface will be nP'.

If the beam were originally straight and horizontal, n would be equal to dx.

P' being supposed small, the effect of the strain nP' at any section, B, is to cause the end A of the chord BA to move vertically through the distance $nP'x$.

If BK and BA (taken equal) are the positions of the chords before and after flexure, $nP'x$ will be the vertical distance between K and A.

By precisely the same kinematical principle the expression $nP'y$ will be the horizontal movement of A in reference to B.

Let $\Sigma nP'x$ and $\Sigma nP'y$ represent summations extending from A to C, then will those expressions be the vertical and horizontal deflections respectively of A in reference to C. It is evident that these operations are perfectly general, and that x and y may be taken in any direction whatever.

The following general but strictly approximate equations relating to the subject of flexure may now be written

$$S = \Sigma P. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$M_1 = \Sigma Pz. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$P' = \frac{M}{EI}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$\Sigma nP' = \Sigma n\frac{M}{EI}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$D_v = \Sigma nP'x = \Sigma\frac{nMx}{EI}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$D_h = \Sigma nP'y = \Sigma\frac{nMy}{EI}. \quad . \quad . \quad . \quad . \quad (6)$$

D_h represents horizontal deflection.

The summation ΣPz must extend from A to a point of no bending, or from A to a point at which the bending moment is M_1'. In the latter case

$$M_1 = \Sigma Pz + M_1'. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

M_1' may be positive or negative.

Art. 19.—The Theorem of Three Moments.

The object of this theorem is the determination of the relation existing between the bending moments which are found in any continuous beam at any three adjacent points of support. In the most general case to which the theorem applies, the section of the beam is supposed to be variable, the points of support are not supposed to be in the same level, and at any point, or all points, of support there may be constraint applied to the beam external to the load which it is to carry; or, what is equivalent to the last condition, the beam may not be straight at any point of support before flexure takes place.

Before establishing the theorem itself, some preliminary matters must receive attention.

If a beam is simply supported at each end, the reactions are found by dividing the applied loads according to the simple principle of the lever. If, however, either or both ends are not simply supported, the reaction in general is greater at one end and less at the other than would be found by the law of the lever; a portion of the reaction at one end is, as it were, transferred to the other. The transference can only be accomplished by the application of a couple to the beam, the forces of the couple being applied at the two adjacent points of support; the span, consequently, will be the lever-arm of the couple. The existence of equilibrium requires the application to the beam of an equal and opposite couple. It is only necessary, however, to consider, in connection with the span AB, the one shown in Fig. 1. Further, from what has immediately preceded, it appears that the force of this couple is equal to the difference between the actual reaction at either point of support and that found by the law of the lever. The bending caused by this couple may evidently be of an opposite kind to that existing in a beam simply supported at each end.

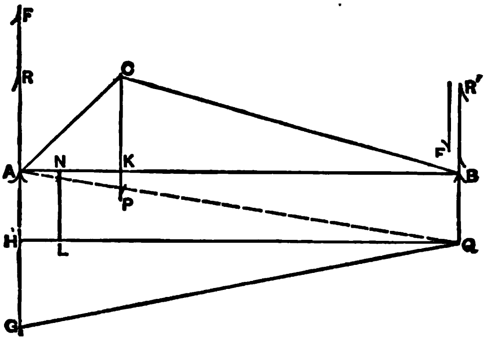


Fig. 1.

These results are represented graphically in Fig. 1. A and B are points of support, and AB is the beam; AR and

BR' are the reactions according to the law of the lever; $RF = R'F$ is the force of the applied couple; consequently

$$AF = AR + RF \quad \text{and} \quad BF = BR' - (R'F = RF)$$

are the reactions after the couple is applied. As is well known, lines parallel to CK, drawn in the triangle ACB, represent the bending moments at the various sections of the beam, when the reactions are AR and BR'. Finally, vertical lines parallel to AG, in the triangle QHG, will represent the bending moments caused by the force $R'F$.

In the general case there may also be applied to the beam two equal and opposite couples having axes passing through A and B respectively. The effect of such couples will be nothing so far as the reactions are concerned, but they will cause uniform bending between A and B. This uniform or constant moment may be represented by vertical lines drawn parallel to AH or LN (equal to each other) between the lines AB and HQ. The resultant moments to which the various sections of the beam are subjected will then be represented by the *algebraic* sum of the three vertical ordinates included between the lines ACB and GQ. Let that resultant be called M. This composition of the resultant moment M will be made clearer by reference to Figs. 2 and 3. Fig. 2 shows the component moment due to the single force F acting with the lever-arm l so that its moment increases directly as the distance from B. Fig. 3, on the other hand, shows the component moment due to the two equal and opposite couples acting at the ends of the span. The resultant moment M is the algebraic sum of the three component moments, shown combined in Fig. 1.

Let the moment GA be called M_a, and the moment

$$BQ = LN = HA = M_b.$$

Also designate the moment caused by the load P, shown by lines parallel to CK in ACB, by M_1. Then let x be any horizontal distance measured from A toward B; l the horizontal distance AB; and z the distance of the point of application, K, of the force P from A. With this notation there can be at once written

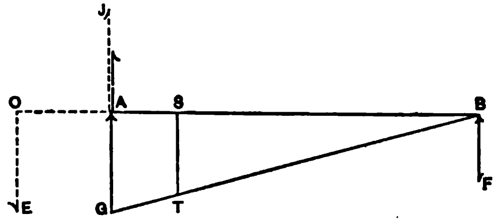


FIG. 2.

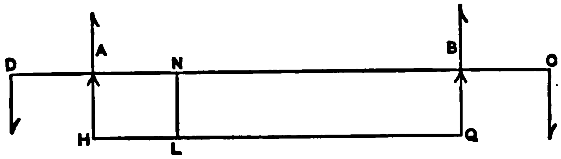


FIG. 3.

$$M = M_a\left(\frac{l-x}{l}\right) + M_b\left(\frac{x}{l}\right) + M_1 . \quad . \quad . \quad . \quad (1)$$

Eq. (1) is simply the general form of eq. (2), Art. 18.

It is to be noticed that Fig. 1 does not show all the moments M_a, M_b, and M_1 to be the same sign, but for convenience they are so written in eq. (1).

The formula which represents the theorem of three moments can now be written without difficulty. The method to be followed involves the improvements added by Prof. H. T. Eddy, and is the same as that given by him in the "American Journal of Mathematics," Vol. I., No. 1.

Fig. 4 shows a portion of a continuous beam, including two spans and three points of support. The deflections will be supposed measured from the horizontal line *NQ*.

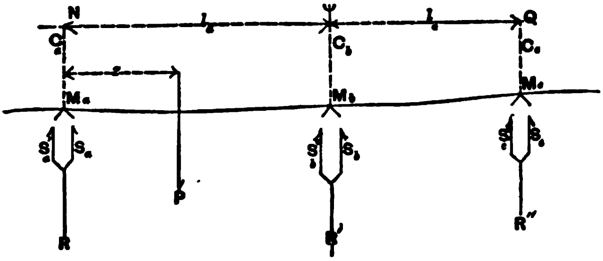


FIG. 4.

The spans are represented by l_a and l_c; the vertical distances of *NQ* from the points of support by c_a, c_b, and c_c; the moments at the same points by M_a, M_b, and M_c, while the letters *S* and *R* represent shears and reactions respectively.

In order to make the case general, it will be supposed that the beam is curved in a vertical plane, and has an elbow at *b*, before flexure, and that, at that point of support, the tangent of its inclination to a horizontal line, toward the span l_a, is *t*, while *t'* represents the tangent on the other side of the same point of support; also let *d* and *d'* be the vertical distances, before bending takes place, of the points *a* and *c*, respectively, below the tangents at the point *b*.

A portion of the difference between c_a and c_b is due to the original inclination, whose tangent is *t*, and the original lack of straightness, and is not caused by the bending; that portion which is due to the bending, however, is, remembering eq. (5), Art. 18,

$$D = c_a - c_b - l_a t - d = \Sigma_b^a \frac{Mxn}{EI}.$$

Fig. 5 will make clear the component parts of the value of D in the preceding equation.

By the aid of eq. (1) this equation may be written:

$$E(c_a - c_b - l_a t - d) = \Sigma_b^a \left[\left\{ M_a\left(\frac{l-x}{l}\right) + M_b\left(\frac{x}{l}\right) + M_1 \right\} \frac{xn}{I} \right]. \quad (2)$$

In this equation, it is to be remembered, both x and z (involved in M_1) are measured from support a toward support b.

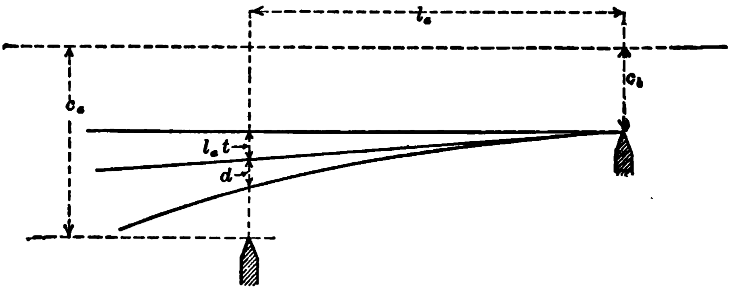


Fig. 5.

Now let a similar equation be written for the span l_c, in which the variables x and z will be measured from c toward b. There will then result

$$E(c_c - c_b - l_c t' - d') = \Sigma_b^c \left[\left\{ M_c\left(\frac{l-x}{l}\right) + M_b\left(\frac{x}{l}\right) + M_1 \right\} \frac{xn}{I} \right]. \quad (3)$$

When the general sign of summation is displaced by the integral sign, n becomes the differential of the axis of the beam, or ds. But ds may be represented by udx, u being such a function of x as becomes unity if the axis of the beam is originally straight and parallel to the axis of x. The eqs. (2) and (3) may then be reduced to simpler forms by the following methods:

In eq. (2) put

$$\Sigma_b^a\left(\frac{l-x}{l}\right)\frac{xn}{I}=\frac{1}{l_a}\int_b^a\frac{u(l_a-x)xdx}{I}=\frac{x_a}{l_a}\int_b^a\frac{u(l_a-x)dx}{I}. \quad (4)$$

Also

$$\frac{x_a}{l_a}\int_b^a\frac{u(l_a-x)dx}{I}=\frac{i_ax_a}{l_a}\int_b^a u(l_a-x)dx. \quad . \quad . \quad (5)$$

Also

$$\frac{i_ax_a}{l_a}\int_b^a u(l_a-x)\,dx=\frac{i_ax_au_a}{l_a}\int_b^a(l_a-x)\,dx=\frac{i_ax_au_al_a}{2}. \quad (6)$$

In the same manner

$$\Sigma_b^a\frac{x^2n}{l_aI}=\frac{1}{l_a}\int_b^a\frac{ux^2dx}{I}=\frac{x_a'}{l_a}\int_b^a\frac{uxdx}{I}. \quad . \quad . \quad (7)$$

Also

$$\frac{x_a'}{l_a}\int_b^a\frac{uxdx}{I}=\frac{i_a'x_a'}{l_a}\int_b^a uxdx. \quad . \quad . \quad . \quad . \quad (8)$$

And

$$\frac{i_a'x_a'}{l_a}\int_b^a uxdx=\frac{i_a'x_a'u_a'}{l_a}\int_b^a xdx=\frac{i_a'x_a'u_a'l_a}{2}. \quad . \quad (9)$$

Again, in the same manner,

$$\Sigma_b^a\frac{M_1xn}{I}=i_{1a}u_{1a}\Sigma M_1x\Delta x. \quad . \quad . \quad . \quad . \quad (10)$$

Using eqs. (4) to (10), eq. (2) may be written:

$$E(c_a-c_b-l_at-d)=\frac{l_a}{2}(M_au_ai_ax_a+M_bu_a'i_a'x_a')$$
$$+u_{1a}i_{1a}\Sigma_b^a M_1x\Delta x. \quad (11)$$

Proceeding in precisely the same manner with the span l_c, eq. (3) becomes

$$E(c_c-c_b-l_ct'-d')=\frac{l_c}{2}(M_cu_ci_cx_c+M_bu_c'i_c'x_c')$$
$$+u_{1c}i_{1c}\sum_b^c M_1x\Delta x. \qquad (12)$$

The quantities x_a and x_c are to be determined by applying eq. (4) to the span indicated by the subscript; while u_a, i_a, u_c, and i_c are to be determined by using eqs. (5) and (6) in the same way. Similar observations apply to u_a', i_a', x_a', u_c', i_c', and x_c' taken in connection with eqs. (7), (8), and (9).

If I is not a continuous function of x, the various integrations of eqs. (4), (5), (7), and (8) must give place to summations (Σ) taken between the proper limits.

Dividing eqs. (11) and (12) by l_a and l_c respectively, and adding the results,

$$\left.\begin{aligned}&E\left(\frac{c_a-c_b}{l_a}+\frac{c_c-c_b}{l_c}-T-\frac{d}{l_a}-\frac{d'}{l_c}\right)\\&\quad=\frac{u_{1a}i_{1a}}{l_a}\sum_b^a M_1x\Delta x+\frac{u_{1c}i_{1c}}{l_c}\sum_b^c M_1x\Delta x\\&\quad+\tfrac{1}{2}(M_au_ai_ax_a+M_bu_a'i_a'x_a'+M_cu_ci_cx_c+M_bu_c'i_c'x_c')\end{aligned}\right\} \qquad (13)$$

in which $T=t+t'$.

Eq. (13) is the most general form of the theorem of three moments if E, the coefficient of elasticity, is a constant quantity. Indeed, that equation expresses, as it stands, the "theorem" for a variable coefficient of elasticity if (ie) be written instead of i; e representing a quantity determined in a manner exactly similar to that used in connection with the quantity i.

In the ordinary case of an engineer's experience $T=0$, $d=d'=0$, *$I=$constant, $u=u_a=u_c=$etc., $=c'=$secant of the inclination for which $t=-t'$ is the tangent;* consequently

$$i_a=i_a'=i_c=i_c'=i_{,a}=i_{,c}=\frac{1}{I}.$$

From eq. (4)

$$x_a=\frac{2l_a}{6}, \qquad x_c=\frac{2l_c}{6}.$$

From eq. (7)

$$x_a'=\frac{4l_a}{6}, \qquad x_c'=\frac{4l_c}{6}.$$

The summation $\Sigma M_1 x\Delta x$ can be readily made by referring to Fig. 1.

The moment represented by CK in that figure is

$$P\left(\frac{l-z}{l}\right).z;$$

consequently the moment at any point between A and K, due to P, is

$$M_1=P\left(\frac{l-z}{l}\right).z.\frac{x}{z}=P\left(\frac{l-z}{l}\right)x.$$

Between K and B

$$M_1'=\left(\frac{l-x}{l-z}\right).CK=P\frac{z}{l}(l-x).$$

Using these quantities for the span l_a,

$$\overset{a}{\underset{b}{\Sigma}} M_1 x\Delta x=\int_0^z M_1 x\,dx+\int_z^{l_a} M_1' x\,dx=\tfrac{1}{6}P(l_a^2-z^2)z.$$

For the span l_c the subscript a is to be changed to c.

Introducing all these quantities eq. (13) becomes, after providing for any number of weights, P:

$$\frac{6EI}{c'}\left(\frac{c_a - c_b}{l_a} + \frac{c_c - c_b}{l_c}\right) = M_a l_a + 2M_b(l_a + l_c) + M_c l_c$$
$$+ \frac{1}{l_a}\overset{a}{\Sigma} P(l_a{}^2 - z^2)z + \frac{1}{l_c}\overset{c}{\Sigma} P(l_c{}^2 - z^2)z. \qquad (14)$$

Eq. (14), with c' equal to unity, is the form in which the theorem of three moments is usually given; with c' equal to unity or not, *it applies only to a beam which is straight before flexure*, since

$$T = t + t' = 0 = d = d'.$$

If such a beam rests on the supports a, b, and c, before bending takes place,

$$\frac{c_a - c_b}{l_a} = -\frac{c_c - c_b}{l_c},$$

and the first member of eq. (14) becomes zero.

If, in the general case to which eq. (13) applies, the deflections c_a, c_b, and c_c belong to the beam in a position of no bending, the first member of that equation disappears, since it is the sum of the deflections *due to bending only* for the spans l_a and l_c, divided by those spans, and each of those quantities is zero by the equation immediately preceding, eq. (2). Also, if the beam or truss belonging to each span is straight between the points of support (*such points being supposed in the same level or not*), $u_a = u_a' = u_{1a} =$ *constant*, and $u_c = u_c' = u_{1c} =$ *another constant*. If, finally, I be again taken as constant, x_a and x_c, as well as $\Sigma M_1 x \Delta x$, will have the values found above.

From these considerations it at once follows that the second member of eq. (14), put equal to zero, expresses the theorem of three moments for a beam or truss straight between points of support, when those points are not in the same level, but when they belong to a configuration of no bending in the beam. Such an equation, however, does not belong to a beam not straight between points of support.

The shear at either end of any span, as l_a, is next to be found, and it can be at once written by referring to the observations made in connection with Fig. 1. It was there seen that the reaction found by the simple law of the lever is to be increased or decreased for the continuous beam, by an amount found by dividing the difference of the moments at the extremities of any span by the span itself. Referring, therefore, to Fig. 4, for the shears S, there may at once be written:

$$S_a = \overset{a}{\Sigma} P \frac{l_a - z}{l_a} - \frac{M_a - M_b}{l_a}. \quad . \quad . \quad . \quad . \quad (15)$$

$$S_b' = \overset{a}{\Sigma} P \frac{z}{l_a} + \frac{M_a - M_b}{l_a}. \quad . \quad . \quad . \quad . \quad . \quad (16)$$

$$S_b = \overset{c}{\Sigma} P \frac{z}{l_c} + \frac{M_c - M_b}{l_c}. \quad . \quad . \quad . \quad . \quad . \quad (17)$$

$$S_c' = \overset{c}{\Sigma} P \frac{l_c - z}{l_c} - \frac{M_c - M_b}{l_c}. \quad . \quad . \quad . \quad . \quad (18)$$

The negative sign is put before the fraction

$$\frac{M_a - M_b}{l_a}$$

in eq. (15) because in Fig. 1 the moments M_a and M_b are represented opposite in sign to that caused by P, while in eq. (1) the three moments are given the same sign, as has already been noticed.

Eqs. (15) to (18) are so written as to make an upward reaction positive, and they may, perhaps, be more simply found by taking moments about either end of a span. For example, taking moments about the right end of l_a,

$$S_a l_a - \overset{a}{\Sigma} P(l_a - z) + M_a = M_b.$$

From this, eq. (15) at once results. Again, moments about the left end of the same span give

$$S_b' l_a - \overset{a}{\Sigma} Pz + M_b = M_a.$$

This equation gives eq. (16), and the same process will give the others.

If the loading over the different spans is of uniform intensity, then, in general, $P = wdz$, w being the intensity. Consequently

$$\Sigma P(l^2 - z^2)z = \int_0^l w(l^2 - z^2)zdz = w\frac{l^4}{4}.$$

In all equations, therefore, for

$$\frac{1}{l_a}\overset{a}{\Sigma} P(l_a^2 - z^2)z$$

there is to be placed the term $w_a\frac{l_a^3}{4}$; and for

$$\frac{1}{l_c}\overset{c}{\Sigma} P(l_c^2 - z^2)z$$

the term $w_c\frac{l_c^3}{4}$. The letters a and c mean, of course, that reference is made to the spans l_a and l_c.

From Fig. 4, there may at once be written:

$$R\ \ = S_a' + S_a. \quad \ldots\ldots \quad (19)$$

$$R' \ = S_b' + S_b. \quad \ldots\ldots \quad (20)$$

$$R'' = S_c' + S_c, \quad \ldots\ldots \quad (21)$$

$$\text{etc.} = \text{etc.} + \text{etc.}$$

Art. 20.—Reaction under Continuous Beam of any Number of Spans.

The general value of the reactions at the points of support under any continuous beam have been given in eqs. (19), (20), (21), etc., of the preceding article. Before those equations, however, can be applied to any particular case, the values of the bending moments, which appear in the expressions S_a, S_b', S_b, etc., for the shears, must be determined. In the application of the theorem of three moments, it is invariably virtually assumed that the continuous beam before flexure is straight between the points of support, and that the latter belong to a configuration of no bending. The moment of inertia I and the coefficient of elasticity E are also assumed to be constant. This is frequently not strictly true, yet it will be assumed in what follows, since the method to be used in finding the moments is entirely independent of the assumption, and remains precisely the same whatever form for the theorem of three moments may be chosen.

Agreeably to the assumption made, eq. (14) of the preceding article takes the following form, which is almost, or quite, invariably used in engineering practice:

$$M_a l_a + 2M_b(l_a + l_c) + M_c l_c = -\frac{1}{l_a}\overset{a}{\Sigma} P(l_a^2 - z^2)z$$

$$-\frac{1}{l_c}\overset{c}{\Sigma} P(l_c^2 - z^2)z. \quad \ldots\ldots\ldots \quad (1)$$

Let Fig. 1 represent a continuous beam of n spans equal or unequal in length. At the points of support,

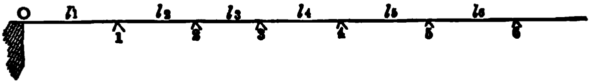


FIG. 1.

0, 1, 2, 3, 4, 5, etc., let the bending moments be represented by M_0, M_1, M_2, M_3, etc. The moment M_0 is always known; it is ordinarily zero, and that will be considered its value.

An examination of Fig. 1 shows that, by repeated applications of eq. (1), the number of resulting equations of condition will be one less than the number of spans. If the two end moments are known (here assumed to be zero), the number of unknown moments will also be one less than the number of spans. Hence the number of equations will always be sufficient for the determination of the unknown moments.

For the sake of brevity let the following notation be adopted:

$$u_1 = -\frac{1}{l_1}\overset{1}{\Sigma}P(l_1^2-z^2)z-\frac{1}{l_2}\overset{2}{\Sigma}P(l_2^2-z^2)z.$$

$$u_2 = -\frac{1}{l_3}\overset{2}{\Sigma}P(l_2^2-z^2)z-\frac{1}{l_3}\overset{3}{\Sigma}P(l_3^2-z^2)z.$$

$$u_3 = -\frac{1}{l_3}\overset{3}{\Sigma}P(l_3^2-z^2)z-\frac{1}{l_4}\overset{4}{\Sigma}P(l_4^2-z^2)z.$$

$$\text{etc.} = \qquad \text{etc.} - \qquad \text{etc.}$$

$$\begin{array}{lll} & a_1 = 2(l_1+l_2); & b_1 = l_2. \\ a_2 = l_2; & b_2 = 2(l_2+l_3); & c_2 = l_3. \\ b_3 = l_3; & c_3 = 2(l_3+l_4); & d_3 = l_4. \\ c_4 = l_4; & d_4 = 2(l_4+l_5); & f_4 = l_5. \\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot \\ p_i = l_i; & q_i = 2(l_i+l_{i+1}); & s_i = l_{i+1}; \end{array}$$

i denoting any number of the series 1, 2, 3, 4, . . . n. It is thus seen that, in general,

$$q_i = 2(p_i + s_i);$$

also that $a_2 = b_1$, $c_2 = b_3$, $d_3 = c_4$, etc. These relations can be used to simplify the final result.

By repeated applications of eq. (1) the following n equations of condition, involving the notation given above, will result:

$$\left.\begin{array}{lr}
a_1M_1 + b_1M_2 & = u_1 \\
a_2M_1 + b_2M_2 + c_2M_3 & = u_2 \\
\qquad + b_3M_2 + c_3M_3 + d_3M_4 & = u_3 \\
\qquad\qquad + c_4M_3 + d_4M_4 + f_4M_5 & = u_4 \\
\qquad\qquad\qquad + d_5M_4 + f_5M_5 + g_5M_6 & = u_5 \\
\qquad\qquad\qquad\quad . \ . \ . \ . \ . \ . \ . & = . \ . \\
\qquad\qquad\qquad\qquad\qquad . \ . \ . & = u_n
\end{array}\right\} . \quad (2)$$

The moment M_{n+1} will also be equal to zero. In consequence of this last condition it is seen that the coefficients of the M's occupy precisely the places of the elements of a determinant of the nth degree. Of the array indicating the determinant, however, there exists only the leading diagonal and one diagonal on each side of it. The determinant for n equations, or $(n+1)$ spans, has, then, the value

$$D = \left\{\begin{array}{l}
a_1, \ b_1, \ 0, \ 0, \ 0, \ 0, \quad . \ . \ . \\
a_2, \ b_2, \ c_2, \ 0, \ 0, \ 0, \quad . \ . \ . \\
0, \ b_3, \ c_3, \ d_3, \ 0, \ 0, \ . \ . \ . \\
0, \ 0, \ c_4, \ d_4, \ f_4, \ 0, \ . \ . \ . \\
0, \ 0, \ 0, \ d_5, \ f_5, \ g_5, \quad . \ . \\
\quad . \ . \ . \ . \ . \ . \ . \ . \\
0, \ 0, \ 0, \ 0, \ 0, \ . \ . \ . \ 0, \ p_n, \ q_n
\end{array}\right\} . \ . \ . \ . \quad (3)$$

Also let D_i represent the value of the determinant D when the column indicated by the ith letter of the series a, b, c, d, f, etc., is replaced by the column u_1, u_2, u_3, u_4, etc. If, for example, $i=3$, the ith letter is c. Hence

$$D_3 = \left\{\begin{array}{llllllllll} a_1, & b_1, & u_1, & 0, & 0, & 0, & . & . & . & . \\ a_2, & b_2, & u_2, & 0, & 0, & 0, & . & . & . & . \\ 0, & b_3, & u_3, & d_3, & 0, & 0, & . & . & . & . \\ 0, & 0, & u_4, & d_4, & f_4, & 0, & . & . & . & . \\ 0, & 0, & u_5, & d_5, & f_5, & g_5, & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ 0, & 0, & u_n, & 0, & 0, & 0, & \ldots & 0, & p_n, & q_n \end{array}\right\} \quad . \quad . \quad . \quad (4)$$

Then, in general,

$$M_i = \frac{D_i}{D}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Eq. (5) will give the value of the bending moment at any point of support, whatever may be the number of spans or the law of loading on any or all the spans.

Precisely the same formulæ are to be used if M_0 and M_n are not zero, but have definite values and are known. In such a case, however, u_1 and u_n would be replaced by

$$u_1' = u_1 - a_0 M_0,$$
$$\mu_n' = u_n - r_n M_{n+1}.$$

The same equations also hold true whatever form of the theorem of three moments may be chosen. It is only to be remembered that the values of the quantities a, b, c, etc., u_1, u_2, u_3, etc., will depend upon the choice.

If all the moments are desired, it will be most convenient to put the vertical column u_1, u_2, u_3, . . . , u_n in place of the vertical column a_1, a_2, 0, 0, . . . 0, in eq. (4), and then find the resulting determinant D_1. Eq. (5) will then give the value of M_1, which, placed in the first of eqs. (2), will enable M_2 to be at once found. M_3 will then result from the second of eqs. (2), M_4 from the third, etc.

So far as the general treatment of the question is concerned, there yet remains to be considered the expansion of the determinants D and D_i.

The expansion of the determinant D is very simple and leads to the following results:

For two spans

$$D = a_1. \quad \ldots \ldots \quad (6)$$

For three spans

$$D = a_1 b_2 - a_2 b_1. \quad \ldots \ldots \quad (7)$$

For four spans

$$D = a_1 b_2 c_3 - a_1 b_3 c_2 - a_2 b_1 c_3. \quad \ldots \quad (8)$$

For five spans

$$D = a_1 b_2 c_3 d_4 - a_1 b_3 c_2 d_4 - a_2 b_1 c_3 d_4 - a_1 b_2 c_4 d_3 + a_2 b_1 c_4 d_3. \quad (9)$$

For six spans

$$\begin{aligned} D = a_1 b_2 c_3 d_4 f_5 - a_1 b_3 c_2 d_4 f_5 - a_2 b_1 c_3 d_4 f_5 - a_1 b_2 c_4 d_3 f_5 + a_2 b_1 c_4 d_3 f_5 \\ - a_1 b_2 c_3 d_5 f_4 + a_1 b_3 c_2 d_5 f_4 + a_2 b_1 c_3 d_5 f_4. \quad \ldots \quad (10) \end{aligned}$$

By the observance of two or three simple rules, the determinant for $(n+1)$ spans, or n points of support, may easily be written.

A series of numbers such as 1, 2, 3, 4, 5, 6, etc., is said to be written in its natural order. Let any permutation of this series, 2, 1, 3, 6, 5, 4, be written, in which 2 is placed before 1, 6 before 5 and 4, and 5 before 4. In this permutation, therefore, there are said to be $(1+2+1)=4$ ***inversions.***

Let (λ_n) represent any letter of the series a, b, c, d, etc., which has the subscript n; also, let $(\lambda_n)_n$ and $(\lambda_n)_{n-1}$ represent the nth and $(n-1)$th letters of the same series which have the subscripts n. In general, the letter inside the parenthesis represents the subscript figure in the determinant, and that outside, the place of the letter in the series a, b, c, d, f, etc.

The nth determinant for $(n+1)$ spans, or n points of support, will then be

$$D_n = D_{n-1}(\lambda_n)_n + D_{n-2}(\lambda_n)_{n-1}(\lambda_{n-1})_n.$$

Now, with the notation taken, if the letters in each term of the determinant are written in their natural order, as $abcdfg$, etc., *the number of inversions in the subscript figures of any term will determine the sign of that term*, i.e., *if the number of inversions is odd, the sign is minus, but if the number is even the sign is plus.*

Since n is the greatest subscript in any term, and since $(\lambda_n)_n$ occupies the most advanced place in the series of letters, no inversions are introduced in multiplying D_{n-1} by $(\lambda_n)_n$. Hence *all terms of $D_{n-1}(\lambda_n)_n$ will have the same signs as the corresponding terms of D_{n-1}.*

Similarly, since n is greater than $(n-1)$, the product $(\lambda_n)_{n-1}(\lambda_{n-1})_n$ involves one inversion. Hence *all terms of*

$$D_{n-2}(\lambda_n)_{n-1}(\lambda_{n-1})_n$$

will have signs contrary to those of the corresponding terms of D_{n-2}.

The number of terms in D_n will evidently be the sum of the numbers of terms in D_{n-1} and D_{n-2}.

An examination of the notation will at once show that

$$(\lambda_n)_n = 2(l_n + l_{n+1}); \quad (\lambda_n)_{n-1} = l_n; \quad \text{and} \quad (\lambda_{n-1})_n = l_n.$$

Hence there will result

$$D_n = 2D_{n-1}(l_n + l_{n+1}) - D_{n-2}l_n^2 . \quad . \quad . \quad . \quad (11)$$

The minus sign before the last term of the second member is on account of the inversion introduced, as already explained.

The general value of the determinant D_i (shown in eq. (4) when $i=3$) can be most easily expanded by considering it the sum of two determinants; and in order to illustrate this method let it be supposed that M_3 is desired. It will then be necessary to expand the determinant D_3, given in eq. (4). As is known from the theory of determinants, D_3 may be written as follows:

$$D_3 = \left\{ \begin{array}{l} a_1,\ b_1,\ 0,\ 0,\ 0,\ 0,\ .\ .\ . \\ a_2,\ b_2,\ u_2,\ 0,\ 0,\ 0,\ .\ .\ . \\ 0,\ b_3,\ u_3,\ d_3,\ 0,\ 0,\ .\ .\ . \\ 0,\ 0,\ u_4,\ d_4,\ f_4,\ 0,\ .\ .\ . \\ 0,\ 0,\ 0,\ d_5,\ f_5,\ g_5,\ .\ .\ . \\ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ . \\ 0,0,0,0,0,0,0,\ .\ .\ . \\ \qquad\qquad .\ .\ .\ p_n,\ q_n, \end{array} \right\} + \left\{ \begin{array}{l} a_1,\ b_1,\ u_1,\ 0,\ 0,\ 0,\ .\ . \\ a_2,\ b_2,\ 0,\ 0,\ 0,\ 0,\ .\ . \\ 0,\ b_3,\ 0,\ d_3,\ 0,\ 0,\ .\ . \\ 0,\ 0,\ 0,\ d_4,\ f_4,\ 0,\ .\ . \\ 0,\ 0,\ u_5,\ d_5,\ f_5,\ g_5,\ .\ . \\ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ . \\ 0,0,u_n,0,0,0,\ .\ .\ . \\ \qquad\qquad .\ .\ .\ p_n,\ q_n, \end{array} \right\} (12)$$

or

$$D_3 = D_3' + D_3'' . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Eq. (12) shows at a glance what D_3' and D_3'' represent.

D_3' is precisely the same in form as D, and is given at once by the eqs. (6) to (11) after writing u_2, u_3, and u_4 for c_2, c_3, and c_4.

In general, D_i' is found by simply writing u_{i-1}, u_i, and u_{i+1} for $(\lambda_{i-1})_i$, $(\lambda_i)_i$ and $(\lambda_{i+1})_i$ in the determinant D.

As a general method, that of alternate numbers is probably as simple as any for the expansion of the determinant D_i''. For example

$$D_3'' = (a_1e_1 + b_1e_2 + u_1e_3)(a_2e_1 + b_2e_2)(b_3e_2 + d_3e_4) \dots$$
$$\dots (u_ne_3 + p_ne_{n-1} + q_ne_n); \quad . \quad (14)$$

in which e_1, e_2, e_3, etc., are the units of the alternate numbers.

The circumstances of any particular case will frequently either furnish a more expeditious method than that of alternate numbers, or allow the expansion of D_i'' to be written at once from an inspection of the array given in eq. (12).

In any case the method of alternate numbers may be used as a check.

Special Method for Ordinary Use.

If the number of spans is large, the expansion of the determinant D_i will, at best, be found somewhat tedious. Special methods may be employed which involve only the determinant D, given in eqs. (6) to (11); and it has already been seen that that determinant admits of a very simple expansion.

Let any one span carry *any load whatever*, while all other spans carry no load. In such a case, P will be zero for every span but one, and, in consequence of the notation employed, all but two quantities in the series u_1, u_2, u_3, u_4, u_5, etc., will also become equal to zero.

If l_i (the ith span) carries the load, there will result

$$u_{i-1} = -\frac{1}{l_i}\overset{i}{\Sigma} P(l_i^2 - z^2)z; \quad . \quad . \quad . \quad . \quad (15)$$

$$u_i = -\frac{1}{l_i}\overset{i}{\Sigma} P(l_i^2 - z^2)z. \quad . \quad . \quad . \quad . \quad (16)$$

All other u's reduce to zero. Although eqs. (15) and (16) have the same form, they are not identical except in special cases, since z is not measured from the same end of the span in both expressions.

Now let u_{i-1} and u_i take the place of those letters in that column of D formed with the ith letter of the series a, b, c, d, etc., which have the subscripts i and $i-1$; u_{i+1} is equal to zero. Or in the notation already employed, let u_{i-1} and u_i take the place of $(\lambda_{i-1})_i$ and $(\lambda_i)_i$, while zero takes the place of $(\lambda_{i+1})_i$. The resulting determinant, D_i, will then be precisely the same as D in general form. The expansion of D_i can then be at once made by simply putting in D the substitutions above indicated. There will then result:

$$M_i = \frac{D_i}{D}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

In order to find M_{i-1}, with the same loading on the same span, u_{i-1} and u_i must take the place of $(\lambda_{i-1})_{i-1}$ and $(\lambda_i)_{i-1}$, respectively, while $(\lambda_{i-2})_{i-1}$ becomes equal to zero. Making these substitutions in the determinant D, there will result the determinant D_{i-1}. Then:

$$M_{i-1} = \frac{D_{i-1}}{D}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

The values of M_i and M_{i-1}, thus obtained, placed in the ith and $(i-1)$th of the eqs. (2) will at once give:

$$M_{i-2} \quad \text{and} \quad M_{i+1}.$$

Similar substitutions in the other equations will give all the moments. Thus the solution is complete, for the span and loading taken, with the use of the expanded determinant D only.

Each span may be treated in the same manner and the same expansion of D will be the only one necessary.

This method is equivalent to splitting the elements u_1, u_2, u_3, u_4, etc., of the general determinant D_i.

In order to determine the bending moment at any point of support, for loading which covers more than one span, or portions of more than one span, it is only necessary to take the algebraic sum of the separate moments (as above determined), at the point of support in question, found for the loading in each single span. The result will be the moment due to the combined action of all the loading.

It is thus seen that the solution of the most general case is made to depend on the one expansion of the determinant D.

Example.

Let there be a continuous beam of six spans, and let any loading rest upon the fourth; it is required to find the expansions of the determinants D_i and D_{i-1}.

The expansion of D is given in eq. (10) and need not be repeated here.

Using the preceding notation,

$$\begin{aligned} i &= 4, & (\lambda_{i-1})_i &= d_3, \\ i-1 &= 3, & (\lambda_i)_i &= d_4, \\ u_i &= u_4, & (\lambda_{i+1})_i &= d_5. \\ u_{i-1} &= u_3, & & \end{aligned}$$

In eq. (10), then, d_4 and d_3 are to be displaced by u_4 and u_3, while zero is to take the place of d_5. Hence

$$D_i = a_1b_2c_3u_4f_5 - a_1b_3c_2u_4f_5 - a_2b_1c_3u_4f_5 - a_1b_2c_4u_3f_5 + a_2b_1c_4u_3f_5. \quad (19)$$

Again,

$$(\lambda_{i-1})_{i-1} = c_3, \qquad (\lambda_i)_{i-1} = c_4.$$

Then, in eq. (10), placing u_3 and u_4 for c_3 and c_4, and placing

$$(\lambda_{i-2})_{i-1} = c_2 = 0,$$

there will result

$$D_{i-1} = a_1 b_2 u_3 d_4 f_5 - a_2 b_1 u_3 d_4 f_5 - a_1 b_2 u_4 d_3 f_5 + a_2 b_1 u_4 d_3 f_5 \\ - a_1 b_2 u_3 d_5 f_4 + a_2 b_1 u_3 d_5 f_4. \quad (20)$$

These values placed in eqs. (17) and (18) will give M_4 and M_3.

The lengths of span may be any whatever; if they are equal, the results will be simplified.

Special Case of Equal Spans.

If all the spans are of equal length, each may be represented by l. There will then result

$$\left.\begin{array}{l} a_2 = b_3 = c_4 = d_5 = \ldots = p_i = b_1 = c_2 = d_3 = f_4 = \ldots s_i = l \\ a_1 = b_2 = c_3 = d_4 = f_5 = \ldots = q_i = 4l. \end{array}\right\} \quad (21)$$

These values of a, b, c, etc., placed in eqs. (6) to (10) give:

For two spans:

$$D = 4l.$$

For three spans:

$$D = 15l^2.$$

For four spans:

$$D = 56l^3.$$

For five spans:

$$D = 209l^4.$$

For six spans:

$$D = 780l^5.$$

Others may be easily and rapidly written by the aid of eq. (11), which now becomes:

$$D_n = 4lD_{n-1} - l^2 D_{n-2}. \quad . \quad . \quad . \quad . \quad . \quad (22)$$

If the determinant for seven (i.e., $n+1$) spans is desired

$$D_{n-1} = 780l^5 \quad \text{and} \quad D_{n-2} = 209l^4.$$

Hence:

$$D_n = D_6 = 3120l^6 - 209l^6 = 2911l^6.$$

Similarly for eight spans:

$$D = 4 \times 2911l^7 - 780l^7 = 10864l^7.$$

For nine spans:

$$D = 4 \times 10864l^8 - 2911l^8 = 40545l^8.$$

For ten spans:

$$D = 4 \times 40545l^9 - 10864l^9 = 151316l^9.$$

The values given in eq. (21) will correspondingly simplify the expansion of the determinant D_i, either in its general form as exemplified in eq. (4) or as given in the special method. As an illustration, eqs. (19) and (20) become, respectively:

$$D_i = 224l^4u_4 - 60l^4u_3,$$

$$D_{i-1} = 225l^4u_3 - 60l^4u_4.$$

These values then give:

$$M_4 = \frac{D_i}{D} = \frac{56u_4 - 15u_3}{195l},$$

$$M_3 = \frac{D_{i-1}}{D} = \frac{15u_3 - 4u_4}{52l}.$$

Then by eqs. (2):

$$M_2 = \frac{u_3}{l} - (4M_3 + M_4),$$

$$M_5 = \frac{u_4}{l} - (M_3 + 4M_4) = -\frac{M_4}{2},$$

$$M_1 = -\frac{M_2}{2}.$$

Thus all the moments are known for this example, i.e., with six spans and loading on the fourth span only.

Reactions.

After the moments are found, either by the general or special method, for any condition of loading, the reactions will at once result from the substitution of the values thus found in the eqs. (15) to (21) of the preceding article, which it is not necessary to reproduce here.

Art. 21.—Deflection by the Common Theory of Flexure.

The deflection or sag of a beam subjected to loading at right angles to its axis is the displacement of the neutral surface in the direction of the loading. Ordinarily the beam is horizontal and the loading vertical, so that the deflection is also vertical. The entire deflection is due both to the lengthening and the shortening of the fibres on the two sides of the neutral surface and to the action of the transverse shear throughout the beam. The equation leading directly to the former portion is eq. (7) of Art., 9, but the equations of Art. 10 must be used to determine the deflection due to shear.

Let x_0 be the coordinate of some point at which the tangent of the inclination of the neutral surface to the axis of x is known; then from eq. (7) of Art. 9

$$\frac{dw}{dx}=\int_{x_0}^{x}\frac{M}{EI}dx. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$\frac{dw}{dx}$ will be at once recognized as the general value of the tangent of the inclination just mentioned, or, in the case of curved beams, as approximately the difference between the tangent, before and after flexure.

Again, let x_1 represent the coordinate of a point at which the deflection w is known, then from eq. (1):

$$w=\int_{x_1}^{x}\int_{x_0}^{x}\frac{M}{EI}dx^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The points of greatest or least deflection and greatest or least inclination of neutral surface are easily found by the aid of eqs. (1) and (2).

The point of greatest or least deflection is evidently found by putting

$$\frac{dw}{dx}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and solving for x. Since $\frac{dw}{dx}$ is the value of the tangent of the inclination of the neutral surface, it follows that *a point of greatest or least deflection is found where the beam is horizontal.*

Again, the point at which the inclination will be greatest or least is found by the equation

$$\frac{d\left(\frac{dw}{dx}\right)}{dx}=\frac{d^2w}{dx^2}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

But, approximately, $\frac{d^2w}{dx^2}$ is the reciprocal of the radius of curvature; hence *the greatest inclination will be found at that point at which the radius of curvature becomes infinitely great*, or, *at that point at which the curvature changes from positive to negative or vice versa.* These points are called points of "contra-flexure." Since:

$$M = EI\frac{d^2w}{dx^2},$$

there is no bending at a point of contra-flexure.

The moment of the external forces, M, will always be expressed in terms of x. After the insertion of such values, eqs. (1) and (2) may at once be integrated and (3) and (4) solved.

The coefficient of elasticity, E, is always considered a constant quantity; hence it may always be taken outside the integral signs. In all ordinary cases, also, I is constant throughout the entire beam. In such cases, then, there will only need to be integrated the expressions:

$$\int_{x_0}^{x} Mdx \quad \text{and} \quad \int_{x_1}^{x}\int_{x_0}^{x} Mdx^2.$$

It is sometimes convenient to express the tangent of inclination of the neutral surface and the deflection in terms of some known intensity k_0 of fibre stress at the distance d from the neutral surface and at a section of the beam where the known external bending moment is M_0. The desired expressions may readily be written by simply transforming eqs. (1) and (2) to the proper shape. It has been shown by eq. (10) of Art. 9 that $k_0 = \frac{M_0 d}{I}$, and hence that $I=\frac{M_0 d}{k_0}$. By substitution of this value of I first in eq. (1) and then in eq. (2), there will result:

$$\frac{dw}{dx}=\frac{k_0}{EM_0 d}\int_{x_0}^{x} M dx \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and

$$w=\frac{k_0}{EM_0 d}\int_{x_1}^{x}\int_{x_0}^{x} M dx^2. \quad . \quad . \quad . \quad . \quad (6)$$

Eqs. (5) and (6) give the desired expressions in which I and d are considered constant in accordance with all ordinary practice. In the use of these last two equations it is supposed that the conditions of any given problems will enable k_0 and M_0 to be computed as known quantities.

The general form of the integral in the second member of eq. (6) is easily determined. The quantities M_0 and M are exactly similar expressions with the same number of terms and of the same degree. The effect of the integration of M twice between the limits indicated is to raise the degree of each term of which it is composed by two, so that the double integration of Mdx^2 divided by M_0 will be a simple product al^2, a being a numerical quantity depending upon the manner of loading, the condition of the ends of the beam, or other attendant circumstances of the same general character. Inserting these results in eq. (6), the expression for the deflection will become

$$w=k_0\frac{al^2}{Ed}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6a)$$

Eq. (6a) is not often used, but there are some practical applications of formulæ in which it must be employed.

Deflection Due to Shearing.

That portion of the deflection due to transverse shearing may be determined as readily as that due to the lengthening and shortening of the fibres of the bent beam. In determining the requisite equations it is necessary to consider only the intensity of shear in the neutral surface, as it is the deflection of that surface which is sought.

Let w' be the deflection due to shearing and let ϕ represent the transverse shearing strain for a unit of length of the beam. The transverse strain for an indefinitely short portion dx of the neutral surface will then be $dw' = \phi dx$. If G represents the coefficient of elasticity for shear, while s represents the intensity of shear, eq. (3) of Art. 2 shows that $\phi = \frac{s}{G}$. There may then be written:

$$dw' = \phi dx = \frac{s}{G} dx. \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

By using the value of s given in eq. (7) of Art. 10,

$$dw' = \frac{Sd^2}{2IG} dx. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The general expressions for the shearing deflection will, therefore, take the form:

$$w' = \frac{d^2}{2IG} \int S dx. \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

The integration required in eq. (9) can be made with ease in any given case, as it is necessary only to express the value of the total transverse shear S in terms of x. The application of that equation to special cases will be made in a later article. Obviously the total deflection in any bent beam will be the sum:

$$w+w'. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Art. 22.—The Neutral Curve for Special Cases.

The curved intersection of the neutral surface with a vertical plane passing through the axis of a loaded, and originally straight, beam may be called the "neutral curve." The neutral curve is the locus of the extremities of the ordinates w of Art. 21; it therefore gives the deflection at any point of the beam due to the direct stresses of tension and compression in it, but not due to the effect of transverse shear, which will be treated in a subsequent article.

The method of finding the neutral curve for any particular case of beam or loading can be well illustrated by the operations in the following three cases:

Case I.

This case is shown in the accompanying figure, which represents a cantilever carrying a uniform load with a single weight W at its free end. As usual, the intensity of the uniform loading will be represented by p.

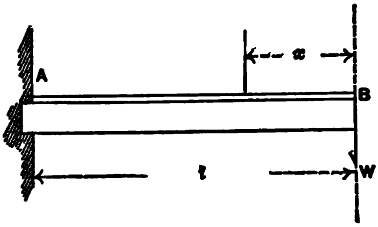


FIG. 1.

Measuring x and w from B, as shown, the general value of the bending moment is

$$M = EI\frac{d^2w}{dx^2} = Wx + \frac{px^2}{2}. \quad . \quad . \quad . \quad . \quad . \quad ①$$

Integrating between x and l, remembering that:

$$\frac{dw}{dx} = 0$$

for $x = l$:

$$EI\frac{dw}{dx} = \frac{W}{2}(x^2 - l^2) + \frac{p}{6}(x^3 - l^3). \quad . \quad . \quad . \quad ②$$

Hence

$$w = \frac{1}{EI}\left\{\frac{W}{2}\left(\frac{x^3}{3} - xl^2\right) + \frac{p}{6}\left(\frac{x^4}{4} - l^3x\right)\right\}. \quad . \quad . \quad ③$$

The greatest deflection, w_1, occurs for $x = l$. Hence

$$w_1 = -\frac{1}{EI}\left(\frac{Wl^3}{3} + \frac{pl^4}{8}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

This value of w_1 is the deflection of B below A. The general value of w in eq. (3) is the vertical distance (deflection) of B below the point located by x; as an ordinate it is measured upward from B as the origin of coordinates.

The greatest moment, M_1, exists at A, and its value is:

$$M_1 = Wl + \frac{pl^2}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

These equations are made applicable to a cantilever with a uniform load by simply making $W = 0$. They then become

$$M = EI\frac{d^2w}{dx^2} = \frac{px^2}{2}, \qquad . \ . \ . \ . \ . \ . \quad (6)$$

$$EI\frac{dw}{dx} = \frac{p}{6}(x^3 - l^3), \ . \ . \ . \ . \ . \ . \ . \ . \quad (7)$$

$$w = \frac{p}{6EI}\left(\frac{x^4}{4} - l^3x\right), \qquad . \ . \ . \ . \ . \ . \quad (8)$$

$$w_1 = -\frac{pl^4}{8EI}, \qquad . \ . \ . \ . \ . \ . \ . \ . \quad (9)$$

$$M_1 = \frac{pl^2}{2}. \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (10)$$

Again, for a cantilever with a single weight only at its free end, p is to be made equal to zero in the first set of equations. Those equations then become:

$$M = EI\frac{d^2w}{dx^2} = Wx, \ . \ . \ . \ . \ . \ . \ . \quad (11)$$

$$EI\frac{dw}{dx} = \frac{W}{2}(x^2 - l^2), \qquad . \ . \ . \ . \ . \ . \ . \quad (12)$$

$$w = \frac{W}{2EI}\left(\frac{x^3}{3} - xl^2\right), \qquad . \ . \ . \ . \ . \ . \quad (13)$$

$$w_1 = -\frac{Wl^3}{3EI}, \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (14)$$

$$M_1 = Wl. \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (15)$$

The general expressions for the shear and the intensity of loading are:

$$S = EI\frac{d^3w}{dx^3} = W + px, \quad . \quad . \quad . \quad . \quad (16)$$

$$EI\frac{d^4w}{dx^4} = p. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Case II.

This case, shown in the figure, is that of a non-continuous beam, supported at each end, and carrying both a uniform load (whose intensity is p) and a single weight W at its middle point. The reaction R, at either end, will then be

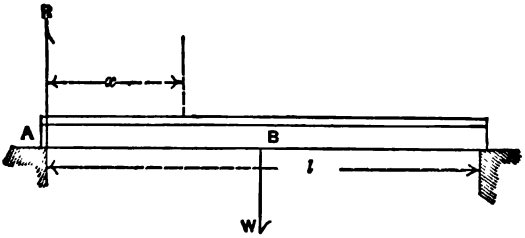


FIG. 2

$$R = \frac{pl + W}{2}.$$

The general value of the moment will then be

$$M = EI\frac{d^2w}{dx^2} = Rx - \frac{px^2}{2}. \quad . \quad . \quad . \quad . \quad ⑱$$

The origin of x and w is taken at A.
Remembering that

$$\frac{dw}{dx} = 0 \quad \text{for} \quad x = \frac{l}{2},$$

and integrating between the limits x and $\frac{l}{2}$,

$$EI\frac{dw}{dx}=\frac{R}{2}\left(x^2-\frac{l^2}{4}\right)-\frac{p}{6}\left(x^3-\frac{l^3}{8}\right). \quad . \quad . \quad . \qquad \text{⑲}$$

Again integrating

$$w=\frac{1}{EI}\left\{\frac{R}{2}\left(\frac{x^3}{3}-\frac{xl^2}{4}\right)-\frac{p}{6}\left(\frac{x^4}{4}-\frac{xl^3}{8}\right)\right\}. \quad . \quad . \qquad \text{⑳}$$

The greatest deflection w_1 occurs at the centre of the span, for which

$$x=\frac{l}{2}.$$

Hence

$$w_1=-\frac{l^3}{48EI}\left\{W+\frac{5}{8}pl\right\}. \quad . \quad . \quad . \quad . \qquad (21)$$

The greatest moment, also, is found by putting

$$x=\frac{l}{2}.$$

It has the value

$$M_1=\frac{l}{4}\left(W+\frac{pl}{2}\right). \quad . \quad . \quad . \quad . \quad . \qquad (22)$$

These formulæ are made applicable to a non-continuous beam carrying a uniform load only, by putting $W=0$. They then become

$$R=\frac{pl}{2},$$

$$M=EI\frac{d^2w}{dx^2}=\frac{px}{2}(l-x), \quad . \quad . \quad . \quad . \qquad (23)$$

$$EI\frac{dw}{dx}=\frac{p}{2}\left(\frac{x^2l}{2}-\frac{x^3}{3}-\frac{l^3}{12}\right), \quad . \quad . \quad . \quad . \quad . \quad (24)$$

$$w=\frac{p}{24EI}(2x^3l-x^4-l^3x), \quad . \quad . \quad . \quad . \quad . \quad (25)$$

$$w_1=-\frac{5pl^4}{384EI}=-\frac{5}{8}\cdot\frac{pl^4}{48EI}, \quad . \quad . \quad . \quad . \quad . \quad (26)$$

$$M_1=\frac{pl^2}{8}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

The formulæ for a beam of the same kind carrying a single weight at the centre are obtained by putting $p=0$ in the first set of equations. Those for the greatest deflection and greatest moment, only, however, will be given. They are

$$w_1=-\frac{Wl^3}{48EI}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

$$M_1=\frac{Wl}{4}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

The general values of the shear and intensity of loading are

$$S=\frac{dM}{dx}=R-px, \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

$$\frac{d^2M}{dx^2}=-p. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

Case III.

The general treatment of continuous beams requires the use of the theorem of three moments. The particular case to be treated is shown in Fig. 3. The beam covers the three spans, DA, AB, and BC, and is continuous over the two points of support, A and B.

$$\left.\begin{aligned} \text{Let } DA &= l_1 \\ \text{"}\quad AB &= l_2 \\ \text{"}\quad BC &= l_3 \end{aligned}\right\} \text{Let } l_2 = nl_1 = n'l_3.$$

Let the intensity of the uniform load on AB be represented by p and let the two single forces P and P' *only*, act

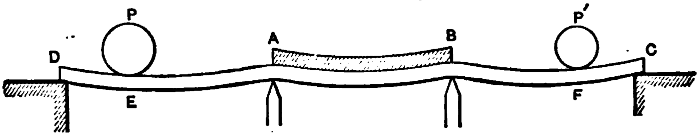


FIG. 3.

in the spans DA and BC respectively. Also let the two distances

$$DE = z_1 = al_1 \quad \text{and} \quad CF = a'l_3$$

be given. *It is required to find the magnitudes of the forces P and P′, if the beam is horizontal at A and B.*

Since the beam is horizontal at A and B, the bending moments over those two points of support will be equal to each other, for the load on AB is both uniform and symmetrical. Let this bending moment, common to A and B, be represented by M_2. As the ends of the beam simply *rest* at D and C, the moments at those two points reduce to zero.

Because the four points D, A, B, and C are in the same level, the first member of eq. (14) of Art. 19 becomes equal to zero.

If that equation be applied to the three points D, A, and B, the conditions of the present problem produce the following results:

$$M_a = 0, \quad M_b = M_c = M_2,$$

and

$$\frac{1}{l_c}\overset{c}{\Sigma} P(l_c^2 - z^2)z = p\frac{l_2^3}{4}.$$

Hence the equation itself will become

$$M_2(2l_1 + 3l_2) + \frac{P}{l_1}(l_1^2 - z_1^2)z_1 + p\frac{l_2^3}{4} = 0. \quad . \quad . \quad (32)$$

$$\therefore \quad M_2 = -\frac{4P(l_1^2 - z_1^2)z_1 + pl_1l_2^3}{4l_1(2l_1 + 3l_2)};$$

$$\therefore \quad M_2 = -l_1\frac{4P(1 - a^2)a + pn^3l_1}{4(2 + 3n)}. \quad . \quad . \quad . \quad (33)$$

$$\therefore \quad \text{Reaction at } D = R_1 = P\frac{l_1 - z_1}{l_1} + \frac{M_2}{l_1}. \quad . \quad . \quad (34)$$

As the origin of z_1 is at D, x will be measured from the same point.

Separate expressions for moments must be obtained for the two portions, DE and EA of l_1, because the law of loading in that span is not continuous.

Taking moments about any point of EA

$$M = EI\frac{d^2w}{dx^2} = R_1x - P(x - z_1). \quad . \quad . \quad . \quad . \quad (35)$$

Remembering that

$$\frac{dw}{dx} = 0$$

for $x = l_1$, and integrating between the limits x and l_1

$$EI\frac{dw}{dx} = \frac{R_1}{2}(x^2 - l_1^2) - \frac{P}{2}(x^2 - l_1^2) + Pz_1(x - l_1). \quad . \tag{36}$$

Again, remembering that $w = 0$ for $x = l_1$, and integrating between the limits x and l_1,

$$EIw = \frac{R_1}{2}\left(\frac{x^3}{3} - l_1^2x + \frac{2l_1^3}{3}\right) - \frac{P}{2}\left(\frac{x^3}{3} - l_1^2x + \frac{2l_1^3}{3}\right) + Pz_1\left(\frac{x^2}{2} - l_1x + \frac{l_1^2}{2}\right). \tag{37}$$

Taking moments about any point in DE

$$EI\frac{d^2w}{dx^2} = R_1x; \quad . \ . \ . \ . \ . \ . \ . \tag{38}$$

$$\therefore \quad EI\frac{dw}{dx} = R_1\frac{x^2}{2} + C. \quad . \ . \ . \ . \ . \ . \tag{39}$$

Making $x = z_1$ in eqs. (36) and (39), then subtracting

$$C = -\frac{R_1}{2}l_1^2 - \frac{P}{2}(z_1^2 - l_1^2) + Pz_1(z_1 - l_1);$$

$$\therefore \quad EI\frac{dw}{dx} = \frac{R_1}{2}(x^2 - l_1^2) - \frac{P}{2}(z_1^2 - l_1^2) + Pz_1(z_1 - l_1). \tag{40}$$

Remembering that $w = 0$ for $x = 0$, and integrating between the limits x and 0,

$$EIw = \frac{R_1}{2}\left(\frac{x^3}{3} - l_1^2x\right) - \frac{P}{2}(z_1^2 - l_1^2)x + Pz_1(z_1 - l_1)x. \tag{41}$$

Making $x = z_1$ in eqs. (37) and (41), then subtracting

$$\frac{R_1l_1^3}{3} - \frac{P}{3}(l_1^3 - z_1^3) + \frac{Pz_1}{2}(l_1^2 - z_1^2) = 0. \quad . \ . \ . \tag{42}$$

Putting the value of M_2 from eq. (33) in eq. (34), then inserting the value of R_1, thus obtained, in eq. (42), after making $z_1 = al_1$,

$$P\left[(1-a)-\frac{a(1-a^2)}{2+3n}-(1-a^3)+\frac{3}{2}a(1-a^2)\right]=\frac{pn^3l_1}{4(2+3n)};$$

$$\therefore\ P=\frac{pn^2l_1}{6a(1-a^2)}=\frac{pnl_2}{6a(1-a^2)}. \quad . \quad . \quad . \qquad (43)$$

This is the desired value of P, which will cause the beam to be horizontal over the two points of support A and B when the span AB carries a uniform load of the intensity p.

By the aid of eq. (43), eq. (33) now gives

$$M_2=-pl_1^2\frac{(2n^2+3n^3)}{12(2+3n)}=-\frac{pn^2l_1^2}{12}=-\frac{pl_2^2}{12}. \quad . \qquad (44)$$

It is to be noticed that M_2 is entirely independent of l_1 or l_2. Eq. (43) also gives

$$p=P\frac{6a(1-a^2)}{n^2l_1}. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (45)$$

Hence

$$M_2=-\frac{Pl_1}{2}(1-a^2)a. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (46)$$

Thus any of the preceding equations may be expressed in terms of p or P.

R_1 also becomes

$$R_1=\frac{pnl_2}{6a(1+a)}-\frac{pnl_2}{12}, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (47)$$

or

$$R_1=P(1-a)[1-\tfrac{1}{2}a(1+a)]. \quad . \quad . \quad . \quad . \quad . \qquad (48)$$

It is clear that there cannot be a point of no bending in *DE*. Hence the point of contra-flexure must lie between *E* and *A*, Fig. 3. In order to locate this point, according to the principles already established, the second member of eq. (35) must be put equal to zero. Doing so and solving for x,

$$x = \frac{P}{P - R_1} z_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

Since P is always greater than R_1, there will always be a point of contra-flexure.

All these equations will be made applicable to the span *BC* by simply writing a' for a, l_3 for l_1, and n' for n.

As an example, let

$$a = \tfrac{1}{2} \quad \text{and} \quad n = 1.$$

Eqs. (43), (44), and (47) then give

$$P = \tfrac{4}{9} pl,$$

$$M_2 = -\frac{pl^2}{12} = -\frac{3Pl}{16},$$

$$R_1 = pl_1(\tfrac{2}{9} - \tfrac{1}{12}) = \tfrac{5}{36} pl = \tfrac{5}{16} P;$$

after writing,

$$l_1 = l_2 = l_3 = l.$$

In general, the span l_1 is called "a beam fixed at one end, simply supported at the other and loaded at any point with the single weight, P."

Let it, again, be required *to find an intensity, "p'," of a uniform load, resting on the span l_1, which will cause the beam to be horizontal at the points A and B.*

Since the load is continuous, only one set of equations will be required for the span. The equation of moments will be

$$EI\frac{d_2w}{dx^2}=R_1x-\frac{p'x^2}{2}. \quad . \quad . \quad . \quad . \quad . \qquad (50)$$

Integrating between the limits x and l_1,

$$EI\frac{dw}{dx}=\frac{R_1}{2}(x^2-l_1^2)-\frac{p'}{6}(x^3-l_1^3). \quad . \quad . \quad . \qquad (51)$$

Integrating between the limits x and 0,

$$EIw=\frac{R_1}{2}\left(\frac{x^3}{3}-l_1^2x\right)-\frac{p'}{6}\left(\frac{x^4}{4}-l_1^3x\right). \quad . \quad . \qquad (52)$$

But, also, $w=0$, when $x=l_1$. Hence

$$R_1\frac{l_1^3}{3}=\frac{p'l_1^4}{8}; \quad \therefore\ R_1=\tfrac{3}{8}p'l_1. \quad . \quad . \quad . \quad . \qquad (53)$$

This equation gives the value R_1 when p' is known. Making $x=l_1$ in eq. (50), and using the value of R_1 from eq. (53),

$$M_2=p'l_1^2(\tfrac{3}{8}-\tfrac{1}{2})=-\frac{p'l_1^2}{8}. \quad . \quad . \quad . \quad . \qquad (54)$$

Adapting eq. (32) to the present case,

$$M_2(2l_1+3l_2)+\tfrac{1}{4}(p'l_1^3+pl_2^3)=0;$$

$$\therefore\ M_2=-\frac{(p'+pn^3)l_1^2}{4(2+3n)}. \quad . \quad . \quad . \quad . \qquad (55)$$

Equating these two values of M_2,

$$p'=\tfrac{2}{3}pn^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (56)$$

Thus is found the desired value of p'. In this case the span l_1 is called "a beam fixed at one end, simply supported at the other and uniformly loaded."

The points of contra-flexure are found by putting the second member of eq. (50) equal to zero and solving for x, after introducing the value of R_1 from eq. (53). Hence

$$\tfrac{3}{4}l_1x - x^2 = 0,$$

or

$$x = 0 \quad \text{and} \quad x = \tfrac{3}{4}l_1.$$

Between the simply supported end and point of contra-flexure the beam is evidently convex *downward*, and convex upward in the other portion of the spans l_1 and l_2, whether the load is single or continuous. Moments of different signs will then be found in these two portions, and there will be a maximum for each sign. The location of the sections in which these greatest moments act may be made in the ordinary manner by the use of the differential calculus; but the *negative* maximum is evidently M_2, given by eqs. (44) and (55). On the other hand, the *positive* maximum is clearly found at the point of application of P in the case of a single load, and at the point

$$x = \tfrac{3}{8}l_1$$

in the case of a continuous load. These conclusions will at once be evident if it be remembered that the portion of the beam between the supported end and point of contra-flexure is, in reality, *a beam simply supported at each end.* These moments will have the values

$$M_1 = Pl_1(1-a)a - l_1\frac{4P(1-a^2)a^2 + pan^3l_1}{4(2+3n)}, \quad . \quad (57)$$

$$M_1' = \tfrac{9}{128}p'l_1^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (58)$$

In case of a single load if P is given, and not p, eq. (45) shows

$$M_1 = Pl_1(1-a)a[1-\tfrac{1}{2}a(1+a)].$$

The points of greatest deflection are found by putting the second members of eqs. (36), (40), and (51) each equal to zero, and then solving for x. They are not points of great importance, and the solutions will not be made.

The following are the general values of the shears for a single load on l_1:

$$\text{In } AE, \quad S = EI\frac{d^3w}{dx^3} = R_1 - P; \quad \text{[from eq. (35)].}$$

$$\text{In } ED \quad S_1 = EI\frac{d^3w}{dx^3} = R_1; \quad \text{[from eq. (38)].}$$

The shear in l_1 for the uniform load p' is

$$S' = EI\frac{d^3w}{dx^3} = R_1 - p'x; \quad \text{[from eq. (50)].}$$

Also

$$\textit{Intensity of load} = EI\frac{d^4w}{dx^4} = -p'.$$

As has already been observed, all the equations relating to the span l_1 may be made applicable to the span l_3 by changing a to a' and n to n'.

The span l_2 remains to be considered.

Since the bending moments at A and B are equal to each other, and since the loading is uniformly continuous, half of it (the load pl_2) will be supported at A and the other half at B. In other words, the vertical shear at an indefinitely short distance to the right of A, also to the left of B, will be equal to $\frac{pl_2}{2}$. Let x be measured to the right and from A. The bending moment at any section x will be

$$EI\frac{d^2w}{dx^2}=M_2+\frac{pl_2}{2}x-\frac{px^2}{2},$$

or

$$EI\frac{d^2w}{dx^2}=M_2+\frac{p}{2}(l_2x-x^2). \quad . \quad . \quad . \quad (59)$$

Integrating between the limits x and 0,

$$EI\frac{dw}{dx}=M_2x+\frac{p}{2}\left(\frac{l_2x^2}{2}-\frac{x^3}{3}\right). \quad . \quad . \quad . \quad (60)$$

Again, integrating between the same limits,

$$EIw=\frac{M_2x^2}{2}+\frac{p}{12}\left(l_2x^3-\frac{x^4}{2}\right). \quad . \quad . \quad . \quad (61)$$

Since

$$\frac{dw}{dx}=0$$

for $x=l_2$, eq. (60) will give M_2 independently of preceding equations. Following this method, therefore,

$$M_2=-\frac{pl_2^2}{12}.$$

This is the same value which has already been obtained. Introducing the value of M_2,

$$EI\frac{d^2w}{dx^2}=\frac{p}{2}\left(l_2x-x^2-\frac{l_2^2}{6}\right), \quad . \quad . \quad . \quad (62)$$

$$EI\frac{dw}{dx}=\frac{p}{2}\left(\frac{l_2x^2}{2}-\frac{x^3}{3}-\frac{l_2^2}{6}x\right), \quad . \quad . \quad . \quad (63)$$

$$EIw=\frac{px^2}{12}\left(l_2x-\frac{x^2}{2}-\frac{l_2^2}{2}\right). \quad . \quad . \quad . \quad (64)$$

The points of contra-flexure are found by putting the second member of eq. ([illegible]) equal to zero. Hence

$$x^2 - l_2 x = -\frac{l_2^2}{6};$$

$$\therefore \quad x = l_2\left(\frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{1}{6}}\right) = \begin{cases} 0.789 l_2. \\ 0.211 l_2. \end{cases}$$

The moment at the centre of the span is found by putting

$$x = \frac{l_2}{2}$$

in eq. (6[illegible]):

$$M_1 = \frac{p l_2^2}{24}.$$

This is the greatest *positive* moment

The general value of the shear is

$$S = EI\frac{d^3 w}{dx^3} = p\left(\frac{l_2}{2} - x\right);$$

and the intensity of load

$$EI\frac{d^4 w}{dx^4} = -p.$$

The span l_2 is generally called "a beam fixed at both ends and uniformly loaded."

It is sometimes convenient to consider a single load at the centre of the span l_2, while the beam remains horizontal at A and B; in other words, to consider "a beam fixed at each end and supporting a weight at the centre."

Let W represent this weight; then a half of it will be the shear at an indefinitely short distance to the right of

A and left of *B*. As before, let x be measured from *A*, and positive to the right. The moment at any point will be

$$EI\frac{d^2w}{dx^2}=M_2-\frac{Wx}{2}. \quad . \quad . \quad . \quad . \quad . \quad (65)$$

Integrating between x and 0,

$$EI\frac{dw}{dx}=M_2x-\frac{Wx^2}{4}. \quad . \quad . \quad . \quad . \quad . \quad (66)$$

If $x=\frac{l_2}{2}$, then will

$$\frac{dw}{dx}=0;$$

hence

$$M_2=\frac{Wl_2}{8}.$$

The general value of the moment then becomes

$$M=EI\frac{d^2w}{dx^2}=\frac{Wl_2}{8}-\frac{Wx}{2}. \quad . \quad . \quad . \quad . \quad (67)$$

If $x=\frac{l_2}{2}$ in this equation, the bending moment at the centre (where W is applied) has the value

$$\textit{Centre moment}=-\frac{Wl_2}{8}.$$

Hence *the bending moments at the centre and ends are each equal to the product of the load by one eighth the span, but have opposite signs.*

A second integration between x and 0 gives

$$w=\frac{1}{EI}\left(\frac{Wl_2x^2}{16}-\frac{Wx^3}{12}\right). \quad . \quad . \quad . \quad . \quad (68)$$

Hence the deflection at the centre has the value

$$\textit{Centre deflection}=\frac{Wl_2^3}{192EI}.$$

By placing $M = 0$, the points of contra-flexure are found at the distance from each end,

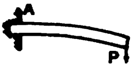
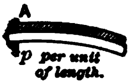
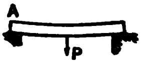
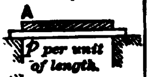
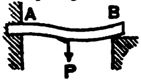
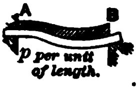
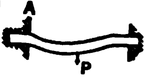
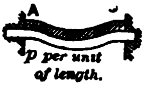
$$x_1 = \frac{l_2}{4}.$$

Addendum to Art. 22.

The formulæ of this article furnish the solutions of many practical questions of maxima deflections and moments. The latter for several ordinary cases are given in the following tabulation:

P is the weight in pounds at *end of beam* or *centre of span*.

p is the load in pounds *per lin. ft.* of beam.

	Beam.	Maximum Moment.	Maximum Deflection.	Point of Contra-flexure.
I		Pl at A	$576\frac{Pl^3}{EI}$ at A	
II		$\frac{1}{2}pl^2$ at A	$216\frac{pl^4}{EI}$ at A	
III		$\frac{1}{4}Pl$ at centre	$36\frac{Pl^3}{EI}$ at centre	
IV		$\frac{1}{8}pl^2$ at centre	$22.5\frac{pl^4}{EI}$ at centre	
V		$-\frac{3}{16}Pl$ at A $\frac{5}{32}Pl$ at centre	$16.16\frac{Pl^3}{EI}$ at $0.447l$ from B	$\frac{3}{11}l$ from B Reaction at B $=\frac{5}{16}P$
VI		$-\frac{1}{8}pl^2$ at A $+\frac{9}{128}pl^2$ at $\frac{3}{8}l$ from B	$9.35\frac{pl^4}{EI}$ at $0.4215l$ from B	$\frac{3}{4}l$ from B Reaction at B $=\frac{3}{8}pl$
VII		$-\frac{1}{8}Pl$ at A $\frac{1}{8}Pl$ at centre	$9\frac{Pl^3}{EI}$ at centre	$\frac{1}{4}l$ from each end
VIII		$-\frac{1}{12}pl^2$ at A $\frac{1}{24}pl^2$ at centre	$4.5\frac{pl^4}{EI}$ at centre	$0.211l$ from each end

l is the length of beam or of span in *feet*.

E is the coefficient of elasticity in pounds per sq. inch.

I is the moment of inertia of the normal section of the beam with all dimensions of section in *inches*.

The "Max. Moments" will be in *foot pounds*, and the "Max. Deflections" will be in *inches*.

In the use of eq. (2), Art. 62, in its many practical applications, it is best to have the moment M in inch-pounds, which will result from simply multiplying the "Max. Moments" of the preceding table by 12.

Case I results from eqs. (14) and (15); Case II from eqs. (9) and (10); Case III from eqs. (28) and (29); Case IV from eqs. (26) and (27). In Case V the reaction is found by putting $a = \frac{1}{2}$ in eq. (48); the point of "Max. Deflection" is found by placing $z_1 = \frac{1}{2}l$ in eq. (40), and the resulting value of $\frac{dw}{dx}$ equal to zero and solving for x, which latter value in eq. (41) will give "Max. Deflection." Case VI results from treating eqs. (53), (51), and (52) in precisely the same manner. Case VII results directly from the formulæ on page 126. Case VIII results directly from the equations on pages 124 and 125.

The preceding cases are those which commonly occur with constant values of E and I. Other cases, such as a single load at any point, or partial uniform load over any part of span, are to be treated by the same general principles.

Art. 23.—Deflection Due to Shearing in Special Cases.

The deflection due to transverse shearing only in all the ordinary cases of loaded beams can readily be computed by aid of the general eq. (9) of Art. 21. If d is the distance of the most remote fibre from the neutral axis of any normal section whose moment of inertia about the same axis is I, and if G and S are the coefficient of elasticity and total transverse shear respectively, the deflection, w', sought is

$$w' = \frac{d^2}{2IG}\int S dx. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The limits of the integration must be indicated for each particular case.

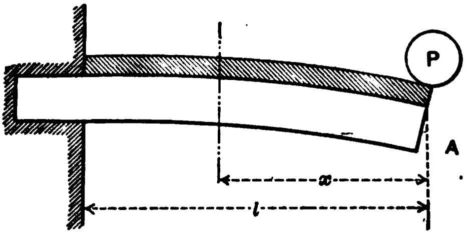


Fig. 1.

In Fig. 1 let the cantilever, whose length is l, carry the single load P at its end, and the uniform load p per linear unit. The shear at any section distant x from A is $S = P + px$. The substitution of this value of S in eq. (1) will give

$$w' = \frac{d^2}{2IG}\int_0^l (P + px)dx = \frac{d^2}{2IG}\left(Pl + \frac{pl^2}{2}\right). \quad . \quad (2)$$

If the uniform load only acts, $P = 0$; and if P only acts, $p = 0$.

Fig. 2 shows the case of a simple beam supported at each end, carrying a uniform load p per linear unit and the single load P at the centre of the span. The reaction $R = \frac{1}{2}(P + pl)$, and the shear $S = R - px$. Hence eq. (1) gives the general value of the deflection

$$w' = \frac{d^2}{2IG}\int_0^x (R - px)dx = \frac{d^2}{2IG}\left\{\frac{x}{2}(P + pl) - \frac{px^2}{2}\right\}. \quad (3)$$

And for the centre ot the span:

$$w' = \frac{d^2}{2IG}\int_0^{l/2}(R - px)dx = \frac{d^2}{8IG}\left(Pl + \frac{pl^2}{2}\right). \quad . \quad (4)$$

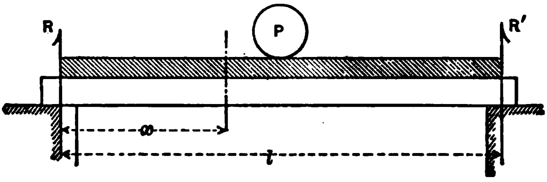
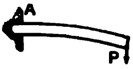
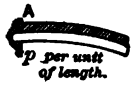
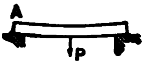
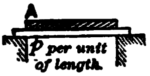
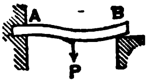
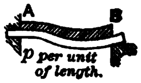
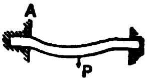
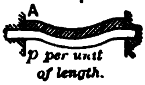


FIG. 2.

The values of the deflection w' may be similarly written for other cases. The following table gives the results for the cases indicated, which are those commonly required.

	Beam.	End Shear.	Shear S.	Section for Deflection w'.	Deflection w'.
I		P	P	l from end	$\frac{d^2Pl}{2IG}$
II		o	px	l " "	$\frac{d^2pl^2}{4IG}$
III		$\frac{1}{2}P$	$\frac{1}{2}P$	$\frac{1}{2}l$ " "	$\frac{d^2Pl}{8IG}$
IV		$\frac{1}{2}pl$	$p(\frac{1}{2}l-x)$	$\frac{1}{2}l$ " "	$\frac{d^2pl^2}{16IG}$
V		$\frac{5}{16}P$ $\frac{5}{16}P$	$\frac{5}{16}P$ $\frac{5}{16}P$	$.447l$ " " $.5l$ " "	$\frac{d^2Pl}{14.32IG}$ $\frac{d^2Pl}{12.8IG}$
VI		$\frac{3}{8}pl$	$p(\frac{3}{8}l-x)$	$.4215l$ " "	$\frac{.0347d^2pl^2}{IG}$
VII		$\frac{1}{2}P$	$\frac{1}{2}P$	$.5l$ " "	$\frac{d^2Pl}{8IG}$
VIII		$\frac{1}{2}pl$	$p(\frac{1}{2}l-x)$	$.5l$ " "	$\frac{d^2pl^2}{16IG}$

The end shears in this table are the reactions taken from the table of the preceding article, the "Beams" in the two tables being the same.

The total deflection for any particular beam is to be found by adding the "Max. Deflection" from the table of the preceding article to the w' found in the above table.

In the notation of the preceding article, if w_1 is the deflection due to the lengthening and shortening of the fibres the total deflection in any case will be

$$w = w_1 + w'. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

These formulæ for shearing deflection, like all the formulæ relating to the distribution of transverse shearing in a bent beam, are more accurately applicable to rectangular or circular sections than to others.

Art. 24.—The Flexure of Long Columns.

A "long column" is a piece of material whose length is a number of times its breadth or width, and which is subjected to a compressive force exerted in the direction of its length. Such a piece of material will not be strained or compressed directly back into itself, but will yield laterally as a whole, thus causing flexure. If the length of a long column is many times the width or breadth, the failure in consequence of flexure will take place while the pure compression is very small.

As with beams, so with columns, the ends may be "fixed," so that the end surfaces do not change their position however great the compression or flexure. Such a column is frequently, perhaps usually, said to have fixed ends. If the ends of the column are free to turn in any direction, being simply supported, as flexure takes place, the column is said to have "round" ends. It is clear that if the column has freedom in one or several directions only, it will be a "round" end column in that one direction, or those several directions, only. It is also evident that a column may have one end round and one end flat or fixed.

In Fig. 1 let there be represented a column with flat ends, vertical and originally straight. After external pressure is imposed at A, the column will take a shape similar to that represented. Consequently the load P, at A, will act with a lever-arm at any section equal to the deflection of that section from its original position. Let y be the general value of that deflection, and at B let $y=y_1$. Let x be measured from A, as an origin, along the original axis of the column. In accordance with principles already established, the condition of fixedness at each of the ends A and C is secured by the application of a *negative* moment, $-M$. It is known from the general condition of the column that the curve of its axis will be convex toward the axis of x at and near A, while it will be concave at and near B (the middle point of the column). Hence, since y is positive toward the left, and since the ordinate and its second derivative must have the same sign when the curve is convex toward the axis of the abscissæ, the general equation of moments must be written as follows:

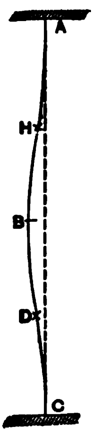


Fig. 1.

$$EI\frac{d^2y}{dx^2}=M-Py. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Multiplying by $2dy$,

$$EI\frac{2dyd^2y}{dx^2}=2Mdy-P2ydy;$$

$$\therefore\ EI\left(\frac{dy}{dx}\right)^2=2My-Py^2+(c=0); \quad . \quad . \quad . \quad (2)$$

$c=0$, because the column has flat ends, and

$$\frac{dy}{dx}=0$$

when $y=0$. Also

$$\frac{dy}{dx}=0$$

when $y=y_1$;

$$\therefore\ M=\frac{Py_1}{2}. \quad \ldots\ldots\ldots \quad (3)$$

Eq. (2) now becomes

$$\sqrt{\frac{EI}{P}}\ \frac{dy}{\sqrt{y_1y-y^2}}=dx;$$

$$\therefore\ x=\sqrt{\frac{IE}{P}}\ \text{versin}^{-1}\frac{2y}{y_1}. \quad \ldots\ldots \quad (4)$$

If $y=y_1$,

$$x=\frac{l}{2}=\pi\sqrt{\frac{EI}{P}}. \quad \ldots\ldots\ldots \quad (5)$$

In this equation l is the length of the column. From eq. (5) there may be deduced

$$P=\frac{4\pi^2EI}{l^2}. \quad \ldots\ldots\ldots \quad (6)$$

It is to be observed that P is *wholly independent of the deflection*, i.e., it remains the same, whatever may be the amount of deflection, after the column begins to bend. Consequently, if the elasticity of the material were perfect, the weight P would hold the column in any position in which it might be placed, after bending begins.

Eq. (6) forms the basis of "Hodgkinson's Formula" for the resistance of long columns, of which more will be given hereafter. It was first established by Euler.

Some very important results follow from the consideration of Fig. 1 in connection with the preceding equations.

The bending moment at the centre, B, of the column is obtained by placing $y=y_1$ in eq. (1); its value is, consequently,

$$M'=-M+Py_1=M. \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Hence *the bending at the centre of the column is exactly the same (but of opposite sign) as that at either end.* Between A and B, then, there must be a point of contra-flexure.

Putting the second member of eq. (1) equal to zero, and introducing the value of M from eq. (3),

$$y=\frac{y_1}{2}.$$

Introducing this value of y in eq. (4), and bearing in mind eq. (5),

$$x=\frac{\pi}{2}\sqrt{\frac{EI}{P}}=\frac{l}{4}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The points of contra-flexure, then, are at H and D, $\frac{1}{4}l$ and $\frac{3}{4}l$ from A.

Hence *the middle half of the column (HD) is actually a column with round ends,* and it is equal in resistance to a fixed-end column of double its length.

Hence writing l' for $\frac{l}{2}$ and putting $2l'$ for l in eq. (6),

$$P=\frac{\pi^2EI}{l'^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Eq. (9) gives the value of P for a round-end column.

Again, either the upper three quarters (AD) or the lower three quarters (CH) of the column is very nearly equivalent to a column with one end flat and one end round, and its resistance is equal to that of a fixed-end column whose length is $\frac{4}{3}$ its own. Putting, therefore,

$$l_1 = \tfrac{3}{4}l,$$

and introducing

$$l = \frac{4}{3}l_1$$

in eq. (6),

$$P = 2.25\frac{\pi^2 EI}{l_1^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

The last case is not quite accurate, because the ends of the columns HC and AD are not exactly in a vertical line.

In reality, the column under compression may be composed of any number of such parts as HD, with the portions HA and CD at the ends, thus taking a serpentine shape, so far as pure equilibrium is concerned. In such a condition the column would be subjected to considerably less bending than in that shown in the figure. In ordinary experience, however, the serpentine shape is impossible, because the slightest jar or tremor would cause the column to take the shape shown in Fig. 1. Hence the latter case only has been considered.

If r is the radius of gyration and S the area of normal section of the column, eqs. (6) and (9) will take the forms

$$\frac{P}{S} = \frac{4\pi^2 Er^2}{l^2} \quad \text{and} \quad \frac{P}{S} = \frac{\pi^2 Er^2}{l^2}.$$

Eq. (10) will, of course, take a corresponding form.

These equations evidently become inapplicable when $\frac{P}{S}$ approaches C, the ultimate compressive resistance of the material in short blocks. The corresponding values of $\left(\frac{l}{r}\right)$ at the limit are

$$\frac{l}{r}=2\pi\sqrt{\frac{E}{C}} \quad \text{and} \quad \frac{l}{r}=\pi\sqrt{\frac{E}{C}} \quad . \quad . \quad . \quad (11)$$

for fixed and round ends respectively; other conditions of ends will be included between those two.

If for structural steel

$$E=30,000,000 \quad \text{and} \quad C=60,000,$$

the above values become 140 and 70, nearly.

Euler's formula, therefore, is strictly applicable only to structural steel columns, with ends fixed or rounded, for which $l \div r$ exceeds 140 and 70, respectively.

If for cast iron

$$E=14,000,000 \quad \text{and} \quad C=100,000,$$

eqs. (11) give

$$\frac{l}{r}=74 \quad \text{and} \quad \frac{l}{r}=37, \text{ nearly.}$$

Euler's formula evidently becomes inapplicable considerably above the limits indicated, since columns in which $\frac{l}{r}$ has those values will not nearly sustain the intensity C.

The analytical basis of "Gordon's Formula" for the resistance of long columns is so closely associated with the empirical that both will be treated together hereafter.

Art. 25.—Graphical Determination of the Resistance of a Beam.

The graphical method is well adapted to the treatment of beams whose normal sections are limited either wholly or in part by irregular curves. In Fig. 1 is represented the normal section of such a beam, the centre of gravity of the section being situated at *C*. The lines *HL*, *AB*, and *DF* are parallel. As is known by the common theory of flexure, the neutral axis will pass through *C*.

Let *aa* be any line on either side of *AB*, then draw the lines *aa'* normal to *AB*, having made *MN* and *HL* equidistant from *AB*. From the points *a'* thus determined draw straight lines to *C*. These last lines will include intercepts, *bb*, on the original lines *aa*. Let every linear element parallel to *AB*, on each side of *C*, be similarly treated. All the intercepts found in this manner will compose the shaded figure.

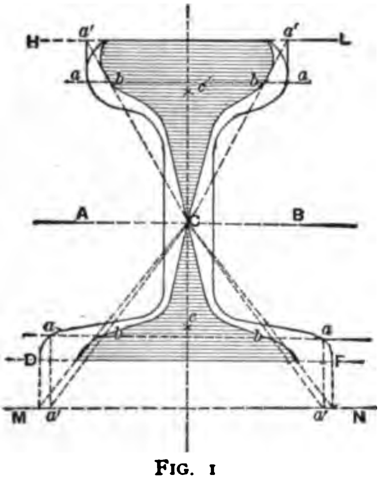


FIG. 1

This operation in reality, and only, determines an amount of stress with a uniform intensity identical with that developed in the layer of fibres farthest from the neutral axis, and equal to the total bending stress existing in the section; this latter stress, of course, having a variable intensity. HL represents the layer of fibres farthest from the neutral surface, consequently MN was taken at the same distance from AB. Any other distance might have been taken, but the intensity of the uniform stress would then have had a value equal to that which exists at that distance from the neutral axis. Again, a different intensity might have been chosen for the stress on each side of AB. It is most convenient, however, to use the greatest intensity in the section for the stress on both sides of the neutral axis; this intensity, which is the modulus of rupture by bending, will be represented, as heretofore, by K.

Let c and c' be the centres of gravity of the two shaded figures. These centres can readily and accurately be found by cutting the figures out of stiff manilla paper and then balancing on a knife-edge. Let s represent the area of the shaded surface below AB, and s' the area of that above AB.

Because this is a case of pure bending, the stresses of tension must be equal to those of compression. Hence

$$Ks = Ks', \quad \text{or} \quad s = s'. \quad . \quad . \quad . \quad . \quad (1)$$

The moment of the compression stresses about AB will be

$$Ks \times c'C.$$

The moment of the tensile stresses about the same line will be

$$Ks \times cC.$$

Consequently the resisting moment of the whole section will be

$$M = Ks(c'C + cC) = Ks \times cc'. \quad . \quad . \quad . \quad . \quad (2)$$

Thus the total resisting moment is completely determined. In some cases of irregular section the method becomes absolutely necessary.

It is to be observed that the centre of gravity, c or c', is at the same normal distance from AB as the centre of the actual stress on the same side of AB with c or c'.

Art. 26.—The Common Theory of Flexure with Unequal Values of Coefficients of Elasticity.

In all cases it has hitherto been assumed that the coefficient of elasticity for tension is equal to the same quantity for compression. In reality this is exactly true for possibly no material whatever, though the error, fortunately, is not serious for the greater portion of the materials used by the engineer. By the aid of the assumptions used in the common theory of flexure, formulæ involving this difference of coefficients may be deduced. As these are of little real value, however, a few general results only will be obtained.

Let E represent the coefficient of elasticity for tension.

Let E' represent the coefficient of elasticity for compression.

As has before been assumed, the normal sections of the beam, which are plane before flexure, will be taken as plane and normal to the neutral surface after flexure. Also, as before (Art. 18), let u represent the rate of strain (strain for unit of length of fibre) at unit's distance from the neutral surface; let the variable width of the section be represented by b, while y represents the variable normal distance of the element bdy from the neutral axis of the section. The element of the tensile stress in the section will be

$$Euy \, . \, bdy.$$

The elementary moment of the same will be

$$Euy^2bdy.$$

In precisely the same manner, the elementary compressive moment will be

$$E'uy^2bdy.$$

Consequently the total resisting moment will have the value

$$M = u\left[E\int_0^{y_1} y^2bdy + E'\int_{-y'}^{0} y^2bdy\right]$$

$$= \frac{K}{y_1}\int_0^{y_1} y^2bdy + \frac{K'}{y'}\int_{-y'}^{0} y^2bdy. \quad . \quad . \quad . \quad (1)$$

The ordinates y and y_1 are those belonging to the extreme fibres of the section, while K and K' represent stress intensities in those fibres. The general value of y is also affected with the negative sign on the compression side of the beam.

It has been shown in Art. 9 that

$$u = \frac{1}{\rho};$$

also, in the case of straight beams, that

$$\frac{1}{\rho} = \frac{d^2w}{dx^2},$$

w being the deflection and x the abscissa measured along the axis of the beam. For the sake of brevity let the quantity in the brackets in the second member of eq. (1) be represented by EJ, in which, consequently, E' will be displaced by nE, n being the ratio between E and E'. Eq. (1) may then take the form

$$M = EJ\frac{d^2w}{dx^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

or

$$\frac{M}{EJ} = \frac{d^2w}{dx^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If M and J are expressed in terms of x, w may at once be found. If, as is usual, the section is uniform, then will J be constant and M, only, will be a function of x.

If the section is rectangular, b will be constant and J will take the following value:

$$J = \frac{by_1^3}{3} + n\frac{by'^3}{3};$$

$$\therefore \; J = \frac{b}{3}(y_1^3 + ny'^3). \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Because the internal tensile stress in any section must equal the internal compressive stress in the same section

$$Eu\int_0^{y_1} bydy = E'u\int_0^{y'} bydy. \quad . \quad . \quad . \quad . \quad (5)$$

Eq. (5) will enable the neutral axis of any section to be located. If the section is symmetrical, the neutral axis will evidently be situated on that side of the centre of gravity of the section on which is found the greatest coefficient of elasticity.

Art. 27.—Greatest Stresses at any Point in a Beam.

If the approximate conditions on which are based the formulæ found in Arts. 9 and 10 are assumed, some interesting and important results may be very easily obtained.

The eqs. (13), (14), and (15) of Art. 2, App. I, lead to the ellipsoid of stress, and hence to all of its special cases and consequences. The equation representing the ellipsoid of stress might first be found, and then the special form relating to the case considered. It will be more simple and direct, however, to use those equations immediately.

If, as in Art. 9, beams carrying loads at right angles to their axes be assumed, in which

$$T_1 = T_3 = N_2 = N_3 = 0,$$

eqs. (13), (14), and (15) of Art. 2, App. I, reduce to

$$\begin{aligned} N_1 \cos p + T_2 \cos r &= P \cos \pi, \\ T_2 \cos p \qquad\qquad &= P \cos \rho. \end{aligned}$$

But since all stress is assumed to be found in planes parallel to ZX

$$\cos r = \sin p \quad \text{and} \quad \cos \rho = \sin \pi.$$

Hence

$$N_1 \cos p + T_2 \sin p = P \cos \pi, \quad . \quad . \quad . \quad . \quad (1)$$

$$T_2 \cos p \qquad\qquad = P \sin \pi, \quad . \quad . \quad . \quad . \quad (2)$$

in which P is the intensity of the *resultant stress* on *any* plane at *any* point; p the angle which the normal to that plane makes with the axis of X (the axis of the beam), and π the angle which the direction of P makes with the same axis.

Let it first be required *to find the plane, at any point, on which the normal or direct stress is the greatest.*

It is known from the theory of internal stress that this greatest normal stress will be the resultant and, hence, a principal stress. Hence the relation $\pi = p$, or

$$N_1 + T_2 \tan p = P, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$T_2 \qquad = P \tan p. \quad . \quad . \quad . \quad . \quad (4)$$

If F is the weight carried by the beam at its end, I the moment of inertia of the beam's cross-section, and d its half depth, or greatest value of z, it has been shown in Art. 10 that

$$N_1 = \frac{Fx}{I}z \quad \text{and} \quad T_2 = \frac{F}{2I}(d^2 - z^2). \quad . \quad . \quad . \quad . \quad (5)$$

Inserting the value of P from eq. (4) in eq. (3)

$$T_2 - T_2 \tan^2 p = N_1 \tan p;$$

$$\therefore \tan^2 p + \frac{N_1}{T_2} \tan p = 1.$$

Solving this quadratic equation and then inserting the values of T_2 and N_1 from eq. (5)

$$\tan p = -\frac{xz}{d^2 - z^2} \pm \frac{\sqrt{(d^2 - z^2)^2 + x^2z^2}}{d^2 - z^2}. \quad . \quad . \quad (6)$$

This value of $\tan p$ put in eq. (3) or eq. (4) will give the greatest value of the direct or normal stress (also resultant) at any point in the beam.

At the exterior surface $d = z$; hence

$$\tan p = 0 \quad \text{or} \quad -\infty.$$

Since for this point $T_2 = 0$, the first value gives, by eq. (3), $P = N_1$. The second value, by eq. (4), gives $P = 0$. These results might have been anticipated.

At the neutral surface $z=0$; hence

$$\tan p = \pm 1 = \tan \pm 45^\circ.$$

Hence *at the neutral surface there are two planes on which the stress is wholly normal, and these planes make angles of* 45° *with the neutral surface, or* 90° *with each other* (i.e., *they are principal planes*).

Since $N_1=0$ at the neutral surface, either of the eqs. (3) or (4) gives

$$P = \pm T_2 = \pm \frac{Fd^2}{2I}. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (7)$$

Hence each of these normal or principal stresses equals in intensity that of the transverse or longitudinal shear at the neutral surface; also, one of these principal stresses is a tension and the other a compression.

$$P = T_2 \cot p = -\frac{F}{2I}\left\{\frac{(d^2-z^2)^2}{xz \pm \sqrt{(d^2-z^2)^2+x^2z^2}}\right\}$$

is the equation of the locus of the point of constant greatest normal intensity of stress, if P be taken constant and equal to any possible value.

Let it next be required to find the plane of greatest shear at any point in the beam and the value of that shear.

The shear on any plane will be

$$P \sin(\pi - p) = T. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (8)$$

Multiplying eq. (1) by $(-\sin p)$ and eq. (2) by $\cos p$, then adding

$$-N_1 \cos p \sin p + T_2(\cos^2 p - \sin^2 p) = P(\sin \pi \cos p$$
$$- \cos \pi \sin p) = P \sin(\pi - p) = T;$$

$$\therefore \quad T = -\frac{N_1}{2}\sin 2p + T_2 \cos 2p. \quad . \quad . \quad . \qquad (9)$$

It is now required to find what value of p will make the general value of T (given by eq. (9)) a maximum. Hence

$$\frac{dT}{dp} = -N_1 \cos 2p - 2T_2 \sin 2p = 0.$$

$$\left.\begin{aligned} \therefore\quad \tan 2p &= -\frac{N_1}{2T_2} = -\frac{xz}{d^2-z^2} \\ \therefore\quad \cos 2p &= \pm\frac{d^2-z^2}{\sqrt{x^2z^2+(d^2-z^2)^2}} \end{aligned}\right\} \quad . \quad . \quad . \quad (10)$$

Eqs. (10) give the value of p which is to be placed in eq. (9), in order to obtain the greatest value of T at any point of the beam.

From eq. (9)

$$T = T_2 \cos 2p \left\{ -\frac{N_1}{2T^2} \tan 2p + 1 \right\};$$

$$\therefore\ T = \pm\frac{F}{2I}\sqrt{x^2z^2-(d^2+z^2)^2}. \quad . \quad . \quad . \quad (11)$$

At the exterior surfaces of the beam

$$z = \pm d.$$

Hence

$$T = \pm\frac{Fxd}{2I} = \pm\frac{N_1}{2}.$$

For this case also $\cos 2p = 0$, or $p = 45°$.

Hence *at the exterior surfaces of the beam the planes of greatest shear make angles of 45° with the axis of the beam, and the intensity of the shear is half that of the direct stress at the same place.*

At the neutral surface $z = 0$. Hence

$$T = \pm\frac{Fd^2}{2I} = T_2 \quad \text{and} \quad \cos 2p = \pm 1.$$

Hence $2p = 0$ or $180°$, or $p = 0$ or $90°$; i.e., the planes of greatest shear are the transverse and longitudinal planes, and the greatest shear itself is, consequently, the transverse or longitudinal shear.

If T is given any possible value and considered constant, eq. (11) will give the locus of the point of constant greatest shear.

The result expressed in eq. (7) is of great value in determining the thickness of the web of flanged beams, as will be seen hereafter.

PROBLEMS FOR CHAPTER III.

✓ Problem 1.—A beam simply supported at each end carries a load of 850 pounds per linear foot over a span of 26 feet. Find the bending moment and transverse shears at the end and centre of span and at 2 points 3 feet and 11 feet 6 inches respectively from the end.

Ans. Moment at end is 0; at 3 feet, 29,325 ft.-lbs.; at 11.5 feet, 70,868.75 ft.-lbs.; at centre, 71,825 ft.-lbs. Shear at end is 11,050 lbs.; at 3 feet, 8500 lbs.; at 11.5 feet, 1275 lbs.

✓ Problem 2.—A beam or girder having a span length of 41 feet carries a uniform load of 1200 pounds per linear foot and a single weight of 1800 pounds at the centre. Find the bending moments and the shears due to the uniform load and the single load separately at the ends and at the centre and at points 6 and 14 feet from the end.

Problem 3.—In Problem 2 find the single weight which placed at the centre of the span will produce the same centre bending moment as the uniform load.

Ans. 24,600 pounds.

✓ Again, find two weights placed 6 feet apart, i.e., one 3 feet either way from the centre, which will produce the same centre bending moment as the uniform load.

Ans. Each of the two weights is 14,406 pounds.

Problem 4.—A beam or girder with a span length of 31 feet carries a uniform load of 300 pounds per linear foot in addition to five loads, the first weighing 7000 pounds at a distance of 3 feet from the end; the second weighing 10,000 pounds 7 feet from the end; the third weighing 11,000 pounds 14 feet from the end; the fourth weighing 17,000 pounds 21 feet from the end, and the fifth weighing 6400 pounds 27 feet from the end.

Construct the shear and moment diagrams for this case, Fig. 2 of Art. 15 and Fig. 2 of Art. 12.

Problem 5.—Find a uniform load for the same beam considered in Problem 4 which will have a centre bending moment equal to the greatest bending moment of that problem; also another uniform load whose end shear shall equal the greatest of the two end shears of Problem 4. Such uniform loads are called "equivalent uniform loads."

Problem 6.—In Problem 2 the moment of inertia I is 3570 (the unit being the inch), while $E = 30,000,000$, the beam being of steel. Find the tangent of inclination of the neutral surface at the end and at 10 feet from the end. Also find the deflection at the centre of span and at 10 feet from the end. Use eqs. (19), (20), and (21) of Art. 22.

Partial Ans. Tangent 10 feet from end is .00344. The deflection at the same point is .53 inch.

Problem 7.—In Problem 6 let it be required to ascertain how much additional deflection is produced by the transverse shear at the centre of the span and at 10 feet from the end. Let the coefficient of elasticity for shear (G) be taken at 12,000,000 pounds, while $I = 3570$ and $d = 14$ inches.

Ans. Deflection at 10 feet is .0054 inch, and at the centre of span .0075 inch.

Problem 8.—Find the loads by Euler's formula which can be carried by a steel solid circular column 3 inches in diameter and 20 feet long with fixed ends if $E = 30{,}000{,}000$ pounds; and by a cast-iron column of the same diameter with one end round and one end fixed, but 13 feet long, if $E = 14{,}000{,}000$ pounds.

CHAPTER IV.

RESILIENCE.

Art. 28.—General Considerations.

THE term resilience is applied to the quantity of work required to be expended in order to produce a given state of strain in a body. If a piece of material is subjected to tension, that state of strain will be simply the stretching of the piece or the amount of compression, if the piece is subjected to compressive stress. In precisely the same manner the resilience of a bent beam is the amount of work performed upon it by its load in producing deflection. There may also be the resilience of shearing or of torsion.

In the ordinary use of the expression, resilience refers to the amount of work expended within the elastic limit, whether of torsion, compression, or tension, but it may properly be extended in its meaning to include the total amount of work required to rupture the material under any one of the preceding conditions of stress. Elastic resilience may easily be computed by means of exact formulæ, but if the total work required to cause rupture in any case is desired, a graphical record of the total strains produced between the elastic limit and failure must be obtained by actual tests. In these articles the formulæ for elastic resilience only will be given; in other subsequent articles the method of computing the total resilience of failure will be illustrated by computations from actual strain records.

Art. 29.—The Elastic Resilience of Tension and Compression and of Flexure.

Let it be supposed that a piece of material whose length is L and the area of whose cross-section is A is either stretched or compressed by the weight or load W applied so as to increase gradually from zero to its full value. If E is the coefficient of elasticity, the elastic change of length will be $\frac{WL}{AE}$. The average force acting will be $\frac{1}{2}W$, hence the work performed in producing the strain will be

$$Resilience = \frac{W^2L}{2AE}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If $\frac{W}{A}$, the intensity of stress in the metal, be represented by t, eq. (1) may be written

$$Resilience = \frac{1}{2}At^2\frac{L}{E}. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Again, eq. (2) may take the following form:

$$Resilience = \frac{1}{2}AE\frac{t^2}{E^2}L = \frac{1}{2}AEl^2L. \quad . \quad . \quad . \quad (3)$$

In eq. (3) the quantity $l^2 = \frac{t^2}{E^2}$ is obviously the square of the strain (stretch or compression) per unit of length.

If a bar of material 1 inch in length and 1 square inch in cross-section be considered, $A = 1$ and $L = 1$ must be inserted in the preceding equations, and there will result

$$\textit{Unit resilience} = \tfrac{1}{2}\frac{t^2}{E} = \tfrac{1}{2}El^2. \quad . \quad . \quad . \quad . \quad (4)$$

The quantity $\frac{t^2}{E} = El^2$ is called the "Modulus of Resilience." The expression is ordinarily employed when t is the greatest intensity of stress allowed in the bar.

The preceding equations are applicable whether the bar or piece of material is in tension or compression, the coefficient of elasticity E being used for either stress, while t represents the intensity of either tension or compression, as the case may be.

Inasmuch as the values of t and E are usually taken in reference to the square inch as the unit of area, it is generally convenient to take L in inches, although any other unit of length may be taken when multiplied by the proper numerical coefficient.

The Resilience of Bending or Flexure.

It has already been shown, in considering the common theory of flexure, as applied to the flexure or bending of beams, that the intensities of the stresses of tension and compression vary from point to point throughout the entire beam. In determining the elastic resilience of flexure, therefore, it is necessary to find the work performed in producing the varying strains corresponding to the stresses in the interior of the beam. The resilience due to the direct stresses of tension and compression will first be considered and then that due to the shearing stresses.

In order to obtain the expression for the work performed by the direct stresses of tension and compression in a beam bent by loads acting at right angles to its axis, a differential of the length, dL, is to be considered at any normal section in which the bending moment is M, the total length of span or beam being L. Let I be the moment of inertia of the normal section, A, about its neutral axis, and let k be the intensity of stress (usually the stress per square inch) at any point distant d from the axis about which I is taken. The elastic change produced in the indefinitely short length dL when the intensity k exists is $\frac{k}{E}dL$. If dA is an indefinitely small portion of the normal section, the average force or stress, either of tension or compression, acting through the small elastic change of length just given, can be written by the aid of a familiar equation of flexure as

$$\frac{1}{2}k.dA=\frac{Md}{2I}.dA. \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Hence the work performed in any normal section of the member, for which M remains unchanged, will be, since $\int k.dA.d=M$,

$$\int\frac{M}{2IE}kd.dA.dL=\frac{M^2}{2EI}dL. \quad . \quad . \quad . \quad . \quad (6)$$

The work performed throughout the entire piece will then be

$$\int\frac{M^2}{2EI}dL. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

The integration indicated in eq. (7) is readily made in all ordinary cases by substituting the value of the bending moment M in terms of the variable horizontal ordinate or abscissa x and the load, it being remembered that dL is precisely the same as dx. If, for example, the beam is non-continuous, simply supported at each end and carries uniformly distributed load p per unit of length throughout the whole span, $M=\frac{p}{2}(Lx-x^2)$. By the insertion of this value of M in eq. (7), there will result

$$Resilience=\frac{1}{2EI}\int_0^L\frac{p^2x^2}{4}(L-x)^2dx=\frac{p^2L^5}{240EI}=\frac{W^2L^3}{240EI}, \quad (8)$$

W representing pL, the entire load on the beam.

This equation gives the value of the total work performed by the direct stresses of tension and compression in the interior of a simple beam uniformly loaded and supported at each end, under the assumption that the moment of inertia I of the cross-section is constant throughout the entire span.

If a single load W rests at the centre of the span, the reaction at each end being $\frac{W}{2}$, the value of the bending moment at any point will be $\frac{W}{2}x$. By inserting this value of M in eq. (7), there will result

$$Resilience=\frac{1}{2EI}\cdot 2\int_0^{\frac{L}{2}}\frac{W^2}{4}x^2dx=\frac{W^2L^3}{96EI}. \quad . \quad . \quad (9)$$

Similarly equations of the elastic resilience of the direct stress of tension and compression in beams loaded in any manner whatever may easily be written. In some cases like the last the deflection at the point of application of a single load may easily be determined. Let that deflection be represented by w; when a single load W rests at the centre of the span the work performed by this load in producing the deflection is $\frac{1}{2}Ww$. Hence that amount of work must be equal to the resilience given by eq. (9), and

$$w=\frac{WL^3}{48EI}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

The Resilience Due to the Vertical or Transverse Shearing Stresses in a Bent Beam.

The work performed by the vertical shearing stresses in a bent beam of any shape of cross-section may readily be found. Let S be the total transverse shear in a normal section, I being the moment of inertia of the latter about its neutral axis, b the width or breadth (constant or variable) of the section, ϕ the unit shearing strain defined in Art. 2, d and d_1 the distances of the extreme fibres from the neutral axis, and G the coefficient of elasticity for shearing. By eq. (6) of Art. 10 the intensity s of the shear at any point in the section at the distance z from the neutral axis will be

$$s=\frac{S}{2I}(d^2-z^2). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Again, by eq. (3) of Art. 2,

$$\phi=\frac{s}{G}=\frac{S}{2GI}(d^2-z^2). \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

The amount of shearing stress on the indefinitely small portion of the section $b.dz$ will be $sb.dz$, and its path in performing the work will be ϕdx, x being the horizontal ordinate of the section of the beam from any convenient origin, as the end or the centre of the span, i.e., in this case the end of the span. The differential work performed in the section will be, by the aid of eqs. (11) and (12),

$$\tfrac{1}{2}(sbdz)\,.\,(\phi dx)=\frac{S^2b}{8I^2G}(d^2-z^2)^2dzdx. \quad . \quad . \quad (13)$$

In this equation it is easy to express the breadth b of the section in terms of z, whatever may be its shape, by the aid of the equation of the perimeter of the section. In all the ordinary and important cases of engineering practice involving this resilience of shearing the shape of the section is rectangular for which b is constant, and it will be so regarded in the following equations. Remembering that x and z are independent variables, and that the first integration will be made in reference to z, that integration will give

$$\frac{b}{8I^2G}\int S^2dx\int_{-d_1}^{d}(d^2-z^2)^2dz=\frac{b(d^5+d_1^5)}{15I^2G}\int S^2dx. \quad (14)$$

As the section is taken to be rectangular in outline, with the breadth b and depth h, $d=d_1=\frac{h}{2}$ and $I=\frac{bh^3}{12}$. Eq. (14) will then become

$$Resilience=\frac{3}{5bhG}\int S^2dx. \quad . \quad . \quad . \quad . \quad (15)$$

The total transverse shear S will have varying values depending upon the amount of loading on the beam and its distribution, i.e., in general it will vary with x, and when not constant it must be expressed in terms of that variable before the remaining integration can be made.

If a single weight W rests on the beam at the distance of l' from one end where the reaction is R', and at the distance l_1 from the other end where the reaction is R_1, the shear S will be constant for each of the segments into which the point of loading divides the span; in one of those segments $S = R'$, and in the other $S = R_1$. The complete integration of eq. (15) will be, therefore,

$$Resilience = \frac{3}{5bhG}(R'^2 l' + R_1^2 l_1).$$

If there be substituted in the parentheses of the second member of the preceding equation the values $R' = W\frac{l_1}{l}$ and $R_1 = W\frac{l'}{l}$, there will result

$$Resilience = \frac{3}{5bhG}\,\frac{l_1 l'}{l}W^2. \quad . \quad . \quad . \quad . \quad (16)$$

If the weight W rest at the centre of the span $l_1 = l' = \frac{l}{2}$ and

$$Resilience = \frac{3}{20Gbh}W^2 l. \quad . \quad . \quad . \quad . \quad (17)$$

Eq. (17) affords a simple method of finding the deflection w_1 of the point of loading due to the transverse shear. As the weight W is supposed to be gradually applied the expended work $\frac{1}{2}Ww_1$ must be equal to the shearing resilience given in eq. (17). Hence

$$w_1 = \frac{3}{10G}\,\frac{Wl}{bh}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

When a non-continuous beam simply supported at each end carries a uniform load over the entire span, it has been shown in Art. 17, eq. (7), that the transverse shear at any section is equal to the load between the centre of span and that section. If, therefore, the origin of x be taken at the centre of span and if p represents the load per unit of length of the beam, $S=px$. By substituting this value of S in eq. (15), and remembering that twice the integral must be taken for the whole beam,

$$Resilience = \frac{3}{5Gbh}\cdot 2\int_0^{\frac{l}{2}} p^2x^2dx = \frac{p^2l^3}{20Gbh} = \frac{W^2l}{20Gbh}. \quad (19)$$

The shearing resilience, therefore, in a non-continuous beam carrying a uniform load is only one third as much as that due to the same load concentrated at the centre of the span.

If, as is usual, G is expressed in pounds per square inch the unit for l, b, and h will be the linear inch.

Other modes of loading than those taken can be treated in precisely the same general manner.

As the intensity of the longitudinal shear at any point of a beam is the same as that of the transverse shear, the total work of the longitudinal shear throughout the beam is the same as the work of the transverse shear. The total work of the shearing stresses in a beam is therefore twice that given in the preceding equations.

The Total Resilience Due to Both Direct and Shearing Stresses.

The general expression for the total resilience of a bent beam due to both shearing and direct stresses will be the sum of the second members of eqs. (7) and (13), expressed by the following equation:

$$Total\ resilience = \int \frac{M^2}{2EI}dL + \int\int \frac{S^2b}{4I^2G}(d^2-z^2)^2dzdx.$$

Or, by eqs. (7) and (15), since $dL=dx$,

$$\textit{Total resilience} = \int \frac{M^2}{2EI}dx + \frac{6}{5bhG}\int S^2 dx. \quad . \quad (20)$$

By the aid of eqs. (8) and (19) the total resilience for a simple non-continuous beam may be as follows:

If the uniform load $pl=W$,

$$\textit{Total resilience} = W^2\left(\frac{l^3}{240EI}+\frac{l}{10Gbh}\right). \quad . \quad (21)$$

For the same beam carrying a single load W at the centre, by eqs. (9) and (17)

$$\textit{Total resilience} = W^2\left(\frac{l^3}{96EI}+\frac{3l}{10Gbh}\right). \quad . \quad . \quad (22)$$

As has been explained, the last two equations are applicable to beams with rectangular sections only.

In a similar manner the total deflection of a beam supported at each end and loaded both with a single weight W at the centre of the span and with a uniform load will be found by the sum of the two expressions given in eqs. (10) and (18):

$$w+w' = W\left(\frac{l^3}{48EI}+\frac{3l}{5Gbh}\right). \quad . \quad . \quad . \quad . \quad (23)$$

Art. 30.—Resilience of Torsion.

The work expended in producing elastic strains of torsion constitutes the resilience of torsion and is a special case of shearing resilience. The twisting moment which produces the angle of torsion α is given by eq. (16) of

Art. 8 and is $M = G\alpha I_p$. When the piece twisted has the length l the total angle of torsion is αl and the differential amount of work performed by the moment M in producing the indefinitely small twist $d(\alpha l) = l.d\alpha$ is $Ml.d\alpha$. Hence

$$Resilience = \int Mld\alpha = GlI_p \int_0^{\alpha_1} \alpha . d\alpha = GlI_p \frac{\alpha_1^2}{2} . . \quad (1)$$

If P and e are the force and lever-arm of the twisting couple, eq. (18) of Art. 8 shows that

$$\alpha_1 = \frac{Pe}{GI_p}.$$

Substituting this value of α_1 in eq. (1),

$$Resilience = \frac{P^2 e^2 l}{2GI_p} . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If the normal section of the piece is circular $I_p = \frac{\pi r^4}{2}$. Hence, for a shaft with circular section,

$$Resilience = \frac{P^2 e^2 l}{\pi G r^4} . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If the section of the shaft is a square, $I_p = \frac{b^4}{6}$, b being the side of the square. Hence, for a square section,

$$Resilience = \frac{3P^2 e^2 l}{Gb^4} . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

In some cases shafts are subjected to combined torsion and bending. In such cases, if it is desired to compute the total elastic resilience it is only necessary to take the sum of the two resiliences, each found as if existing independently of the other.

The resilience of torsion beyond the elastic limit or between the elastic limit and the ultimate resistance must be determined, as in all cases of distortion beyond the elastic limit, from an actual strain record, as given by the testing machine when the piece is strained up to any given degree of permanent stretch or to rupture.

Art. 31.—Suddenly Applied Loads.

A load is considered suddenly applied when its full amount acts instantly upon any piece of material loaded by it. In the preceding articles relating to resilience the loads are treated as being gradually increased from zero to their full values. In such cases the amount of external loading at any instant is supposed to be equal only to the internal stress or stresses opposing it, so that the work performed is equivalent to one half the total load multiplied by the total resulting strain. When the loads are suddenly applied, on the other hand, the internal stresses produced are exactly equal to the external forces only when the strains corresponding to the latter are reached, and the work performed up to that point is just double the work expended when the loads are gradually applied. It follows from this last consideration that the strains produced by the suddenly applied loads will be double those found under gradual application. Inasmuch as the elastic strains are proportional to the corresponding stresses, it further follows that the stresses produced by suddenly applied loads will be double in intensity those which are produced by the same loads gradually applied.

The work expended by a suddenly applied load up to the point of strain corresponding to its amount being double the work performed by the internal stresses, the total stress induced in the material at the limit of the final strain produced by such a load will be double the amount of the latter. The internal stresses in the piece will, therefore, cause it to recover from its strained condition and vibrations will result, the treatment of which constitutes an important branch of the theory of elasticity in solid bodies. Some general features of that treatment will be given in Art. 12, App. I, but as they are seldom used in engineering practice they will not be considered here. It is only important at this point to note carefully the distinction between the effects of a given load gradually applied and suddenly applied, the strains and stresses in the latter condition being double those in the former.

Again, it is also important to distinguish between loads suddenly applied, and shocks, as they are called in engineering practice. A shock is produced when the load falls freely before acting upon a piece of material sustaining it. The cause of shock, therefore, is a suddenly applied load with the effect of a free fall of the latter superimposed. These matters must be carefully taken into account and allowed for in such structures as bridges carrying rapidly moving trains, and those allowances are incorporated in the provisions of specifications covering bridge construction.

PROBLEMS FOR CHAPTER IV.

Problem 1.—A 6-inch by 1.75-inch steel eye-bar 48 feet long is subjected to a stress of 117,500 pounds. If that load is gradually applied what is the work performed in the total length of the bar, if $E = 30,000,000$ pounds? Also what is the unit resilience?

$t = \frac{117,500}{10.5} = 11,190.$ $L = 48 \times 12 = 576$ inches. Eq. (2) of Art. 29 then gives

$$\text{Resilience} = \text{work performed} = \frac{10.5 \times (11,190)^2 \times 576}{2 \times 30,000,000}$$
$$= 12,621 \text{ in.-lbs.}$$

Eq. (4) of Art. 29 gives

$$\text{Unit resilience} = \frac{(11,190)^2}{2 \times 30,000,000} = 2.09 \text{ in.-lbs.}$$

Problem 2.—A cast-iron column 18 feet long having an area of cross-section of 40.8 sq. in. carries a load of 245,000 pounds. If the coefficient of elasticity E is 14,000,000 pounds, how much work is performed in compressing the column if the load is gradually applied.

Problem 3.—A 30-pound 10-inch rolled steel I beam carries a uniform load of 1000 pounds per linear foot in addition to its own weight with a span of 16 feet. What will be the resilience or work performed in the material of the beam under the gradual application of that total load of 1030 pounds per linear foot, the moment of inertia I of the beam being 134.2 and $E = 30,000,000$ pounds? Eq. (8) of Art. 29 is to be used, in which L is 192 inches. Incidentally, what will be the greatest intensity of stress, k, in the extreme fibres?

Ans. Resilience = 1987 in.-lbs; $k = 15,000$ lbs. per square inch.

Problem 4.—In Problem 3 if the thickness of the web of the 10-inch rolled beam is .5 inch, find the resilience of the vertical or transverse shearing stresses in the beam, the coefficient of shearing elasticity, G, being taken at 12,000,000 pounds. The remaining data are $l = 192$ inches; $h = 10$ inches; $b = 0.5$ inch, and $W = 16,480$ pounds, and they are to be used in eq. (19) of Art. 29.

Problem 5.—In Problems 6 and 7 at the end of Chapter II find the work performed in twisting the two steel shafts, i.e., the resilience for 25 feet length in the one case and 13 feet in the other. Use equations of Art. 30.

Problem 6.—In Problem 3 suppose the load suddenly applied, what will be the resulting resilience and greatest intensity of extreme fibre stress?

CHAPTER V.

COMBINED STRESS CONDITIONS.

Art. 32.—Combined Bending and Torsion.

PROBABLY the most important case of combined bending or flexure and torsion is that of the ordinary crankshaft represented in Fig. 1.

The centre of the thrust of a connecting-rod is at A, on the crank-pin journal against which the connecting-rod bears. The centre of the shaft-bearing is at B. If the thrust at A is represented by P, then the actual resultant moment about the centre of the bearing B will be $P \times AB$. The problem is to determine the maximum stresses developed by this resultant moment in the section of the shaft at B. Two methods may be employed in both of which the resultant moment of P multiplied by the lever arm AB is resolved into its two components, one of which is the ordinary bending moment represented by $M = P \times CB$, and the other is the twisting moment $M' = P \times AC$. The latter produces torsion in the journal at B and the former produces pure flexure or bending at the same section.

Let CB be represented by l while e represents AC. The moment of pure bending at B will be

$$M = Pl. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The twisting moment producing pure torsion will be

$$M' = Pe. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If *d* represents the distance of the most remote fibre in the section *B* from the neutral axis of the latter, and if *k* is the greatest intensity of bending stress at the distance *d* from the neutral axis, while *I* is the moment of inertia of the normal section of the shaft at *B* about the same neutral axis, the following will be the value of *k*:

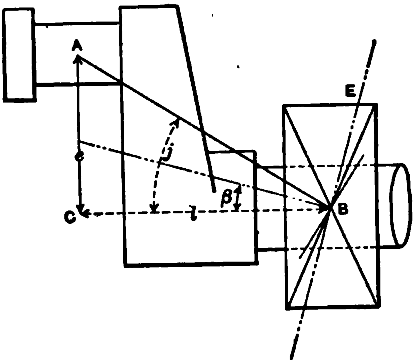


Fig. 1.

$$k = \frac{Md}{I} = \frac{Pld}{I}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Again, if *T* is the greatest intensity of torsional shear in the normal section of the shaft at *B*, at the greatest distance *r*, in the perimeter, from the centre of gravity or the centroid of the same section, the value of the maximum intensity *T* will be

$$T = \frac{M'r}{I_p} = \frac{Per}{I_p}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

In eq. (4) I_p is the polar moment of inertia of the normal section at *B*.

First Method.

In this method it is only necessary to consider the intensities k given by eq. (3) and T given by eq. (4), the greatest allowed working stresses of direct tension and of shearing respectively. k would have the value of the greatest tensile working stress of the material of the shaft for the reason that if tested to failure the shaft would yield first on the tension side.

It being understood, therefore, that k and T represent the greatest allowed working intensities of stress, usually expressed in pounds per square inch, eq. (3) will give

$$\frac{I}{d}=\frac{M}{k}=\frac{Pl}{k}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Under the same conditions eq. (4) will give

$$\frac{I_p}{r}=\frac{M'}{T}=\frac{Pe}{T}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

For the circular section

$$I=\frac{\pi r^4}{4} \quad \text{and} \quad I_p=\frac{\pi r^4}{2}. \quad . \quad . \quad . \quad . \quad (7)$$

For a square section

$$I=\frac{b^4}{12} \quad \text{and} \quad I_p=\frac{b^4}{6}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

b being the side of the square. In eq. (5) for a circular section $d=r$ and for a square section $d=\frac{b}{\sqrt{2}}$. In eq. (6), $r=r$ for the circular section, but for the square section

$r = \frac{b}{\sqrt{2}}$. Making those substitutions in eqs. (5) and (6) for the circular section, there will result, D being the diameter of the shaft,

$$\left.\begin{array}{l} \text{For bending} \ldots r = \frac{D}{2} = \sqrt[3]{\frac{4Pl}{\pi k}} = 1.08\sqrt[3]{\frac{Pl}{k}} \\ \text{For torsion} \ldots r = \frac{D}{2} = \sqrt[3]{\frac{2Pe}{\pi T}} = .86\sqrt[3]{\frac{Pe}{T}} \end{array}\right\} \quad . \quad (9)$$

In the practical use of eqs. (9) that one of the two values of r should be taken which is the greatest. This will insure that both the direct stress of tension and the shearing stress shall not exceed the prescribed values of k and T.

The substitution of the values of I and I_p for the square section in eqs. (5) and (6) will give, remembering that d and r are each one half the diagonal of the square,

$$\left.\begin{array}{l} \text{For bending} \ldots b = \sqrt[3]{\frac{6\sqrt{2}Pl}{k}} = 2.04\sqrt[3]{\frac{Pl}{k}} \\ \text{For torsion} \ldots b = 1.62\sqrt[3]{\frac{Pe}{T}} \end{array}\right\} \quad . \quad (10)$$

In eq. (10), also, the greatest value of b given by the application of the two formulæ is to be taken, so that, as in the case of the circular section, neither of the two intensities k and T shall exceed the values prescribed for them.

This method involves only the consideration of the simple formulæ of the common theories of flexure and torsion.

Second Method.

The second method of treatment of this case of the crank-shaft consists in determining the greatest intensity of the direct stress of tension in the section B of the shaft at the journal-bearing. This resultant maximum intensity is produced by the combination of the same component moments, $M = Pl$ and $M' = Pe$, as in the preceding method. With the sections of shafting always employed the maximum intensity of bending stress k and the maximum intensity of torsional shear T exist at the same point and on the same plane, i.e., the plane of normal section of the shaft. The existence of the shear T on the normal section at the distance r from its centre of gravity carries with it the same intensity of shear at the same point on a longitudinal plane passing through the axis of the shafting. At the point considered, therefore, on two indefinitely small planes at right angles to each other, one normal to the axis of the shaft and the other parallel to it, there exist the direct intensity of tension k on the first, and the intensity of shear T on the second. The problem is to determine at the same point the greatest intensity of the direct stress of tension on any plane whatever, and the angle β between the direction of that stress and the axis of the shaft. Reference may best be made to the general formulæ of internal stresses in a solid body for its solution, and those are eqs. (3) and (4) of Art. 27. Those equations are adapted to this case by making $N_1 = k$, $T_2 = T$, $\tan p = \tan \beta$, and $P = t$, the latter quantity being the greatest intensity of tension desired. These substitutions give the following two equations:

$$k + T \tan \beta = t. \quad . \quad . \quad . \quad . \quad . \quad (11)$$

$$T = t \tan \beta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

By the elimination of tan β between eqs. (11) and (12)

$$k+\frac{T^2}{t}=t. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

The solution of eq. (13) gives

$$t=\sqrt{\frac{k^2}{4}+T^2}+\frac{k}{2}. \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Eq. (14) gives the expression for the greatest intensity of direct tension in the shaft in terms of the known stresses k and T.

By the elimination of T between eqs. (11) and (12)

$$t(1-\tan^2\beta)=k;$$

also by eq. (12),

$$2 \tan \beta=\frac{2T}{t}.$$

Dividing this last equation by the one which precedes it,

$$\tan 2\beta=\frac{2T}{k}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

By eq. (15) the position of the plane or section of the shaft on which the maximum intensity t exists may at once be found. Inasmuch as β is the angle between the direction of the stress t and the axis of the shaft, the angle between the plane on which t acts and the axis of the shaft will be $90°+\beta$.

Under this method of treatment it would be necessary to design the shaft so that t should not exceed the greatest prescribed tensile working stress for the material employed.

The greatest intensity of compressive stress in the shaft would be found by giving the negative sign to the radical in the second member of eq. (14).

The preceding formulæ have been established in a manner to make them applicable to any form of shaft section or any values of k and T. It is only necessary to insert in those formulæ any intensities of those stresses that may exist. If, for example, it were considered desirable to add the shear $\frac{P}{\pi r^2}$ due to the thrust P to the torsional shear it would only be necessary to take $T+\frac{P}{\pi r^2}$ for T wherever the latter quantity occurs.

If a shaft is circular in section, as is almost universally the case, so that $I_p=2I$, and if the shearing effect of P in the section at B, Fig. 1, be omitted, useful and extremely simple relations may be deduced. In that case $D=2r$, being the diameter of the shaft, and j the angle ABC of Fig. 1, M as before being the resultant moment, or $M=P \times AB$:

$$k=\frac{rM\cos j}{I} \quad \text{and} \quad T=\frac{rM\sin j}{2I}. \quad . \quad . \quad (16)$$

By the substitution of these values in eq. (14),

$$t=\frac{rM}{2I}(1+\cos j)=\frac{5.1M}{D^3}(1+\cos j). \quad . \quad . \quad (17)$$

Hence

$$D=1.72\sqrt[3]{\frac{M}{t}(1+\cos j)}. \quad . \quad . \quad . \quad . \quad (18)$$

Eq. (17) gives, by the aid of the first of eqs. (16),

$$t=\frac{rM}{2I}(1+\cos j)=\frac{k}{2}(\sec j+1). \quad . \quad . \quad . \quad (19)$$

The second of eqs. (16) gives, after substituting the value of $\frac{r}{2I}=\frac{16}{\pi D^3}$,

$$T=\frac{5.1 M \sin j}{D^3}. \quad . \quad . \quad . \quad . \quad . \quad (20)$$

The substitution of the values of T and k from eqs. (16) in eq. (15) gives

$$\tan 2\beta=\frac{2T}{k}=\tan j; \quad \therefore \ \beta=\tfrac{1}{2}j. \quad . \quad . \quad . \quad (21)$$

This last set of results relating to circular shafts will, in all ordinary cases, supply everything required for the operations of design or of investigations regarding conditions of stress in existing shafts.

Eqs. (16), first of (17), (19), and (21) apply as they stand to square shafts.

The first method involves simpler considerations than the second, not only analytically, but also in respect to empirical quantities required to be used. The test pieces from which the ultimate resistance of the material is determined are always taken parallel to the axis of the shaft, but the greatest intensity of stress t found in the second method has a direction inclined to that axis by the angle β. In general, therefore, it will probably be found more practicable to use the first method rather than the second.

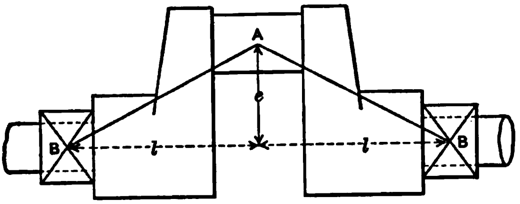


FIG. 2.

In the case of the double crank-shaft shown in Fig. 2, it is only necessary to treat each half precisely as if it were the single crank-arm in Fig. 1.

Art. 33.—Combined Bending and Direct Stress.

There are a considerable number of practical problems of combined flexure and direct stress of sufficient importance to merit careful examination, and among them is the flexure of long columns treated in Art. 24. In this place the particular cases to be considered are those in which the bending is produced by a uniform load at right angles to the axis of the member, or by eccentricity of longitudinal loading, the direct stress (or external force) being applied in a direction parallel to the same axis. Lower chord eye-bars and other horizontal or inclined chord members of pin bridges belong to this class.

Let M_1 represent the bending moment in the member at that section where the deflection is greatest, produced by loading at right angles to the member's axis or by eccentricity in the application of the longitudinal loading; let w' represent the greatest deflection resulting from the total bending moment and direct stress; also, let P be the total direct stress acting upon the member whose length is l, while k represents the greatest intensity of stress due to bending alone and at the distance d of the most remote fibre from the neutral axis of the section at which the deflection w' is found. Finally, let A be the area of cross-section of the member which, together with the moment of inertia I, is supposed to be constant throughout the entire length; and let $q = \frac{P}{A}$, the intensity of uniform stress in the member due to the direct stress or force P.

The resultant maximum bending moment in the member will then be

$$M = M_1 \pm Pw'. \quad \dots \dots \quad (1)$$

If P is tension it will tend to pull the member straight, thus producing a moment opposite to M_1. In the second member of eq. (1), therefore, the negative sign is to be used for a member in tension and the positive sign for a member in compression.

The greatest resultant intensity of stress, t, in the member will then take the value

$$t = \frac{P}{A} + \frac{Md}{I} = \frac{1}{A}\left(P + \frac{Md}{r^2}\right). \quad \dots \dots \quad (2)$$

The quantity r is the radius of gyration, so that $I = Ar^2$.

When the intensity t is prescribed, the required area of section A is

$$A = \frac{1}{t}\left(P + \frac{Md}{r^2}\right). \quad \dots \dots \quad (3)$$

These equations are perfectly general and may be applied to all cases of combined bending and direct stress.

Art. 34.—The Eye-bar Subjected to Bending by Its Own Weight or Other Vertical Loading.

Let Fig. 1 represent a lower chord eye-bar of a pin-connected bridge with the length l and carrying the total tension P. The depth of the bar is h and the thickness b, so that the area of the normal section is bh. The bar acts as a beam carrying its own weight as a uniform load over the span l. That load deflects the bar as a beam while the direct stress of tension (P) decreases that deflection by tending to pull the bar straight. The problem is to determine the greatest stress in the bar and incidentally its centre deflection.

There are several methods of procedure. The first and simplest method is approximate in its results, although sufficiently close for some purposes. It consists in treating the bending and direct stresses as existing independently, so that results are obtained by simply adding the bending to the direct intensities. This method will be treated first.

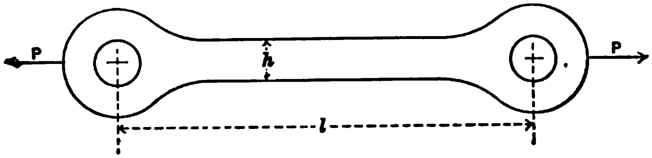


Fig. 1.

The more exact method consists in recognizing the bending moment as the resultant of those due to the transverse load acting on the bar as a simply supported beam, and to the direct stress P acting with the greatest deflection as its lever-arm.

Approximate Method.

Although reference will be made to Fig. 1, the formulæ as written will be equally applicable to compression members in which P would be the total force of compression.

If the total weight of the bar or compression member is W, and if I is the moment of inertia of its cross-section about the neutral axis, while k is the greatest intensity of bending stress at the distance d from the same axis, the theory of flexure gives

$$M_1=\frac{Wl}{8}=\frac{kI}{d}; \therefore k=\frac{Wld}{8I}. \quad . \quad . \quad . \quad . \quad (1)$$

If the area of cross-section is represented by A, while the radius of gyration is r, $I=Ar^2$. Again, the quantity $I \div d$ is called the "section modulus," and tabulated values of it for rolled sections may be found in hand-books. Let m be that modulus, then eq. (1) may take the form

$$k=\frac{Wld}{8Ar^2}=\frac{Wl}{8m}. \quad \ldots \ldots \quad (2)$$

The intensity of direct tension is

$$q=\frac{P}{A}. \quad \ldots \ldots \quad (3)$$

Obviously k will be tension on the lower side of the bar or other member and compression on the upper side. The greatest intensity of stress in the piece will be the sum of q and k. Eqs. (2) and (3) will, therefore, give the value of that greatest intensity, t, of stress as follows:

$$t=q+k=\frac{1}{A}\left(P+\frac{Wld}{8r^2}\right). \quad \ldots \ldots \quad (4)$$

When the greatest value of t is prescribed, the required area of section, A, can be at once written from eq. (4)

$$A=\frac{1}{t}\left(P+\frac{Wld}{8r^2}\right). \quad \ldots \ldots \quad (5)$$

In the case of an eye-bar with the cross-section bh, $d=\frac{h}{2}$ and $\frac{d}{r^2}=\frac{6}{h}$. Hence

$$t=\frac{1}{bh}\left(P+\frac{3Wl}{4h}\right) \quad \ldots \ldots \quad (6)$$

and

$$bh=\frac{1}{t}\left(P+\frac{3Wl}{4h}\right). \quad \ldots \ldots \quad (7)$$

If the bar carries any other uniform load than its own, it is only necessary to make W represent the total uniform load, including the weight of the bar itself.

Finally the direct force P may act with the eccentricity e. In this case the moment Pe produces uniform bending throughout the length of the bar, and it is only needful to write $\left(\frac{Wl}{8} \pm Pe\right)$ for $\frac{Wl}{8}$ in the preceding formulæ, the double sign showing that Pe may act either with or against the moment of the uniform load.

The formulæ of this article are not sufficiently exact for the usual cases of engineering practice.

Art. 35.—The Approximate Method Ordinarily Employed.

The method commonly employed in practical work for the treatment of compound bending and direct stress is a much closer approximation than the preceding method, although not exact. Its chief feature is the introduction of the bending moment produced by the direct or longitudinal force multiplied by the actual maximum deflection. In the same manner the moment due to the eccentricity of the line of action of that force is introduced wherever necessary.

Eq. (6a) of Art. 21 gives the following expression for the deflection w' due to pure bending and in terms of the greatest intensity of bending stress k, a being a constant depending, among other things, upon the distribution of loading:

$$w' = a\frac{kl^2}{Ed}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If the deflection as given in eq. (1) be placed equal to each of the two parts of the deflection given in eq. (21)

of Art. 22, it will be found for a beam simply supported at each end and loaded uniformly, that $a=\frac{5}{48}$, and for the same beam loaded by a single weight only at the centre of the span, $a=\frac{4}{48}$. The cases which occur in practice conform nearly to that of a load uniformly distributed over the length l. Hence for such a beam there is ordinarily taken

$$w'=\frac{kl^2}{10Ed}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The moment produced by the direct force or stress P acting with the lever arm w' will have the opposite sign to that of M_1 (the moment due to transverse loading or to eccentricity), if the member is in tension, but if the member is in compression those two moments will have the same sign. The resultant equation of moments may, therefore, be written

$$M=\frac{kI}{d}=M_1\pm Pw'. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

As stated, the plus sign is to be used for a compression member and the negative sign for a tension member.

If the value of w', given by eq. (2), be substituted in eq. (3), the following value of k will result:

$$k=\frac{M_1 d}{I\mp\frac{Pl^2}{10E}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

In eq. (4) the plus sign is to be used for tension members and the minus sign for compression members. This equation is general and adapted to all forms of cross-section under the conditions virtually assumed. Although not explicitly stated, it is essentially assumed that the ends of the member remain absolutely fixed in distance apart. This is frequently not the case, especially in the lower chord eye-bar of a pin-connected bridge subjected to direct tension and to bending due to its own weight, the bar usually being horizontal.

If the ends of the beam or member, uniformly loaded, are fixed, $a=\frac{1}{16}$, when k is the greatest intensity of bending stress at the mid-point of the member, or $\frac{1}{32}$ if k is the intensity of the bending stress at the fixed ends. One of those values (usually $\frac{1}{32}$) is to be substituted therefore for $\frac{1}{10}$ in the formulæ which follow when the fixed-end condition exists.

The resultant maximum intensity of stress t in the member will obviously be

$$t=k+q, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

in which equation q is the uniform intensity $P \div A$.

Eq. (4) will be immediately applicable to any particular case by substituting in it the values of I and M_1 for that special case.

If the case of the lower chord eye-bar mentioned in a preceding paragraph be considered, the total weight of the bar being W, while b and h represent its thickness and depth respectively, $I=\frac{bh^3}{12}$ and $M_1=\frac{Wl}{8}$. These values substituted in eq. (5) will give the desired value of the resultant intensity, as follows:

$$t=\frac{P}{bh}+\frac{W\frac{l}{h}}{\frac{4}{3}bh+\frac{8Pl^2}{5Eh^2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Eq. (6) gives the value of the maximum intensity of tension in the extreme lower fibres of the eye-bar when subjected to the total direct tension P and to the bending due to its own weight.

The greatest intensity of bending stress in the bar is evidently the second term of the second member of eq. (6), and it has the following value if the weight of the bar per unit of length is $\frac{W}{l}=g$, or if the weight of a cubic unit of the metal is i:

$$k=\frac{\frac{5}{8}\frac{gE}{b}}{\frac{5}{6}E\frac{h^2}{l^2}+\frac{P}{bh}}=\frac{\frac{5}{8}ihE}{\frac{5}{6}E\frac{h^2}{l^2}+q}. \quad . \quad . \quad . \quad . \quad (7)$$

It is frequently important to observe what depth of bar with a constant area of cross-section, subjected to a prescribed working stress, will give the maximum bending stress due to its own weight when the length is fixed. That depth can readily be determined by taking the first derivative of k, as given by eq. (7), with h as the variable. By performing that operation and placing $\frac{dk}{dh}=0$, there will at once result

$$h=\frac{l}{\sqrt{\frac{5}{6}E}}\sqrt{q}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The value of h resulting from an application of eq. (8) gives the depth of bar which, with a given value of l, will under the conditions of the case yield the greatest intensity of bending stress k; it indicates, therefore, a limit of depth to be avoided as far as practicable.

Steel is the usual structural material for eye-bars for which E may be taken at 29,000,000. For this value of E, h will become, by eq. (8),

$$h=\frac{l}{4900}\sqrt{q}.$$

In this equation $q=\frac{P}{A}$ is the intensity of uniform stress in the bar, or the "working stress."

By placing the value of h, as given by eq. (8), in the value of k, eq. (7), there will result the maximum possible bending stress in a bar of given length l and given area of cross-section A:

$$k=\tfrac{1}{3}i\sqrt{E}\frac{l}{\sqrt{q}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (9)$$

If $E=29,000,000$ and $i=.286$ lb. per cubic inch for steel, eq. (9) will take the value, for the corresponding values of h in the equation preceding eq. (9),

$$k=\frac{510l}{\sqrt{q}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (10)$$

The following table shows at a glance the greatest possible fibre stresses in eye-bars of different lengths and depths when the working tensile stresses in pounds per square inch are those given in the extreme left-hand column of the table:

Working Tensile Stresses in Pounds per Square Inch.	Length of Eye-bars in Feet.											
	15		20		25		30		35		40	
	Depth.	Fibre Stress.	Depth.	Fibre Stress.	Depth.	Fibre Stress.	Depth.	Fibre Stress.	Depth.	Fibre Stress.	Depth.	Fibre Stress.
	Ins.	Lbs. p. Sq. In.	Ins.	Lbs. p Sq. In.	Ins.	Lbs. p. Sq. In	Ins.	Lbs. p. Sq. In.	Ins.	Lbs. p. Sq. In.	Ins.	Lbs. p. Sq. In.
8,000	3.3	1030	4.4	1370	5.5	1710	6.6	2050	7.7	2400	8.8	2740
10,000	3.7	920	4.9	1220	6.1	1530	7.3	1840	8.6	2140	9.8	2450
12.000	4.0	840	5.4	1120	6.7	1400	8.0	1680	9.4	1960	10.7	2240
14,000	4.3	780	5.8	1030	7.2	1290	8.7	1550	10.1	1810	11.6	2070
16,000	4.6	730	6.2	970	7.7	1210	9.3	1450	10.8	1690	12.4	1940

In using the preceding formulæ it is to be remembered that the ordinary unit of length, as well as the unit of cross-section, is the linear inch, and that the weight i of a cubic unit will then be the weight of a cubic inch. This investigation will yield results sufficiently accurate for all the usual cases of engineering practice, although it does not provide for the straightening effect of the pull P, except as producing a bending moment opposite to that of the uniformly distributed load W.

Allowance for any other distributed loading than the weight of the bar itself, and for any eccentricity of the line of action of P that may exist, are made precisely as explained in the two paragraphs following eq. (7) of Art. 34.

Art. 36.—Exact Method of Treating Combined Bending and Direct Stress.

In this method of finding the results of direct stress combined with bending it is necessary to determine an expression for the centre deflection of the bar, or compression member, considered as simply supported at each end. As the line of action of the direct stress P is supposed to coincide with the original centre line or axis of the bar, if g is the weight per linear unit of the latter, the bending moment M_1 in the second member of eq. (1), Art. 33, becomes

$$M_1 = \frac{g}{2}x(l-x).$$

As this case is one in which P is tension the general eq. (1) of Art. 33 will take the following form by the aid of eq. (7) of Art. 9:

$$\frac{d^2w}{dx^2} = \frac{1}{EI}\left(\frac{g}{2}x(l-x) - Pw'\right). \quad . \quad . \quad . \quad . \quad (1)$$

In this equation $g = \frac{W}{l}$ is the weight per linear inch, or other unit, of the bar or member producing a bending moment opposite to that induced by the direct stress P acting with the lever-arm w'. The integration indicated in eq. (1) may be completed, but as it is not a simple integration it will not be made here. As the greatest bending stress is found at the centre of span the centre deflection only is needed and a different procedure may be followed.

Let w_1 represent the centre deflection of the member considered, a beam simply supported at each end and carrying its own weight only, or any other total weight W uniformly distributed. It is necessary to use the expression for the work performed, or resilience of the beam in being deflected at the centre by the amount w_1. Eq. (8) of Art. 29 gives that resilience as

$$\textit{Resilience} = \frac{W^2 l^3}{240 EI}. \quad \ldots \ldots \quad (2)$$

In producing the centre deflection w_1 the centre of gravity of the weight W will descend through the distance w_0 found by placing Ww_0 equal to the resilience given in eq. (2). Hence

$$w_0 = \frac{Wl^3}{240 EI}. \quad \ldots \ldots \quad (3)$$

Also, since by eq. (26) of Art. 22 $w_1 = \dfrac{5Wl^3}{384EI}$,

$$\frac{w_0}{w_1} = \frac{8}{25}; \quad \therefore \ w_0 = \tfrac{8}{25} w_1. \quad \ldots \ldots \quad (4)$$

Hence the resilience becomes

$$\textit{Resilience} = W \tfrac{8}{25} w_1. \quad \ldots \ldots \quad (5)$$

If the value of W in terms of w_1 be taken from eq. (26) of Art. 22 and substituted in eq. (5),

$$Resilience = \frac{3072EI}{125l^3} w_1^2. \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Hence *the resilience of a bent beam varies as the square of the centre deflection.*

If the actual centre deflection of the bar or member considered be w', the resilience of the beam when deflected to that extent will be

$$Resilience = \left(\frac{w'}{w_1}\right)^2 \frac{W^2l^3}{240EI}. \quad . \quad . \quad . \quad . \quad (7)$$

The curvature of the bar or member being slight, the lengths (equal to each other) of the neutral surface with the deflections w' and w_1 will be, if l' and l_1 are the corresponding lengths of span or horizontal projections of the neutral surface,

$$l'\left(1 + \frac{8}{3}\frac{w'^2}{l'^2}\right) = l_1\left(1 + \frac{8}{3}\frac{w_1^2}{l_1^2}\right). \quad . \quad . \quad . \quad (8)$$

Hence

$$l' - l_1 = \frac{8}{3}\left(\frac{w_1^2}{l_1} - \frac{w'^2}{l'}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

The difference $l' - l_1$ represents the movement toward or from each other of the two ends of the bar or member under the action of the direct stress or force P.

In the case of the eye-bar, the pull of the force P removes a part of the deflection w_1, and in so doing performs work in aiding to lift the weight W of the bar, the remainder of the work of lifting W being performed by the elastic efforts of the bar to straighten itself from the deflection

w_1 to w', the latter portion of the work being represented by the quantity $\frac{W^2l^3}{240EI}\left(1-\frac{w'^2}{w_1^2}\right)$. Hence the following equation of work may be written,

$$\frac{P}{2}\frac{8}{3}\left(\frac{w_1^2-w'^2}{l}\right)+\frac{W^2l^3}{240EI}\left(1-\frac{w'^2}{w_1^2}\right)=W\frac{8}{25}(w_1-w'). \quad (10)$$

The conditions under which the work represented by eq. (10) is performed are such that either (w_1-w') or (w_1+w') may be written in the second member. The resulting numerical value of w' will be the same in both cases but affected by different signs. As the equation is written the numerical value of w' will be negative.

In eq. (10) there is taken $l'=l_1=l$, the length of panel, which may be done with essential accuracy.

Dividing both sides of eq. (10) by (w_1-w') and solving for w',

$$w'=-\frac{w_1}{1+\frac{6}{25}\frac{Wl}{P}\frac{1}{w_1}}. \quad . \quad . \quad . \quad . \quad (11)$$

The deflection $w_1=\frac{5Wl^3}{384EI}$ appearing in eq. (11) is a known quantity.

After w' is determined, the resultant bending moment at the centre of the bar will be

$$M'=\frac{Wl}{8}-Pw'. \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

If the area of cross-section of the bar is A, the maximum intensity of stress t in it will be, by eq. (2) of Art. 33,

$$t=\frac{1}{A}\left(P+\frac{M'd}{r^2}\right). \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Or if the maximum value of t is specified

$$A = \frac{1}{t}\left(P + \frac{M'd}{r^2}\right). \quad . \quad . \quad . \quad . \quad . \quad (14)$$

If the section is rectangular, so that $A = bh$ and $d = \frac{h}{2}$,

$$t = \frac{1}{bh}\left(P + \frac{6M'}{h}\right) \quad . \quad . \quad . \quad . \quad . \quad (15)$$

and

$$bh = \frac{1}{t}\left(P + \frac{6M'}{h}\right). \quad . \quad . \quad . \quad . \quad . \quad (16)$$

When the depth of the bar is small in comparison with the length l, it may happen that the resultant or final deflection w' will be such as to make the bending moment M' equal to zero. Or

$$M' = \frac{Wl}{8} - Pw' = 0; \quad \therefore \ w' = \frac{Wl}{8P}. \quad . \quad . \quad (17)$$

When w' found by eq. (17) is less than w' given by eq. (11), eq. (17) is to be employed. This result shows that the bar will be subject to no bending, but that it will hang like a flexible cable. The conditions thus developed are those which indicate when a horizontal or inclined bar stressed in tension ceases to act partially as a beam and becomes purely or wholly a tie.

These formulæ are perfectly general for all cases of bars or members in tension, even for such small sections as wire. Their application to individual cases will show that excessive intensities will not exist where simple tension members are held under stress in a nearly horizontal position.

Art. 37.—Combined Bending and Direct Stress in Compression Members.

If the ordinary approximate method of Art. 35 be employed, eq. (4) of that article is immediately applicable, using the minus sign in the denominator, P being the total direct stress of compression and M_1 the bending moment due to the uniform transverse load and to eccentricity of the line of action of P, if there be any. The greatest intensity of bending stress as represented by that formula would then be

$$k=\frac{M_1 d}{I-\frac{Pl^2}{10E}}. \qquad (1)$$

In this equation, d is the distance from the neutral axis of the section to the extreme fibre in which the intensity k exists.

If e be the eccentricity of the line of action of P, and if W be the weight of the compression member whose length is l,

$$M_1=\frac{Wl}{8}\pm Pe. \qquad (2)$$

When the moment of P produces bending of the same sign with the transverse load W, the plus sign is to be used in eq. (2), and the minus sign when those moments are opposite. If the line of action of P coincides with the axis of the member, the moment Pe disappears from eq. (2). Again, if the member is vertical, so that there is no transverse bending due to the load W, when the line of action of P has the eccentricity e,

$$M_1=Pe. \qquad (3)$$

This latter case exists very frequently in the columns of buildings.

Eq. (1) is thus seen to represent the greatest intensity of bending stress with M_1 taken from either eq. (2) or eq. (3) for the cases of transverse loading, no transverse loading, eccentric longitudinal loading, or any combination of those cases.

The resultant intensity of stress, i.e., the greatest intensity of compressive stress in the entire compression member, will be

$$t=\frac{P}{A}+\frac{M_1 d}{I-\frac{Pl^2}{10E}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

As A is the area of cross-section, $I=Ar^2$, r being the radius of gyration of the cross-section of the compression member. If $q=\frac{P}{A}$, eq. (4) will take the form

$$t=\frac{P}{A}+\frac{M_1 d}{Ar^2-\frac{Pl^2}{10E}}=q+\frac{M_1 d}{Ar^2-\frac{Pl^2}{10E}}. \quad . \quad . \quad . \quad (5)$$

In the use of this equation, the intensity q must obviously never exceed the working value given by the column formula employed. Indeed, if there is suitable eccentricity q may be much less than that working long column value.

In practical operation the principal use of eq. (5) may be the determination of the area of cross-section A with some prescribed value of t. It is usually feasible to assign general outside dimensions of the proposed column section and that will enable a close approximate value of r to be assigned. If, at the same time, an approximate value of q may also be taken, the resolution of the first and third members of eq. (5) will at once give

$$A=\frac{P}{10E}\frac{l^2}{r^2}+\frac{M_1}{t-q}\frac{d}{r^2}. \quad . \quad . \quad . \quad . \quad (6)$$

If, on the other hand, such an assignment of q may not be made, it will be necessary to solve the first and second members of eq. (5), as a quadratic equation, for A. Bringing both terms of the second member of eq. (5) over a common denominator and solving the resulting equation of the second degree in the usual manner, the following general value of A will be found:

$$A=\frac{1}{2}\left(\frac{P}{10E}\frac{l^2}{r^2}+\frac{P}{t}+\frac{M_1}{t}\frac{d}{r^2}\right)$$
$$\pm\sqrt{\frac{1}{4}\left(\frac{P}{10E}\frac{l^2}{r^2}+\frac{P}{t}+\frac{M_1d}{tr^2}\right)^2-\frac{P}{10Et}\frac{l^2}{r^2}}. \quad (7)$$

Frequently there may be written $d=\frac{h}{2}$ and $r=.4h$. Hence

$$\frac{d}{r^2}=\frac{3}{h}\text{ (nearly).}$$

If, again, $d=\frac{h}{2}$ and $r=.35h$,

$$\frac{d}{r^2}=\frac{4}{h}\text{ (nearly).}$$

The preceding values of the radius of gyration r represented in terms of the depth h of the compression member are closely approximate for practical design work.

Eqs. (6) and (7) will give the desired area of section of the compression member carrying both direct stress and bending produced by transverse loading under the assumptions of the method ordinarily employed. Those formulæ are sufficiently accurate for their purposes, but it may be desirable to use the more exact formulæ given in the next section.

Exact Method for Combined Compression and Bending.

The exact procedure for combined compression and bending is identical with that used in Art. 36, the formulæ determined there simply being adapted to a compressive longitudinal force instead of a force of tension. It is to be observed, as in the case of the tension member, that the compression member may be horizontal or inclined, so as to be subjected to bending either from its own weight or from some other form of loading in addition to that weight. The member may also be subjected to uniform bending throughout its length by the eccentric application of the longitudinal force P concurrently with the preceding cross bending, or, as in the case of a vertical column carrying eccentric loading, by that force P alone.

It is essential to recognize in this connection that while the columns may occasionally be in the pin-end condition, usually their ends are essentially in a condition of at least partial fixedness, although the degree of fixedness is indeterminate. It will conduce to simplicity of treatment if the transverse bending, either from distributed loading or by the eccentricity of application of the column load, be treated as if the ends of columns are hinged. It has been shown in Art. 22 that the centre deflection of a beam of given length and cross-section with ends simply supported and with the loading uniformly distributed is five times as great as when the ends of the same beam are fixed. In the following analysis, therefore, the bending from both the sources named may be considered as produced in a column with hinged ends by a total uniformly distributed load W, sufficient in amount to cause one fifth of the actual bending moment acting on the column with ends fixed. In this manner the fixed or constrained end condition of the actual column is provided for, while the simplicity of the hinged end computations is retained. The bending moment produced by P, acting with the lever-arm of the greatest deflection, will concur with the bending moment produced by the own weight of the member or other vertical uniform loading, instead of being opposed to it, as was the case with the tension member of Art. 36. The work performed, therefore, by P and the uniform loading W will be equal to the resilience or elastic work performed in the member in changing the deflection from w_1 to w', it being remembered, in this case, that w' may be less than w_1. Under these conditions, then, eq. (10) of Art. 36, expressing the work done on the beam in changing the deflection from the w_1 to w' will become the following, the second member representing the resilience or the work done by the elastic stresses throughout its volume:

$$\frac{8}{3}\frac{P}{2}\left(\frac{w'^2-w_1^2}{l}\right)+\frac{8}{25}W(w'-w_1)=\frac{W^2l^3}{240EI}\left(\frac{w'^2}{w_1^2}-1\right). \qquad (8)$$

Dividing both members of this equation by $(w'-w_1)$, then solving for w', the following value of the latter will immediately result:

$$w'=\frac{w_1}{\frac{6}{25}\frac{Wl}{P}\frac{1}{w_1}-1}, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (9)$$

in which

$$w_1=\frac{5Wl^3}{384EI}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (9a)$$

Having found the deflection w', the general equation for the resultant maximum bending moment, eq. (1) of Art. 33, will take the following form, in which the coefficient c is introduced to provide for fixedness of ends in the manner shown in Prob. 4, at the end of this chapter. If the ends are hinged, corresponding to the end condition of a beam simply supported, $c=1$, but if the ends are fixed, c may be taken as .5:

$$M=c\left(\frac{Wl}{8}\pm P(e\pm w')\right). \quad . \quad . \quad . \quad . \quad (10)$$

In this equation care must be exercised in using the double signs, observing that both plus signs are to be taken together as are both minus signs; also, that the eccentricity e in a vertical column is taken in a direction opposite to the deflection w', in which case e is to be considered positive and the lever-arm of P is $(e+w')$. In the upper chord of bridges e may be given such value that

$$M=\frac{Wl}{8}-P(e-w')=0 \text{ (nearly)}. \quad . \quad . \quad . \quad (11)$$

In the case of vertical columns, like those in buildings, ordinarily the term $\frac{Wl}{8}$ disappears, leaving the bending moment in the column:

$$M=P(e+w'). \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

In the great majority of cases w' is so small in comparison with e as to make it negligible, so that

$$M=Pe. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

These various values of the bending moment M cover all that usually occur in practical operations.

If, in accordance with the preceding notation, t is the maximum resultant intensity of stress in the member, there will result

$$t=\frac{P}{A}+\frac{Md}{I}=\frac{1}{A}\left(P+\frac{Md}{r^2}\right). \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Evidently the uniform intensity of compressive stress $\frac{P}{A}$ must not exceed the intensity of working stress given by a suitable long column formula. When the greatest working intensity t is prescribed, the desired area of cross-section of the compression member will be

$$A=\frac{1}{t}\left(P+\frac{Md}{r^2}\right). \quad . \quad . \quad . \quad . \quad . \quad (15)$$

The closely approximate values of $\frac{d}{r^2}$ given immediately following eq. (7) may be used in a precisely similar manner in eq. (15), so as to simplify the practical use of that equation.

Problems for Chapter V.

Problem 1.—A steel eye-bar 8 ins. by 1½ ins. in section and 32 feet long sustains, in a horizontal position, a tensile stress of 144,000 pounds, i.e., 12,000 pounds per square inch. Find the greatest bending tensile intensity of stress and the resultant intensity of tensile stress at its centre section by the ordinary approximate method of Art. 35, and by the exact method of Art. 36. In this problem, $E=30{,}000{,}000$; $l=32\times12=384$ ins.; $P=144{,}000$ and $W=40.8\times32=1306$ pounds. Also $I=\frac{1.5\times8\times8\times8}{12}=64$.

By eq. (6) of Art. 35 the resultant intensity of tensile stress required is

$$t = 12{,}000 + \frac{1306 \times 48}{16 + 17.7} = 12{,}000 + 1860 = 13{,}860 \text{ lbs. per sq. in.}$$

The centre deflection w_1, due to own weight only, used in the exact method, is $w_1 = .5$ inch. Hence, by eq. (11) of Art. 36, the centre deflection under tensile stress is

$$w' = \frac{.5}{1 + 1.67} = .19 \text{ inch.}$$

The resultant intensity of tensile stress at the centre section of the eye-bar, is therefore,

$$t = 12{,}000 + \frac{6 \times 35{,}328}{1.5 \times 8 \times 8} = 12{,}000 + 2208 = 14{,}208 \text{ lbs. per sq. in.}$$

The approximate method, therefore, gives an intensity $2208 - 1860 = 348$ pounds per sq. in. too small.

Problem 2.—A horizontal square 2 in. by 2 in. steel bar 30 ft. long is subjected to a tensile stress of 48,000 pounds, i.e., 12,000 pounds per square inch. Find the same quantities as in Prob. 1. $E = 30{,}000{,}000$; $l = 360$ inches; own weight, $W = 408$ pounds, and $P = 48{,}000$ pounds.

By eq. (6) of Art. 35

$$t = 12{,}000 + 835 = 12{,}835 \text{ lbs. per sq. in.}$$

In the exact method the centre deflection due to own weight is

$$w_1 = 6.2 \text{ inches.}$$

Eq. (11) of Art. 36 gives

$$w' = 5.55 \text{ inches.}$$

On the other hand, the criterion, eq. (17) of Art. 36, gives

$$w' = \frac{408 \times 360}{8 \times 48,000} = .3825 \text{ inch.}$$

The bar, therefore, will be subject to no bending and its stress will be simply that of tension, the centre deflection being .3825 inch. If the deflection were sufficient to give the bar sensible inclination, it would be necessary to multiply the horizontal force $P = 48,000$ by the secant of that inclination to obtain the actual tensile stress in the bar.

The results given by the ordinary approximate method are thus seen to be quite erroneous.

Problem 3.—A 1.5-inch round steel bar 48 ft. long, carrying a tensile stress of 10,000 pounds per square inch, is inclined at an angle of 51° to the horizontal. Will it be subjected to any bending, and what will be its centre deflection at right angles to its axis if $\alpha = 51°$? The component of the bar's weight producing the deflection named is $W \cos \alpha$, in which $W = 288$ pounds is the bar's weight; $W \cos \alpha = 224$ pounds; $l = 48$ ft. $= 576$ ins. By the usual formula, $w_1 = 74$ ins. Eq. (11) of Art. 36 then gives $w' = 72$ ins.; but eq. (17) of Art. 36 gives

$$w' = \frac{224 \times 576}{8 \times 17,700} = .91 \text{ inch.}$$

Hence this latter deflection is the true value and the bar is subjected to no bending in its stressed condition.

Problem 4.—A steel column 18 ft. long sustains a load of 240,000 pounds and carries a transverse load (i.e., perpendicular to its axis) of 300 pounds per linear foot, the latter total being 5400 pounds. The column has a section like that shown as "top chord latticed," Page 476, Art. 59, and it is composed of two 15-in. by $\frac{1}{2}$-in. web plates, two 3-in. by 3-in. 7-lb. angles, two 3-in. by 4-in. 14-lb. angles, and one 18-in. by $\frac{7}{16}$-in. top plate. The sectional area is 35 sq. ins. The moment of inertia I is 1255, and the radius of gyration r is 6. The loading is applied to the latticed side of the column, so that the eccentricity of application is 8.5 inches. It is required to find the deflection at the centre of the column length, the bending moment and greatest intensity of stress at the same section. Also, if the area of section were not given, find that area if the greatest allowed intensity of compression is 12,000 pounds per square inch. The details of the column at top and bottom are first to be assumed such as to make those ends essentially fixed and then hinged.

If the ends of the column were hinged, the centre bending moment would be

$$M = \frac{5400 \times 216}{8} + 240,000 \times 8.5 = 2,185,800 \text{ in.-lbs.}$$

As the ends of the column are first to be taken as fixed, and as the deflection in that condition will be but one fifth of that existing with ends hinged, it will be necessary to take one fifth of the preceding bending moment and place it equal to the expression for the centre bending moment produced by a uniformly distributed load acting on a column supposed to be with hinged ends. If W represents that uniformly distributed load,

$$\frac{Wl}{8} = 437,160 \text{ in.-lbs.}$$

Hence

$$W = 16,200 \text{ pounds.}$$

By eq. (9a) of Art. 37, E being 30,000,000 and $l = 216$ ins.,

$$w_1 = .0565 \text{ in.}$$

Remembering that $P = 240,000$, eq. (9) then gives

$$w' = .00093 \text{ in.}$$

These deflections are so small in comparison with $e = 8.5$ inches, that they will have no sensible effect upon the result and they may be neglected.

In consequence of the elastic motions of the members of a steel structure it is difficult to estimate accurately the effect of such degree of fixedness of the ends of a column as may be attained in an actual structure, but it is probable that the resulting bending moment at the centre of the column due to eccentricity and lateral loading is not less than one half that existing with hinged ends, and that ratio will be employed. In eq. (10) of Art. 37, therefore, $c = .5$, and the bending moment will be

$$M = \frac{1}{2}\left(\frac{5400 \times 216}{8} + 240,000 \times 8.5\right) = 1,092,900 \text{ in.-lbs.}$$

Hence by eq. (14) of the same Article the greatest intensity of compression will be

$$t = \frac{240,000}{35} + \frac{1,092,900 \times (d = 8.5)}{1255}$$
$$= 6857 + 7402 = 14,259 \text{ lbs. per sq. in.}$$

This computation shows the serious effect of eccentric application of loading.

If the greatest allowed intensity of compression is 12,000 pounds per square inch, eq. (15) of Art. 37 shows that the area of cross section required is

$$A = \frac{1}{12,000}\left(240,000 + \frac{1,092,900 \times 8.5}{36}\right) = 41.3 \text{ sq. ins.}$$

The moment of inertia I will now become 1481 instead of 1255.

These results may be compared with those of the ordinary approximate method by finding the greatest intensity of compression, t, by eq. (4) of Art. 37, as follows, after displacing $\frac{1}{10}$ by $\frac{1}{32}$ on account of the fixed end condition:

$$t = \frac{240{,}000}{41.3} + \frac{1{,}092{,}900 \times 8.5}{1481 - 12} = 5811 + 7570$$
$$= 12{,}135 \text{ lbs. per sq. in.}$$

There is, therefore, no material discrepancy.

Results corresponding to the preceding, but under the supposition that the ends of the column are hinged, may readily be found as follows:

$$\frac{Wl}{8} = 2{,}185{,}800; \quad \therefore\ W = 81{,}000 \text{ pounds.}$$

Hence

$$w_1 = .2825 \text{ in.}$$

and

$$w' = .0047 \text{ in.}$$

While these deflections are five times as large as before, w' is still too small to affect sensibly the results and it will be neglected. The bending moment at the centre of the column will then be

$$M = \frac{5400 \times 216}{8} + 240{,}000 \times 8.5 = 2{,}185{,}800 \text{ in.-lbs.}$$

and the greatest intensity of compression

$$t = \frac{240{,}000}{35} + \frac{2{,}185{,}800 \times 8.5}{1255}$$
$$= 6857 + 14{,}800 = 21{,}657 \text{ lbs. per sq. in.}$$

If the greatest allowed intensity of compression is 12,000 pounds per square inch, the area of cross-section becomes

$$A = \frac{1}{12,000}\left(240,000 + \frac{2,185,800 \times 8.5}{36}\right) = 63 \text{ sq. ins.}$$

The moment of inertia I will now become 2259 instead of 1255.

Comparing these results with those of the ordinary approximate method by finding the greatest intensity of compression, t, by eq. (4) of Art. 37,

$$t = \frac{240,000}{63} + \frac{2,185,800 \times 8.5}{2259 - 37} = 12,170 \text{ lbs. per sq. in.}$$

This result is a close agreement with the other.

Problem 5.—The pin of a crank-shaft like that shown in Fig. 1 of Art. 32 sustains a maximum thrust, P, of 32,000 pounds, the length of crank, e, being 20 inches, and the axial distance, l, between the centre of the thrust and shaft bearings being 18 inches. Find the diameter of the steel shaft at the bearing B if the greatest allowed bending tension, k, is 10,000 lbs. per sq. in. and the greatest allowed torsional shear, T, is 7000 lbs. per sq. in.

In using the formulæ of Art. 32 the data will be as follows:

$e = 20$ ins.; $l = 18$ ins.; $AB = 26.9$ ins.; $\tan j = \frac{20}{18} = 1.111$; $j = 48°$; $\cos j = .669$; $\sin j = .743$; $P = 32,000$ lbs.; $k = 10,000$ lbs. per sq. in.; $T = 7000$ lbs. per sq. in.; bending moment $M = 576,000$ in.-lbs.; twisting moment $M' = 640,000$ in.-lbs.

The first method of Art. 32 gives for bending, by using the first of eqs. (9),

$$D = 2.16\sqrt[3]{\frac{32,000 \times 18}{10,000}} = 8.34 \text{ ins.}$$

and for torsion, by the second of eqs. (9),

$$D = 1.72\sqrt[3]{\frac{32{,}000 \times 20}{7000}} = 7.75 \text{ ins.}$$

The bending requirement being the larger the diameter would be taken not less than 8.34 inches or 8.5 inches.

The employment of the second method leads to the following results. The direction of the resultant t will be inclined to the axis of the shaft; hence it will be made a little less than 9000 pounds, i.e., 8500 pounds per square inch; the resultant moment M will be

$$M = 32{,}000 \times 26.9 = 860{,}800 \text{ in.-lbs.}$$

Eq. (18) of Art. 32 then gives

$$D = 1.72\sqrt[3]{\frac{860{,}800}{8500}(1 + .669)} = 9.51 \text{ ins.}$$

Remembering that sec $j = 1.494$, eq. (19) of Art. 32 gives the following value of the greatest bending tension or compression in the section B:

$$k = \frac{2 \times 8500}{1 + 1.494} = 6800 \text{ lbs. per sq. in.}$$

The second method, therefore, requires a larger diameter of shaft with reduced intensities of bending and torsional stresses.

PART II.—TECHNICAL.

CHAPTER VI.

TENSION.

Art. 38.—General Observations.—Limit of Elasticity.

HITHERTO certain conditions affecting the nature of elastic bodies and the mode of applying external forces to them, have been assumed as the basis of mathematical operations, and from these last have been deduced the formulæ to be *adapted* to the use of the engineer. These conditions are never realized in nature, but they are approached so closely that, by the introduction of empirical quantities, the formulæ give results of sufficient accuracy for all engineering purposes; at any rate, they are the only ones available in the study of the resistance of materials.

In determining the quantity called the "coefficient of elasticity," it is supposed that the body is perfectly elastic, i.e., that it will return to its original form and volume when relieved of the action of external forces, also that this "coefficient" is constant. There is reason to believe that no body known to the engineer is either perfectly elastic or possesses a perfectly constant coefficient of elasticity. Yet within certain not well defined limits, the deviations from these assumptions are not sufficiently great to vitiate their great practical usefulness.

The "not well defined" limit for any one given material is called its "limit of elasticity," or "elastic limit." The "limit of elasticity," then, may be defined as *that degree of stress within which the coefficient of elasticity is essentially constant and equal to the stress divided by the strain.*

In some materials, like many grades of wrought iron and steel, the limit of elasticity approximates, to a greater or less degree, to the condition of a well-defined point. If a piece of such a material is subjected to stress in a testing machine, at the elastic limit, the amount of strain caused by a given increment of stress will be observed to increase rapidly. This increase may be uniform for a considerable range of stress, but it finally becomes irregular, after which failure takes place.

In other materials, there seems to be no simple relation between stress and strain for any condition of stress whatever. For such a material it obviously is impossible to assign either any definite elastic limit or coefficient of elasticity.

Between these limits all grades of material are found.

It should be stated that some authorities have given arbitrary definitions of the elastic limit, and that these definitions have been much used. Wertheim and others have considered the elastic limit to be that force which produces a permanent elongation of 0.00005 of the length of a bar. Again, Styffe defines, as the limit of elasticity, a much more complicated quantity. He considers the external load to be gradually increased by increments, which may be constant, and that each load, thus attained, is allowed to act during a number of minutes given by taking 100 times the quotient of the increment divided by the load. Then the "limit of elasticity" is "that load by which, when it has been operating by successive small increments as above described, there is produced an increase in the permanent elongation which bears a ratio to the length of the bar equal to 0.01 (or approximates most nearly to 0.01) of the ratio which the increment of weight bears to the total load.". (Iron and Steel, p. 30.)

The most natural value, however, seems to be that stress which exists at the point where the ratio between stress and strain ceases to be essentially constant, though the assignment of the precise point be difficult in many cases and impossible in some; and in that sense it is here used, though seldom in ordinary testing.

Again, in the common theory of flexure, modes of application of external forces and a constitution of material are assumed, which are never realized; yet the resulting formulæ are of inestimable value to the engineer.

Finally, it will be shown in the first section of Art. 41 that it is in general impossible to produce a uniform intensity of stress in a normal cross-section of a body subjected to pure tension, and, consequently, that the ultimate resistance, as experimentally determined, is a mean intensity which may be, and usually is, considerably less than the maximum sustained by the test piece.

In the ordinary testing of materials, as done in the practice of civil engineering, the elastic limit, i.e., the limit of constant ratio of stress over strain, is rarely ever determined. The point usually termed the elastic limit is commonly found by observing the intensity of stress at which the material of the test-specimen "breaks down," as indicated by the failure of the lever arm of the testing machine to remain horizontal as the stress in the specimen is increased. This point is actually the "stretch limit." It is the point at which the strain for a given increment of stress suddenly and greatly increases, even for a small increment of stress. The intensity of stress, called the stress limit, is sensibly higher than the elastic limit, although not greatly so. The difference between the two values changes much in different materials, and, indeed, has not a constant value even for the same material. Approximately the stretch limit may be from 2000 to 3000 pounds per square inch higher than the elastic limit for the ordinary grades of structural steel, and possibly less than that for wrought iron. In materials like ordinary grades of cast iron, where the relations between stress and strain are erratic, it is obviously impossible to determine any well-defined difference between the elastic limit and the stretch limit.

These general observations are to be carefully borne in mind in connection with all that follows.

Art. 39.—Ultimate Resistance.

After a piece of material, subjected to stress, has passed its elastic limit, the strains increase until failure takes place. If the piece is subjected to tensile stress, there will be some degree of strain, either at the instant of rupture or somewhat before, accompanied by an intensity of stress greater than that existing in the piece in any other condition. This greatest intensity of internal resistance is called the "Ultimate Resistance."

In very ductile materials this point of greatest resistance is found considerably before rupture; the strains beyond it increasing very rapidly while the resistance decreases until separation takes place.

These phenomena are very marked in ductile materials like wrought iron and structural steel, particularly in the latter. In such cases if the application of stress to the test piece is carefully controlled a considerable stretching of the piece may be produced beyond the point of ultimate resistance without actually separating the metal, the load per square inch of original section of the piece decreasing rapidly. It is not difficult to obtain such results with soft or mild steel.

The ultimate resistances of different materials used in engineering constructions can only be determined by actual tests, and they have been the objects of many experiments.

It has been observed in these experiments that many influences affect the ultimate resistance of any given material, such as mode of manufacture, condition (annealed or unannealed, etc.), size of normal cross-section, form of normal cross-section, relative dimensions of test piece, shape of test piece, etc. In making new experiments or drawing deductions from those already made, these and similar circumstances should all be carefully considered.

Art. 40.—Ductility.—Permanent Set.

One of the most important and valuable characteristics of any solid material is its "ductility," or that property by which it is enabled to change its form, beyond the limit of elasticity, before failure takes place. It is measured by the permanent "set," or stretch, in the case of a tensile stress, which the test piece possesses after fracture; also, by the decrease of cross-section which the piece suffers at the place of fracture.

In general terms, i.e., for any degree of strain at which it occurs, "permanent set" is the strain which remains in the piece when the external forces cease their action. It will be seen hereafter that in many cases, and perhaps all, permanent set decreases during a period of time immediately subsequent to the removal of stress. Indeed, in some cases of small strains it is observed to disappear entirely.

Some experimenters, with the aid of very delicate measuring apparatus, have observed permanent set even within what is ordinarily termed the limit of elasticity, and have been led to believe that a very small permanent set exists with any degree of stress whatever. In such cases, however, it is probable that the greater part or all of the permanent set disappears after the lapse of a few hours.

Art. 41.—Wrought Iron.—Coefficient of Elasticity.

At the present time wrought iron as a structural material has been completely displaced by the various grades of soft and mild structural steel, although the former metal is still used in small quantities for some special purposes. At the same time, many structures are still standing which were built either wholly or partially of wrought iron. Again, there are many characteristics of ductility possessed by wrought iron similar in their nature to those found in structural steel, although less in amount and less satisfactory in their action as qualities of a structural material. A study of many of the physical features of wrought iron, therefore, will be equally applicable to all ductile materials, and particularly to soft and mild structural steel. For these reasons wrought iron is here considered at greater length than its displacement by structural steel would seem at first sight to justify. The determination of the elastic limit, the stretch limit, the ultimate resistance, final stretch, final contraction, and other necessary data in connection with wrought iron, illustrates accurately and completely corresponding procedures in connection with all ductile material, so that the greater portion of the consideration of the qualities of wrought iron bears directly upon corresponding studies in connection with structural steel and other similar materials.

Before considering the experimental results which are to follow, it will be interesting as well as important to examine some of the circumstances which attend the experimental determination of the coefficient of elasticity.

If the tensile stress is uniformly distributed over each end of a test piece, it will not be so distributed over any other normal section. Since lateral contraction takes place, the exterior molecules of the piece must move towards the centre; but if this motion exists, the molecules in the vicinity of the centre must be drawn farther apart, or suffer greater strains, than those near the surface.

Hence the stress will no longer be uniformly distributed, but the greatest intensity will exist at the centre and the least at the surface of the piece. These effects will evidently increase, for a given form of cross-section, with its area. But the stretch, or strain, from which the coefficient of elasticity is computed, is measured on the surface of the piece, and corresponds, as has just been shown, to an intensity of stress less than the mean, while the latter is actually used in the computation. In the notation of eq. (1), Art. 2, p is too great and l too small; hence E will be too large.

As these effects increase with the area of the cross-section, while other things are the same, *larger bars should give greater coefficients of elasticity than smaller ones.*

These effects will evidently be intensified, also, if the external force is applied with its greatest intensity near, or at, the centre of the bar, as is the case in testing eye-bars.

Again, on the other hand, if the ends of the test piece are gripped on the surface, or skin, as is usually the case with small pieces, these effects will be very much modified, and possibly entirely counteracted, so that the greatest intensity will exist at the surface. In the latter case, the resulting coefficient would be too small.

Between these extreme cases, all grades will be found.

From these considerations, it is clear that the manner of gripping the test piece, length, character, and area of cross-section all affect the value of the coefficient of elasticity, and should be given in connection with the latter.

These conclusions apply to any other material, as well as to wrought iron.

Table I gives the results of some experiments made by the Phœnix Iron Co., ot Phœnixville, Penn., on some flats and rounds of the dimensions shown in the column headed "Size."

TABLE I.

No. of Bars.	Size.	Length.		Stretch.	p.	E.
	Inches.	Ft.	In.	Inches.	Pounds.	Pounds.
12	$4 \times 1\frac{3}{8}$	35	0	0.2692	20,000	31,203,000
9	$4 \times 1\frac{3}{16}$	27	6	0.2033	"	32,464,700
24	$3\frac{1}{2} \times 1\frac{1}{4}$	35	0	0.2500	"	33,600,000
24	$3\frac{1}{2} \times 1\frac{1}{8}$	35	0	0.2617	"	32,098,000
23	$3 \times \frac{7}{8}$	35	0	0.2587	"	32,470,000
24	$3 \times \frac{3}{4}$	35	0	0.2633	"	31,902,000
24	2×1	24	$9\frac{1}{2}$	0.1948	"	30,544,000
36	$2\frac{7}{8}$O	11	9	0.0953	"	29,380,000
68	$2\frac{3}{4}$O	11	11	0.0998	"	28,056,000
120	$2\frac{5}{8}$O	11	9	0.0947	"	29,567,000
48	$2\frac{1}{2}$O	11	9	0.0955	"	29,319,000
72	$2\frac{3}{8}$O	11	9	0.0940	"	29,787,000
48	$2\frac{1}{8}$O	11	9	0.1008	"	27,777,777

The column "p" is the intensity per square inch which caused the stretches shown in the column headed "Stretch."

From eq. (1) of Art. 2

$$E = \frac{p}{l}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

In this case, for any individual bar,

$$l = \frac{stretch}{length},$$

remembering that the stretch and length must be reduced to the same unit.

Let the above formulæ be applied to the twenty-four bars $3 \times \frac{3}{4}$ inches $\times$ (35 ft. = 420 ins.) long:

$$E = \frac{20,000 \times 420}{0.2633} = 31,902,000 \textit{ pounds}.$$

The other values are found in precisely the same way. The quantities in the column E are the averages of the number of experiments given in the extreme left-hand column. The fact that the results are the averages of a great number of experiments gives the table peculiar value. This table is taken from "Useful Information for Architects and Engineers," published by the Phœnix Iron Co. The following reference to the table is taken from the same source: "The annexed table gives the results attained in testing, with the proof load of 20,000 pounds per square inch, a number of bars for the International Bridge over the Niagara River, near Buffalo, N. Y. The recovery of each bar, after the removal of the load, was perfect, no permanent set occurring at less than 25,000 pounds. It will be observed that the stretch per foot of the flat bars is less than that of the rounds, giving them higher moduli of elasticity." It is interesting and important to observe this last point.

It is to be observed, finally, that these coefficients of elasticity are determined for one intensity of stress only, i.e., 20,000 pounds per square inch. It is probable that values a little different might be given by other intensities.

Table II contains coefficients of elasticity for tension in pounds per square inch, computed from data given in "Report of Tests of Metals" for 1881, made on the government testing machine at Watertown, Mass. These results, like those in the preceding table, derive additional interest and importance from the fact that they belong to full-size bars and such as are ordinarily used in engineering practice.

TABLE II.

Number.	Size of Bar. Inches.	Gauged Lengths. Inches.	Stretch in Gauged Length. Inches.	*E* Pounds in per Sq. In.
S 1	3.03×1.01	80	.029	27,586,240
S 2	3.03×1.01	80	.0279	28,673,760
S 3	3.03×1.01	80	.0272	29,411,760
S 4	3.03×1.01	80	.0238	33,613,440
D 5	3.03×1.01	80	.0281	28,469,760
D 6	3.03×1.01	80	.0215	37,209,300
D 7	3.03×1.01	80	.0273	29,304,000
D 8	3.03×1.01	80	.0260	30,769,200
S 9	5.05×1.28	80	.0275	29,090,880
S 10	5.04×1.27	80	.029	27,586,240
S 11	5.03×1.27	80	.0278	28,776,960
D 12	5.02×1.26	80	.026	30,769,200
D 13	5.03×1.26	80	.0256	31,250,000
D 14	5.03×1.26	80	.0277	28,880,860
S 15	3.05×1.01	80	.028	28,571,430
S 16	3.05×1.01	80	.0284	28,168,960
S 17	3.05×1.00	80	.0279	28,673,760
D 18	3.05×1.00	80	.0285	28,070,160
D 19	3.05×1.00	80	.032	25,000,000
D 20	3.05×1.02	80	.0315	25,396,800
S 21	5.08×1.26	80	.025	32,000,000
S 22	5.08×1.26	80	.028	28,571,430
S 23	5.09×1.26	80	.0256	31,250,000
D 24	5.05×1.25	80	.026	30,769,200
D 25	5.06×1.26	80	.0272	29,411,760
D 26	5.06×1.25	80	.027	29,629,600

Those bars whose numbers are preceded by an "S" are of single rolled material, while those preceded by a "D" are double rolled.

The values of "*E*," the coefficient of elasticity, were computed by eq. (1), *l* representing the stretch in 80 inches divided by 80. The stretch in every case was measured at a stress of 10,000 pounds per square inch, which is the limit quite generally specified as a maximum in railway bridges of ordinary length. It is seen that the values thus determined are not on the whole very different from those shown in Table I for double this intensity of stress.

Bar No. 9 was slightly warped and No. 25 not originally straight, but the coefficients do not seem to be appreciably affected.

A comparison between the results for single and double rolled iron shows that there is no appreciable difference between them either in uniformity or magnitude. In the aggregate, the values run from 25,000,000 to 37,209,300 pounds, giving a variation of fifty per cent. of the lowest amount. The latter value is too high and is doubtless due to some error in observation. The same observation can be applied to all values averaging about 29,000,000.

Double rolling, which materially increased the cost of the metal, is thus seen to give it no elastic advantage.

Table III gives the results of the experiments of Mr. Eaton Hodgkinson on the tensile elasticity and permanent set of two wrought-iron bars. The coefficients of elasticity E have been computed from the data contained in the first three columns as given by Mr. B. B. Stoney in his "Theory of Strains in Girders and Similar Structures." The following is the notation used:

p = pounds per square inch;
Ll = total elongation, or strain, for the bar;
"*Sets*" = permanent set;
E = coefficient of tensile elasticity $= p \times L \div Ll$.

These experiments show some very interesting results.

In the first place permanent sets were observed with the low intensities of stress of 8003 and 3786 pounds, and it becomes a question whether permanent sets would not have been observed with lower intensities and more deli-

TABLE III.

TENSILE EXPERIMENTS ON TWO ANNEALED "BEST" WROUGHT-IRON BARS TEN FEET LONG AND ONE INCH SQUARE.

Bar No. 1.

p.	*Ll.*	*Sets.*	*E.*
	Inches	Inches.	
2,668	.00986	——	32,457,000
5,335	.02227	——	28,198,000
8,003	.03407	.000305	28,180,000
10,670	.04556	.000407	28,101,000
13,338	.05705	.000509	28,056,000
16,005	.06854	.000610	28,020,000
18,673	.07993	.000813	28,033,000
21,340	.09193	.001525	27,855,000
24,008	.10485	.003966	27,475,000
26,676	.12163	.009966	26,308,000
29,343	.15458	.031424	22,782,000
32,011	.26744	——	14,361,000
	.28271 in 5 minutes	.13566	
34,678	.5148	.36864	8,083,000
37,346	1.095	1.01695	4,077,000
Repeated	1.1949	1.02966	——
40,013	1.220 in 5 minutes	1.093	3,924,000
Repeated and left on	1.411 after 1 hour	——	——
"	1.424 after 2 hours	——	——
"	1.433 after 3 hours	——	——
"	1.434 after 4 hours	——	——
"	1.436 after 5 hours	——	——
"	1.437 after 6 hours	——	——
"	1.443 after 7 hours	——	——
"	1.443 after 8 hours	——	——
"	1.443 after 9 hours	——	——
"	1.443 after 10 hrs.	——	——
42,681	2.148 in 5 minutes	1.983	2,384,000
Repeated	2.339 in 5 minutes	——	——

Bar No. 2.

p.	*Ll.*	*Sets.*	*E.*
	Inches.	Inches.	
1,262	.00520	——	29,125,000
2,524	.01150	——	26.347,000
3,786	.01690	.00050	26,851,000
5,047	.02240	.00060	26,990,000
6,309	.02772	.00050	27,312,000
7,571	.03298	.00045	27,551,000
8,833	.03790	.00050	27,953,000
10,095	.04300	.00050	28,198,000
11,357	.04854	——	28,077,000
12,619	.05370	.00070	28,199,000
13,880	.05950	——	27,984,000
15,142	.06480	——	28,041,000
16,404	.06980		28,186,000
17,666	.07530	.00130	28,153,000
18,928	.08170	——	27,794.000
20,190	.08740	.00270	27,734,000
21,452	.09310	——	27,644,000
22,713	.09920	.00410	27,474.000
23,795	.10570	——	27,213.000
25,237	.11250	.00680	26,919,000
26,499	.12040	——	26,420,000
27,761	.12880	.0120	25,872,000
29,023	.14500	——	23,986,000
30,285	.1991	——	18,244,000
30,285	.2007 after 5 min.	——	18,030,000
30,285	.2018 after 10 min.	.0736	——
30,285	.2045 after 15 min	.0774	——
Repeated	.2080 after 20 min	.0796	——
"	.2096 after 1 hour	.0814	——
"	.2366 after 17 hrs.	.1082	——
31,546	.242 after 5 min.	.1083	15,617,000
Repeated	.2449 after 5 min.	.1111	——
32,808	.5506	.4141	7,132,000
Repeated	.7024 after 5 min.	.5635	——
"	.70966 after 10 min.	.6558	——

TABLE III.—*Continued.*

Bar No. 1.				Bar No. 2.			
p.	*Ll.*	*Sets.*	*E.*	*p.*	*Ll.*	*Sets.*	*E.*
	Inches.	Inches.			Inches.	Inches.	
Repeated	2.383 in 10 min.	2.212	——	Repeated	1.014 after 15 min.	.866	——
"	2.428 after 46 hrs.	2.237	——	34,070	1.346 after 1 min.	——	2,839,000
45,348	2.580 after 5 min.	2.377	2,109,000	34,070	1.400 after 2 min.	——	——
Repeated	2.605 after 1 hour	——	——	34,070	1.600	1.44	——
"	2.606 after 2 hours	——	——	Repeated	1.65 after 1 min.	——	——
"	2.606 after 19 hrs.	2.403	——	"	1.786 after 1 hour	1.628	——
48,016	2.975 after 5 min.	2.733	1,936,000	35,332	2.04 after 5 min.	1.874	2,078,000
Repeated	3.019 after 1 hour	——	——	Repeated	2.18 after 5 min.	2.01	——
"	3.029 after 11 hrs.	——	——	"	2.254	2.08	——
50,684	4.195 in 10 min.	3.941	1,448,000	36,594	2.54 after 6 min.	——	1,743,000
Repeated	4.226	——	——	37,856	2.894	——	1,571,000
"	4.227 in 7 hours	——	——				
"	4.227 in 12 hours	——	——				
53,351	Broke	——	——				

cate apparatus, at least for a short time after the material is subjected to stress.

In both bars the largest value of *E* is found for the smallest intensity of stress. In bar No. 1 the values of *E* decrease, with one exception, regularly from the greatest. In bar No. 2, however, greater irregularity is observed; there are two maxima, one for the intensity 1262 pounds, and the other for about 12,000, with nearly regular gradations from these values.

Considering the whole range in both bars, *E* may be considered nearly constant until an intensity of about 26,000 pounds per square inch is reached in each case; it then begins to fall off very rapidly. 26,000 pounds per square inch, then, may be considered about the limit of elasticity for both bars.

It is very important to observe the increase of strain with the lapse of time after the limit of elasticity has been considerably passed.

Values of the coefficient of elasticity, therefore, mean little after that limit is exceeded.

The results of the experiments on bar No. 1 are shown graphically in Fig. 1. The values of "p" are laid off vertically through O to a scale of 20,000 pounds to the inch; the tensile strains are the horizontal coordinates of the curve laid down at full size. The essentially straight portion of

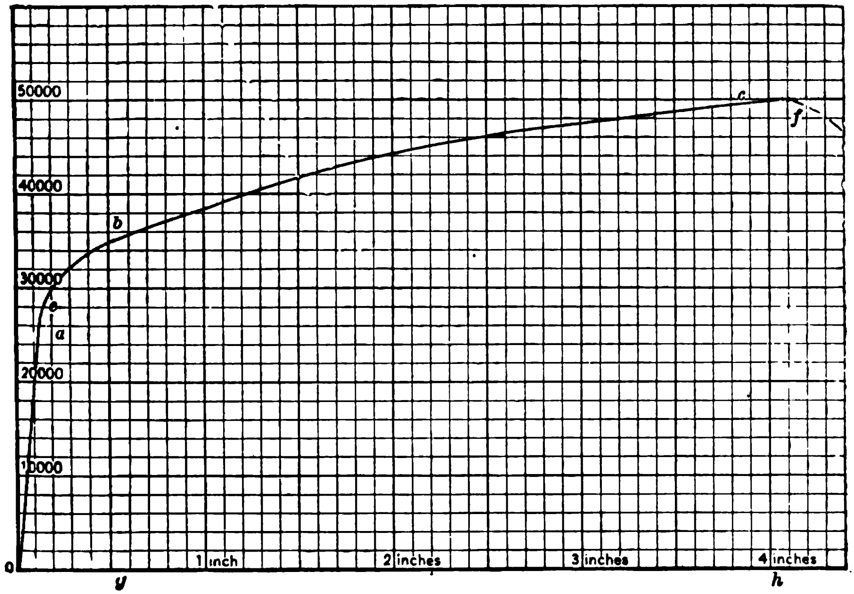


FIG. 1.

the curve between O and a is within what is ordinarily known as the "elastic limit."

The equation for this portion of the line is

$$p = El,$$

E being assumed constant if Oa is considered straight.

The point a is at a vertical distance above O indicating about 26,000 pounds per square inch, i.e., about the elastic limit. Above this point the curvature of the line is very sharp, indicating a rapid fall in the value of E and a rapid rise in the values of the strains l or Ll. For "p" $= 27{,}000$ (nearly) the table shows $E = 23{,}000{,}000$ (nearly) and $Ll = 0.12$ inch; while for "p" $= 37{,}000$, $E = 4{,}100{,}000$ and $Ll = 1.095$ inches (nearly). These phenomena are always characteristic of the limit of elasticity.

Above the point b the curvature is slight, indicating (what the table shows) a comparatively slow change in the values of E.

The table shows that bar No. 2 would exhibit a curve of precisely the same character but with a more rapid decrease to E above the elastic limit. The tests of this bar were not carried to failure on account of the breaking of one of the holding details.

Within the elastic limit the mean values of E may be taken about as follows:

For bar No. 1,

$$E = 28{,}000{,}000 \text{ pounds.}$$

For bar No. 2,

$$E = 27{,}000{,}000 \text{ pounds.}$$

Reviewing the preceding values, therefore, it would appear that the coefficient of tensile elasticity for good wrought iron may be ordinarily taken to lie between

25,000,000 to 30,000,000 pounds per square inch, with extreme values arising from variation of mode of manufacture, chemical constitution, size of bar, etc., lying some distance either side of those limits.

Since $E=\frac{p}{l}$, if $p=1$, $l=\frac{1}{E}$ will be the elongation or tensile strain for each unit of stress; hence, *the coefficient of elasticity is the reciprocal of the strain for a unit of stress.* For an intensity of stress of 20,000 pounds, for example, then,

$$l=\frac{20,000}{25,000,000} \text{ to } \frac{20,000}{30,000,000}$$

$$=\frac{1}{1250} \text{ to } \frac{1}{1500},$$

or a bar of wrought iron will be stretched $\frac{1}{1250}$th to $\frac{1}{1500}$th of its length.

Ductility and Resilience.

Curves showing the intensities of stresses and the strains corresponding to them, like that of Fig. 1, exhibit with much completeness the physical characteristics of the specimens under test. In that figure, for instance, the straight portion *oa* of the strain curve belongs to the purely elastic behavior of the metal, and the point *a* indicates the elastic limit at 26,000 pounds per square inch. The point *e* indicates the stretch limit at about 29,000 pounds per square inch. Between *a* and *e* there is no constant proportionality between constant increments of stress and the corresponding increments of strain, but there is not a perceptibly great increase of strain for a given small increment of stress until the stretch limit *e* is passed. As the intensity of stress increases beyond *e*, however, the material is said to break down; in other words, constant and comparatively great increases of strain take place for a given small increment of stress, and that condition may in some cases hold until the test specimen parts in failure. In other cases, like that shown in Fig. 1, the latter part of the strain curve, such as *bc* approaches more nearly a straight line not greatly inclined to a horizontal. This indicates, as shown in the figure, that although the load on the test-piece is not much increased, the piece itself is stretching rapidly. The highest point *c* of the curve indicates the ultimate resistance. In the present instance the test-piece parted at the ultimate resistance, but in such highly ductile materials as structural steel there would be a continuation of the strain curve, as shown by the broken portion *cf* which is inclined downward at a rapid rate, showing that although the metal is not actually parted the load which it carries is rapidly decreasing.

The work performed in rupturing the test-piece can also be readily computed from Fig. 1. The actual strain shown by the horizontal coordinate of any point of the curve is the entire path of the force or load carried by the test specimen up to the degree of stress indicated by the vertical ordinate of the curve. In other words, the amount of work performed on the specimen up to the point *b* is indicated in inch-pounds by the area of the figure *oebg*, just as the entire work performed in rupturing the specimen is shown by the area of the complete figure *oebch*. The strain at the elastic limit of 26,000 pounds per square inch is .11744 inch. The average force acting upon the specimen up to the elastic limit would be one half of 26,000 pounds, or 13,000 pounds, as the cross-sectional area of the test specimen is one square inch. The elastic resilience or work performed upon the specimen up to the elastic limit is, therefore,

$$.11744 \times \frac{26,000}{2} = 1527.6 \text{ inch-pounds.}$$

Inasmuch as the test specimen was 120 inches long, the elastic resilience of the bar would be 12.73 inch-pounds per cubic inch of its volume. Similarly, the area of the rectangular figure *oebch* is 4.97 square inches, and as the scale of force is 20,000 pounds per linear inch, that figure represents $4.97 \times 20{,}000$ pounds $= 99{,}360$ inch-pounds of work; or $\frac{99{,}360}{120} = 828$ inch-pounds of work per cubic inch of volume of the test specimen. If this test-bar, therefore, were to be broken by a falling weight of 100 pounds, that weight would be required to fall through a height of

$$\frac{99{,}360}{100} = 993.6 \text{ inches.}$$

It is clear from the figure that if the metal possessed little ductility so that its strain curve extended no further than the point *b*, the work required to be expended in breaking it would be very small compared with that needed for rupturing the actual wrought-iron piece. The effect of a falling weight may represent a shock or blow, or be taken as the equivalent of what is usually called a suddenly applied load. These considerations show why a ductile material requiring so much more work to be performed to break it is much better adapted to sustain shock than a non-ductile or brittle material. The latter class of materials can be strained so little before failure that little work is required to be expended to break them.

Computations of resilience or work to be expended in straining a material either up to the elastic limit or even up to the point of failure are very rarely made in civil engineering practice, although the value of a metal possessing high values of resilience is always recognized. Such strain curves as those shown in Fig. 1 disclose completely the ductile properties and resilience of any metal to which they belong.

Ultimate Resistance and Elastic Limit.

It has been found by experiment that bars of wrought iron which are apparently precisely alike in every respect, except in area of normal section, *do not give the same ultimate tensile resistance per square inch.* Other things being the same, *bars of the smallest cross-section give the greatest intensity of ultimate tensile resistance.*

Aside from the absence of uniform distribution of stress in the interior of the bar, as was shown in the section "*coefficient of elasticity,*" and the intensified effects of the processes of production on pieces with comparatively small cross-sections, this result is to be expected from the circumstances which attend fracture. When a piece of material is subjected to tension to the point of rupture, not only a tensile strain of essentially uniform character, from end to end, takes place, but also a very considerable local transverse strain or contraction at the place of fracture. This latter manifests itself only shortly before rupture as a short "neck" in the piece. A given *percentage* of "local" contraction in the case of a large section involves a much larger absolute lateral movement of the molecules than in the case of a small section, and it is evident that this absolute lateral movement will exert a much more potent influence toward severing the molecules sufficiently for rupture than the percentage of contraction. Hence the degree of local and lateral movement required by rupture will be reached with a less mean intensity of stress in the cases of large sections than in those of small ones. This is equivalent to a greater intensity of ultimate resistance for the small sections, and, as has been indicated, this conclusion is verified by experiment.

The same considerations result in the additional conclusion that, other things being equal, the smaller sections will give the *greater final contraction*, and a greater intensity of ultimate resistance with greater final contraction involves a *greater final stretch* for the same length of piece.

These last two conclusions will also be found to be hereafter verified by experiment.

Again, it is found independently of the effects of the processes of production, as might be anticipated, that the length in terms of the lateral dimensions of the test piece, within certain limits, affects very perceptibly the ultimate resistance.

If a specimen of the shape shown in Fig. 2 be broken by a tensile stress, it will, of course, fail in the reduced section *MN*. Before failure takes place, the reduced portion will be considerably elongated and the normal section correspondingly reduced in consequence of the shearing strains in the oblique planes shown by the dotted lines. (See Arts. 3 and 4.) When the reduced portion in the vicinity of *MN* is very short in comparison with its lateral dimensions, it includes the *whole* of very few of these oblique planes, if any at all; consequently very little movement of these oblique layers over each other can take place; in other words, little or no reduction of section can take place before rupture. In this latter case, then, a greater area of metal section will offer its resistance to the external tensile force, at the instant of failure, than in the former, and a correspondingly greater intensity of ultimate resistance will be found. Thus the shape and dimensions of the test piece will considerably influence the ultimate

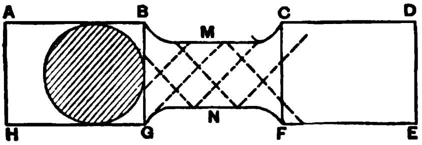


FIG. 2.

resistance and strains, as will soon be shown by experimental results.

All the preceding conclusions, though given in connection with wrought iron, are independent of the nature of the material, and apply equally to steel and cast iron.

Since the reduction of area of the fractured section and the elongation of the bar are true measures of the ductility of the iron, these are or should be always measured with care.

Table IV exhibits in a very plain manner the decrease of ultimate tensile resistance with the increase of sectional area of round bars; it is taken from the "Report of the Committees of the U. S. Board appointed to Test Iron, Steel, and other Metals," etc., by Commander L. A. Beardslee, U.S.N.

This decrease is probably partly due to the effect produced upon the iron by the rolls as it passes through them, the bars of smaller sections being more "drawn" and at a lower temperature in consequence of the less mass cooling more quickly.

The notation of the table is the following:

"*Dia.*" = diameter of the round bar in inches;
"*T.*" = ultimate tensile resistance;
"*E. L.*" = elastic limit.

It will be observed that the ultimate resistance per square inch varies between widely separated limits, in some cases, for the same diameter of bar. This is due to the fact that the different bars, even of the same diameter, were from a number of different mills, and consequently involved different treatment in manufacture, chemical constitution, etc. A general view of the table, however, shows in a marked and satisfactory manner the decrease of *T* with the increase of the diameter or area of normal section. The last fourteen bars of the table are of the same manufacture and show a decrease in *T* as nearly uniform as could be expected.

TABLE IV.

ULTIMATE RESISTANCE AND ELASTIC LIMIT IN POUNDS PER SQUARE INCH OF ORIGINAL NORMAL SECTION.

Dia.	*T.*	*E. L.*	Dia.	*T.*	*E. L.*	Dia.	*T.*	*E. L.*
½	59,885	——	1⅜	53,016	35,379	1 13/16	50,969	30,814
⅝	54,090	40,980	"	51,296	31,992	"	50,307	29,767
¾	62,700	——	"	50,594	34,940	"	48,953	——
"	59,000	——	1½	57,052	38,417	1⅞	55,803	31,031
"	57,700	——	"	56,505	32,496	"	53,100	32,074
"	55,400	——	"	55,131	33,771	"	52,875	35,641
"	52,275	39,126	"	54,540	——	"	52,505	32,312
13/16	55,450	——	"	55,415	32,869	"	51,459	27,816
"	52,050	——	"	54,354	34,617	"	50,363	——
"	57,660	——	"	54,544	33,027	"	51,039	33,067
⅞	51,546	35,933	"	53,512	——	"	49,744	35,615
15/16	50,630	33,931	"	52,819	34,840	"	48,670	23,250
1	61,727	——	"	52,736	34,901	2.0	60,213	31,441
"	57,363	37,415	"	52,700	35,880	"	52,914	31,198
"	57,807	39,230	"	52,155	27,708	"	49,164	——
"	56,790	36,885	"	51,994	32,054	"	51,684	33,104
"	51,921	31,300	"	51,456	34,591	"	52,127	32,461
"	52,819	32,267	"	51,047	——	"	52,011	34,702
"	51,400	34,600	1⅝	56,344	35,889	"	51,146	28,567
1⅛	60,458	37,344	"	57,402	35,701	"	50,000	36,184
"	57,470	31,900	"	56,227	33,207	"	50,171	28,983
"	57,498	41,311	"	54,334	32,163	"	47,812	35,864
"	55,927	37,250	"	53,339	33,540	"	48,249	31,413
"	54 644	34,695	"	53,614	30,664	"	46,151	36,050
"	53,900	26,787	"	52,675	33,745	2 1/16	51,559	——
"	53,035	34,410	"	52,314	29,364	"	49,422	——
"	52,267	32,019	"	52,401	34,012	2⅛	50,481	——
1¼	59,461	36,501	"	51,205	33,318	"	51,225	——
"	57,897	32,469	"	50,970	33,625	"	48,382	30,459
"	55,782	35,596	1 11/16	56,595	38,310	2 3/16	51,666	——
"	56,334	33,921	"	54,114	——	2¼	51,530	——
"	55,253	34,784	"	57,789	34,160	"	49,290	32,163
"	53,893	32,712	1¾	57,874	——	"	48,898	——
"	53,247	32,520	"	54,410	31,354	"	46,866	28,241
"	53,752	——	"	53,846	36,573	2½	48,475	28,932
"	52,970	32,075	"	55,018	34,283	"	47,428	29,941
"	53,022	——	"	53,264	——	"	47,344	29,758
"	50,040	30,730	"	53,154	35,323	2¾	46,446	26,333
1⅜	58,926	37,548	"	51,509	29,404	3.0	47,761	26,400
"	58,021	32,152	"	50,395	36,254	3¼	47,014	24,591
"	54,949	31,030	"	50,547	35,954	3½	47,000	24,961
"	54,277	33,622	"	49,816	31,214	3¾	46,667	23,636
"	52,733	34,606	"	50,129	32,271	4.0	46,322	23,430
"	53,557	33,650			——			
"	52,537	34,469	1 13/16	56,577				

In the words of the Report, as given by Wm. Kent, C.E., in the abridgment, "The elastic limit as given is not from perfectly accurate data; it is simply the amount of stress which produced the first perceptible change of form, divided by the bar's area."

TABLE V.

RECTANGULAR BARS.

No.	Kind of Iron.	Size of Bar.	Stress in Lbs. per Sq. In.		Per Cent. of	
			Elastic Limit.	Ultimate.	Final Elongation in 80 Inches.	Final Contraction.
		Inches.				
1	Single Refined	3×1	29,000	52,470	18.0	31.0
2	Double "	3×1	31,000	53,550	16.0	27.7
3	Single "	5×1¼	27,330	50,410	16.6	24.1
4	Double "	5×1¼	27,170	50,920	19.0	25.7
5	Single "	3×1	28,330	48,700	13.1	27.1
6	Double "	3×1	29,170	51,370	22.2	35.6
7	Single "	5×1¼	24,830	49,240	16.0	18.1
8	Double "	5×1¼	27,170	51,010	19.7	29.5

Table V shows the results of some tests in the U. S. Govt. machine during 1881, at Watertown, Mass. Nos. 1 and 2 are means of four tests; the others are means of three. Nos. 1, 2, 3, and 4 are for bars from the Elmira Iron and Steel Rolling Mill Co.; Nos. 5, 6, 7, and 8 are from the Passaic Rolling Mill Co. As a rule the large bars give the least elastic limit and ultimate resistance.

It is also important to observe that the double refined iron, with two exceptions, gives the highest results of all kinds.

It appears from an examination of the tables that although there are irregularities the elastic limit may be taken for all practical purposes as half the ultimate resistance.

The ultimate resistance, it is to be particularly observed, is given in pounds per square inch of *original* sectional area. On account of the reduction of the fractured section the ultimate resistance should be specifically referred either to its own section (to be noticed hereafter) or to the original section. The customary reference is to the latter, though it is frequently interesting to make an accompanying reference to the former.

The committee also made some valuable experimental investigations with the object of ascertaining the influence of the relative dimensions of the test-piece, already remarked upon in connection with Fig. 2. Eighteen specimens were prepared, of which Figs. 3 and 4 represent types.

Fig. 3 represents a specimen whose middle portion is turned down to a uniform diameter. Seventeen of the specimens were of this kind, with lengths of cylindrical portions varying from ½ inch to 10 inches. Fig. 4 represents the eighteenth specimen, with simply a groove in the centre, in which, at *ab*, the fracture took place. In this latter specimen the reduction of area at the section of failure must necessarily be much less than in those like Fig. 3; hence the ultimate resistance will be correspondingly greater.

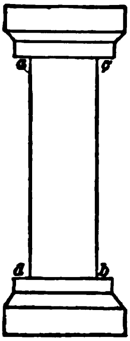


FIG. 3.

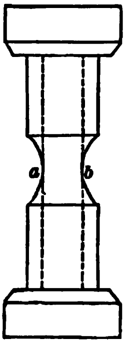


FIG. 4.

Table VI is taken from the Report already cited, and contains the results of the experiments on the eighteen specimens prepared in the manner indicated above.

L = original length in inches;
l = per cent of elongation;
a = per cent of contraction of fractured area;
t_x = stress in pounds per square inch at first stretch;
T = ultimate tensile resistance in pounds per square inch of original section.

TABLE VI.

No.	*L.*	*l.*	*a.*	*t.*	*T.*	Remarks.
1	10	23.1	38.2	29,678	54,888	Slight seam.
2	9½	24.3	36.5	28,011	55,288	
3	9	21.5	31.1	29,345	55,355	
4	8½	22.0	31.2	29,345	55,622	
5	7½	25.0	39.9	30,840	54,890	Slight seam.
6	7	25.8	38.6	30,412	55,488	
7	6½	22.1	40.0	28,562	51,800	Bad seam.
8	6	22.3	34.7	30,600	55,418	
9	5½	25.4	39.3	29,475	55,333	
10	5	21.2	32.2	29,278	55,887	Slight seam.
11	4	25.7	37.4	29,705	55,532	
12	3½	26.7	36.6	31,817	55,482	
13	3	27.0	38.3	31,123	56,190	
14	2	27.0	36.2	33,428	56,428	Seamy.
15	1½	26.0	34.0	42,249	57,096	"
16	1	37.0	34.3	34,288	58,933	
17	½	30.0	37.9	57,565	59,388	Seamy.
18	Groove	....	20.6	45,442	71,300	

The diameters at the section of failure were nearly uniform and originally about 0.97 inch.

The values of l, a, and T are as nearly uniform as could be expected until the length decreases to about 4 diameters (2 inches).

For the grooved specimen t and T are very large and a very small.

Other experiments on a still softer iron were made with the same general results.

"In conclusion," states the committee, "our results lead us to the decision that, in testing iron, no test piece should be less than one half inch in diameter, as inaccuracy is more probable with a small than with a large piece, and the errors are more increased by reduction to the square inch; that the length should not be less than four times the diameter in any case; and that, with soft, ductile metal five or six diameters would be preferable."

It has long been the impression that there exists a considerable difference between the ultimate tensile resistance of the "skin" of a bar of iron and that of the portion of the bar underneath the skin. The U. S. Committees, therefore, broke a number of bars first with the skin on, or "in the rough," and then with the skin turned off. In a large majority of the cases the rough bars gave the highest ultimate resistance per square inch by a small amount, while in a few cases the results were of the opposite character. On the whole, however, "the accumulated evidence indicates that the strength of the skin of the bar is greater in proportion to its area than that of the rest of the bar."

All the tests, of which the results have hitherto been given, were made on round bars or on specimens turned from them. Results of tests on other iron will now be detailed, and it will be convenient to use the following and customary symbols for the various kinds of "shape" irons:

L, for angle irons;
⊥, for tee irons;
⊏, for channel bars;
I, for eye beams;
◘, for rectangular bars or "flats";
O, for rounds;
✚, for star sections;

in short, any shape iron or steel is represented by a skeleton of its section.

Table VII contains the results of tests of a wide range of full-sized eye-bars as ordinarily manufactured for bridge-building purposes. Some were made and tested in 1887.

TABLE VII.

Section. Inches.	Length. Inches.	*E. L.*	*T.*	Cont.	Strain.	Remarks.	
4×1	237	27,500	51,000	0.25	0.143	"Strain" for 17 ft.	(*a*)
4×1	66	25,318	48,070	0.24	0.117	" " 4 ft.	(*a*)
$3 \times \frac{15}{16}$	230	32,590	51,870	0.365	0.216	" " 16 ft.	(*b*)
$3\frac{1}{2} \times 1\frac{1}{4}$	173	30,360	49,040	0.326	0.208	" " 13 ft.	
$4 \times 1\frac{1}{8}$	251	33,000	50,350	0.336	0.208	" " 18 ft.	
$5 \times 1\frac{3}{8}$	251	28,010	47,440	0.17	0.175	" " 18 ft.	
$5 \times 1\frac{1}{4}$	370	26,500	51,200	0.16	0.11	" " 29 ft.	(*a*)
$6 \times 2\frac{1}{16}$	304	26,750	49,000	0.211	0.129	" " 23 ft.	(*c*)
$6 \times 2\frac{1}{16}$	304	26,000	48,000	.....	0.124	" " 23 ft.	(*d*)
$8 \times 1\frac{1}{2}$	300	24,200	41,280	0.097	0.061	" " 23.3 ft.	(*e*)
$8 \times 1\frac{1}{2}$	300	24,030	41,860	0.099	0.056	" " 23.3 ft.	(*f*)

E. L. = elastic limit in pounds per sq. in.

T = ultimate resistance in pounds per sq. in. of original section.

Cont. = ultimate unit contraction or reduction of original area.

Strain = ultimate unit stretch in length given under "Remarks."

All bars except *e* and *f* were rolled and manufactured by the Phœnix Iron Co. The bars *a* were of single-rolled iron, while all the others were of double-rolled material. The bars *e* and *f* were made by the Central Bridge Co. of Buffalo, N. Y., and were tested at the U. S. Arsenal, Watertown, Mass., as sample bars selected from those used in the Niagara Cantilever bridge; and the results are taken from the paper on that structure by Mr. C. C. Schneider, Ch. Engr., presented to the Am. Soc. of C. E., March 4, 1885.

Bars *a* were tested at the works of the Phœnix Iron Co.;

bars *b* at the works of the Keystone Bridge Co., and bars *c* and *d* at the works of the Union Bridge Co., at Athens, Pa.

The bar *d* broke through the eye, for the reason that about $\frac{3}{8}$ inch in thickness of material was planed off the face of the head in order to get it into the testing machine.

The expressions "single rolled" and "double rolled" as applied to the production of wrought iron had considerable significance during the first portion of the period in which wrought iron was used for structural purposes. It means that the iron was twice rolled from the muck bar. The muck bar was the product of rolling the white-hot spongy mass of iron produced by the puddling furnace. These muck bars were frequently from 12 to 14 feet long and about 4 or 5 inches by 1 inch in cross-section. They were cut into lengths of perhaps 4 or 5 feet and piled together so as to make approximately a rectangular volume of dimensions depending upon the bar it was desired to produce. This pile made from muck bars was heated in a suitable furnace to a welding temperature and then passed through rolls until it was sufficiently reduced in cross-section and correspondingly lengthened so as to produce the finished bar. This bar was called "single rolled," i.e., it was once rolled from the muck bar. When it was desired to improve the quality of the iron these single-rolled bars were cut into pieces from 3 to 5 or 6 feet long and piled as were the portions of the original muck bars The size of the pile depended upon the size of bar which it was desired to produce. This pile was again raised to a wielding temperature in a heating furnace and then put through a second series of rolls, thus producing the double-rolled bar; that is, twice rolled from the original muck bar. At first this double rolling sensibly improved the quality of the iron, but as the processes for producing wrought iron were also improved the advantage of double rolling for all structural purposes practically disappeared and specifications calling for it were ultimately omitted.

Considered as a comparison between single- and double-rolled bars, the table possesses much interest. The first three of the bars *b* show unusually high elastic limits, which may have been the result of the approximate methods used in the observation. On the whole, however, the elastic limits of the single-rolled bars are not essentially different from those determined for the double-rolled material.

With the exception of the last two ultimate resistances, which are very low, even for large bars, that column shows much more nearly uniform results for the two grades of bars than found among the elastic limits, and the values for the single rolled metal are fully equal to the best of the double rolled. As was to be expected, the smaller bars gave results appreciably in excess of those belonging to the larger ones.

The percentages of contraction for the single rolled bars are seen to be on the whole somewhat smaller than those belonging to the others, although the advantage is not maintained by the latter throughout the entire table. The last preceding observation holds, but less markedly, in the column of ultimate stretch or ''strains.'' While the decision of such questions should be made only on a far greater number of tests than given in the table, it is proper to say that the latter shows precisely what is found in extended experience, i.e., that double rolled iron, as produced by the most reputable iron companies, possessed only a possible small and unimportant advantage in ductility and uniformity, but with less welding properties. Its cost was from twenty to twenty-five per cent. over that of single rolled iron; which was out of all proportion to the very small advantage gained.

No class of materials used by engineers possesses more widely varying characteristics of a physical nature than plates used in bridge construction. The very wide plates forming the webs of large plate girders gave a high elastic limit, comparatively low ultimate resistance, final stretch and contraction; these were always sheared plates. Narrow plates, either rolled in grooves or universal mill, approximate more nearly in character to bars in all respects of elastic and ultimate resistances and final stretch and elongation.

TABLE VIII.

BRIDGE PLATE SPECIMENS.

Original Plate. Inches.	Test Specimen.		Lbs. per Sq. In. Orig. Sec.		Per Cent. of Final	
	Section. Inches.	Length. Inches.	Elastic Limit.	Ultimate Resistance.	Contraction.	Stretch in 8 Inches.
$8\times\frac{1}{4}$	1×0.30	8	31,900	47,700	22.4	19.8
$8\times\frac{1}{4}$	1×0.3	8	36,000	52,800	23.4	17.5
$10\times\frac{1}{4}$	1×0.26	8	35,110	52,670	20.6	16.3
$10\times\frac{1}{4}$	1×0.26	8	34,600	51,540	21.9	16.8
$10\times\frac{5}{16}$	1×0.32	8	37,000	53,700	27.2	19.1
$10\times\frac{7}{16}$	1×0.45	8	32,810	51,670	30.8	26.3
$10\times\frac{7}{16}$	1×0.45	8	32,960	52,250	28.3	21.8
$10\times\frac{1}{2}$	1×0.50	8	27,160	49,500	25.96	21.0
$10\times\frac{9}{16}$	1×0.57	8	27,400	49,220	29.67	23.0
$12\times\frac{3}{8}$	1×0.40	8	30,230	50,380	20.91	15.0
$12\times\frac{5}{8}$	1×0.63	8	34,700	51,200	22.5	18.2
$14\times\frac{3}{8}$	1×0.39	8	33,850	50,390	28.9	19.8
$14\times\frac{3}{8}$	1×0.39	8	34,870	54,100	44.1	23.0
$20\times\frac{7}{16}$	1×0.45	8	36,860	53,930	18.9	13.3
$20\times\frac{7}{16}$	1×0.45	8	36,580	53,720	25.6	20.8
$24\times\frac{5}{16}$	1×0.31	8	30,290	46,900	18.6	11.0
$24\times\frac{5}{16}$	1×0.31	8	28,100	46,920	18.1	15.0
$27\times\frac{3}{8}$	1×0.39	8	39,470	52,370	28.2	18.0
$30\times\frac{5}{16}$	1×0.32	8	33,330	49,210	15.2	12.5
$30\times\frac{5}{16}$	1×0.31	8	34,290	49,680	15.1	11 5
$69\times\frac{3}{8}$	1×0.40	8	35,800	49,870	25.6	11.9
$72\times\frac{3}{8}$	1×0.39	8	44,380	53,060	14.5	7.0
$72\times\frac{3}{8}$	1×0.35	8	44,030	55,960	23.0	9.8

In consequence of the mode of production of wrought iron the spongy character of the mass of iron with its included slag produced in the puddling furnace caused the finished wrought-iron product, as drawn out between the rolls, to be fibrous, the fibres obviously extending in the direction of rolling. In the use of wrought-iron plates, therefore, it was necessary in order to secure the greatest ultimate resistance to stress them along the fibres and not across the latter. This was frequently if not usually specified, as it was an important feature in the employment of wrought-iron plates. The data given in Table VIII, therefore, belong to the tests of plate specimens which were loaded in the direction of the fibres. In fact, practically all specimens of plates were tested along the fibres, although tests would occasionally be made across the fibres in case it was expected to use the plates in that way, so as to ascertain the diminished ultimate resistance relatively to the results obtained by loading parallel to the fibres. It is not necessary to reproduce tests made of wrought-iron plates across the fibres, but the ultimate resistance found in that way was frequently not more than 90 per cent. of that determined along the fibres and sometimes less. In considering the ultimate resistance of wrought-iron plates, therefore, it is always necessary to specify clearly whether resistance along the fibres or across the fibres is intended. The mode of loading iron plates was practically always in the direction of the fibres, i.e., in the direction of rolling.

Table VIII gives results for bridge plates throughout a great range of width. All the specimens were of the same thickness as the original plates; hence two sides were as they came from the rolls and two were machine finished. Although these results are somewhat irregular, the lowest ultimate resistances and relatively highest elastic limits are found with the greatest widths, with the exception of the 72-inch plates. It is altogether probable that both the high elastic limit and ultimate resistance for those were produced by the addition of steel to the piles from which the plates were rolled. This practice had obtained some footing in order to meet the extreme requirements of some very exacting specifications based upon insufficient knowledge regarding the actual capacities of plate iron in great widths.

TABLE IX.

ANGLE IRON SPECIMENS.

Size of Original Angle. Inches.	Test Specimen.		Lbs. per Sq. In. Orig. Sec.		Per Cent. of Final	
	Section. Inches.	Length. Inches.	Elastic Limit.	Ultimate Resistance.	Contraction.	Stretch in 8 Inches.
Pounds.						
6 ×4 —71	1×0.7	8	26,800	49,420	25.6	24.5
6 ×4 —46	1×0.46	8	28,570	49,460	24.6	20.0
6 ×4 —46	1×0.46	8	28,850	49,340	27.3	21.3
6 ×3½—36	1×0.38	8	29,890	47,090	19.5	11.8
6 ×3½—36	1×0.38	8	30,520	48,420	18.2	13.0
5 ×3 —25	1×0.34	8	29,200	46,900	22.7	14.4
5 ×3 —25	1×0.34	8	29,370	47,770	20.7	15.6
4 ×3 —34	1×0.49	8	28,250	48,860	26.8	22.8
4 ×3 —34	1×0.47	8	29,780	49,360	25.1	17.8
4 ×3 —34	1×0.49	8	34,440	52,900	18.5	18.8
4 ×3 —34	1×0.49	8	31,680	50,620	31.9	21.3
3½×3 —23	1×0.38	8	26,960	45,790	26.8	13.75
3½×3 —23	1×0.38	8	28,600	49,080	28.1	23.0
3 ×3 —17	1×0.30	8	30,660	51,000	34.3	22.3
3 ×3 —21	1×0.38	8	30,260	51,050	30.8	23.1
3 ×3 —21	1×0.37	8	29,030	48,920	29.0	26.0
3 ×3 —28	1×0.46	8	31,080	51,300	35.4	27.3
3 ×3 —18	1×0.34	8	32,220	50,300	30.1	20.8
3 ×3 —18	1×0.33	8	33,230	52,430	36.3	22.8
3 ×2½—17	1×0.33	8	33,030	51,350	27.6	23.0
3 ×2½—17	1×0.33	8	32,320	51,360	34.7	23.3
3 ×2½—18	1×0.35	8	29,760	50,280	32.1	21.8
3 ×2½—18	1×0.35	8	30,430	50,140	29.5	19.3

Table IX gives the results of tests of specimens cut from all sizes of angles used in ordinary bridge work. These angles were all produced by the Phœnix Iron Company, and the tests were made at the works of that company in 1887. The results are most excellent, as well as being typical for the shapes tested. Although the results at one or two points are a little irregular, on the whole the elastic and ultimate resistances, as well as the final contraction and elongation, increase with considerable uniformity from the heaviest sections to the lightest, showing clearly the improved qualities in the smaller angle bars. The table demonstrates in a very marked manner the varying characteristics which always accompany varying dimensions of bars of the same kind, even when produced of uniform material in the original piles.

Effect of Annealing.

The operation of annealing is performed by first heating wrought iron or other metal in such a way as not to change its chemical condition by any external influences, or, in other words, by heating in a non-oxidizing furnace or other similar appliance to a cherry-red color and then gradually cooling it, usually in air and sometimes covered with ashes or sand. The purpose of the operation is to soften the material just enough to permit the molecules to flow a little and so eliminate internal stresses in the material and without sensible distortion or change of shape or dimensions of the piece. The effect of annealing is slowly to soften the wrought iron, thus increasing its ductility but decreasing its ultimate resistance. In the case of wrought-iron bars used for structural purposes annealing would not usually decrease the ultimate resistance more than perhaps 5 to 8 per cent., but in the case of wrought-iron wire or other cold-drawn metal in which the ultimate resistance would be abnormally high it might be reduced more than one third and sometimes one half its original value.

A committee of the Franklin Institute in Philadelphia appointed to investigate certain physical qualities of metals in 1837 found that the ultimate tensile resistance of a wrought-iron wire .156 inch in diameter was 89,162 pounds per square inch in its natural state, but that it fell to 48,144 pounds per square inch and 50,889 pounds per square inch after the process of annealing. These results have frequently been confirmed as to their general character by many investigations since that early date.

In practically all cases where the metal has been made hard and stiff by cold drawing or other treatment it will be softened and made ductile by the process of annealing, but its ultimate resistance will also be reduced. The effects of annealing are much more marked as a whole in steel than in wrought iron. The annealing temperature is usually 1000 to 1200 degrees Fahr.

The Effect of Hardening on the Tensile Resistance of Iron and Steel.

The hardening of iron and steel is a process having in general the opposite results to those produced by annealing; it usually raises the ultimate resistance and the elastic limit, but decreases, sometimes greatly, the ductility of the metal. Hardening is produced by raising to a high temperature as in annealing and then suddenly decreasing the temperature by quenching in water, oil, or other liquid or semi-liquid. If the quenching is done in water the general result is considerable hardening with the corresponding increase in ultimate resistance, frequently of erratic value in consequence of the production of irregular internal stresses. These results, like those of annealing, are much more marked in steel than in wrought iron.

If the quenching of steel at a high temperature is arrested at some point intermediate between that of ordinary air and the temperature to which the metal was raised, various degrees of tempering are reached. There are many degrees or grades of temper depending upon the manipulation of the operation of quenching and its arrest at various temperatures. These degrees of temper are indicated by the colors of the oxidized coatings formed on the metal. In some of the useful grades of tempering the metal is much more hardened and toughened than in others. If the metal is steel and quenched in oil, a very useful degree of hardening is produced accompanied by remarkable toughness, so that hardening in oil is a process of much value for certain purposes. A great variety of degrees of hardening and tempering may be produced by the choice of liquid or semi-liquid and the character of the manipulation accompanying the quenching or partial quenching. As a rule these operations all increase ultimate resistance and elastic limit, but decrease the ductility.

Variation of Ultimate Resistance with Increase of Temperature.

The decrease of ultimate resistance with the increase of temperature is a matter of considerable importance in connection with the use of structural steel or other materials in some cases, as in the fire-boxes of boilers or portions of buildings where high temperatures are sometimes liable to exist. Many investigations have been made from time to time, therefore, to ascertain if possible the relation between high temperatures and ultimate resistances of wrought iron and steel; the former metal only will be considered in this section. As wrought iron has been completely displaced both for boiler purposes and structural members in building frames, it will only be necessary to state general results in connection with this metal.

In some of the older investigations the ultimate resistance of wrought iron was found to increase with the temperature up to about 200° Fahr., while in other cases the ultimate resistance remained essentially unchanged until the temperature of over 570° Fahr. was reached. This was the case in the investigations of the old committee of the Franklin Institute, to which reference has already been made. Above a temperature of 570° Fahr. that committee found the decrease in ultimate resistance below the greatest to vary approximately as the 2.6 power of (*Temp.* $-80°$).

TABLE X.

Temperature.		Fibrous Iron.	Fine-grained Iron.	Bessemer Steel.
Cent.	Fahr.			
0°	32°	100	100	100
100	212	100	100	100
200	392	95	100	100
300	572	90	97	94
500	932	38	44	34
700	1292	16	23	18
900	1652	6	12	9
1000	1832	4	7	7

In the London *Engineering* of July 30, 1880, is given a synopsis of some German experiments by Herr Kollmann, which is reproduced in the table. The resistance of the materials at zero degrees Cent. or 32 degrees Fahr. is taken as 100 and that at other temperatures is the proper proportional part of that number. It will be noticed that these German tests show a much earlier decrease of resistance than some others.

In his "Useful Information for Engineers," Second Series, Sir William Fairbairn gives the results of numerous tests of specimens of plate and rivet iron at different temperatures, showing that there was no sensible decrease in ultimate resistance up to a temperature of about 400° Fahr., but that a decrease of about 20 per cent. in ultimate resistance takes place at a "scarcely-red" heat and a decrease of about one third the original ultimate at a "dull-red" heat.

It is quite probable if not certain that the effect of temperature will depend a little upon the chemical constitution of the metal, so that the results of these investigations naturally show some irregularities. It may be accepted conclusively, however, that there is no essential decrease in the ultimate tensile resistance of wrought iron below about 500° Fahr., but that at about 1000° Fahr. the metal may lose more than one half of its ultimate resistance.

Effect of Low Temperature.

When wrought iron was used largely and for some time entirely for railroad rails, the effect of low temperature upon its ultimate resistance or upon its capacity to resist shock was a matter of much importance which the question has now lost, since wrought iron has been displaced by steel for railroad rails for a considerable period of years. At the same time there are some features of this question which are common both to wrought iron and to steel, and for that reason it may be well to set forth some of the earlier conclusions as to the effect of low temperature on wrought iron. This matter derived its importance from the breakage of wrought-iron railroad rails in the winters of comparatively high latitudes. It was, therefore, studied in such localities as Norway and Sweden and by the State Railroad Commission of Massachusetts. An early Swedish authority of prominence, Knut Styffe ("Iron and Steel"), reached the following conclusions as the result of his experiments:

1. "That the absolute strength of iron and steel is not diminished by cold, but that even at the lowest

temperature which ever occurs in Sweden it is at least as great as at the ordinary temperature (about 60° Fahr.). . . .

3. "That neither in steel nor in iron is the extensibility less in severe cold than at the ordinary temperature. . . .
4. "That the limit of elasticity in both steel and iron lies higher in severe cold. . . ."

He concluded from his experiments that the common impression of increased weakness and brittleness with a low degree of temperature is entirely erroneous. His tests, however, were wholly with tension gradually applied, and could support no conclusion in regard to other conditions.

Christer P. Sandberg, also a prominent authority on iron and steel, who translated Styffe's works, reached directly opposite conclusions, as follows:

1. "That for such iron as is usually employed for rails in the three principal rail-making countries (Wales, France, and Belgium), the breaking strain, as tested by sudden blows or shocks, is considerably influenced by cold. such iron exhibiting at 10° Fahr. only one third to one fourth of the strength which it possesses at 84° Fahr.
2. "That the ductility and flexibility of such iron is also much affected by cold, rails broken at 10° Fahr. showing on an average a permanent deflection of less than one inch, whilst the other halves of the same rails, broken at 84° Fahr., showed a set of more than four inches before fracture.
3. "That at summer heat the strength of the Aberdare rails was 20 per cent. greater than that of the Creusot rails; but that in winter the latter were 30 per cent. stronger than the former."

All these experiments were made previous to 1869, and with iron rails.

The State Railroad Commission of Massachusetts reported in 1874 that in their opinion neither iron nor steel attained any greater degree of brittleness or became any more ''unreliable for mechanical purposes'' at low temperatures than at ordinary. They did not observe as a ''rule that the most breakages'' occurred ''on the coldest days.'' They further state that ''the introduction of steel in the place of iron rails has caused an almost complete cessation of the breaking of rails.''

It would appear from the preceding investigations and observations that the subject was hopelessly involved in confusion, but this is largely if not entirely due to the fact that the effect of certain chemical impurities in both iron and steel were not recognized. It is practically certain that where the older investigators found no prejudicial results at low temperatures, the percentage of phosphorus was low, and that it was high where low temperature was found to affect the metal prejudicially. It was a well-known fact established by many experiments that the brittleness of wrought iron which carried a comparatively high percentage of phosphorus was much increased by chilling, as in a freezing mixture, and it is also practically certain that phosphorus has the same effect upon steel and cast iron. The decreased breakage of steel rails in cold weather is doubtless largely due to the improved processes under which they are manufactured with comparatively low percentages of phosphorus.

Modern experiences all go to show that at such low temperatures as those found even in the highest latitudes where railroads are built neither iron nor steel will suffer sensibly decreased ultimate resistance, ductility, or capacity to resist shock with the comparatively low percentages of prejudicial elements now found in those metals.

Iron Wire.

Iron wire, like steel wire, is a product manufactured by the process of cold drawing. The small rods from which the wire is made are drawn through steel dies at ordinary temperatures, the hole in each die being a little smaller than the preceding. This operation, like all other similar processes of cold working, temporarily raises the temperature of the metal to some extent, but not enough to have any sensible effect upon the physical qualities of the finished product. It is evident from what has been said that iron wire should have an ultimate resistance much higher than that of wrought iron in bars.

Mr. John A. Roebling, who built the old railroad suspension bridge across the Niagara gorge in 1852–53, used English iron wire for the cables. This wire was about .145 inch in diameter and gave an ultimate tensile resistance of about 98,500 pounds.

In the United States Report, "Tests of Metals and Other Materials," for 1897 will be found tests of wrought-iron round rods .8 inch in diameter giving an ultimate resistance of 50,000 pounds per square inch, and the tests of iron wire .01 inch in diameter having an ultimate resistance of 110,000 pounds per square inch. In the same report will be found the tests of galvanized iron wire .2 inch in diameter giving ultimate resistances of 65,280, 65,060, and 64,710 pounds per square inch of original section. As a rule wrought-iron wire having diameters lying between about .2 and .1 inch will have ultimate resistances running from about 70,000 to 90,000 pounds per square inch. As is usual in such material, while the ultimate resistance is high the ductility will in general be low and shown by a comparatively small amount of final stretch.

TABLE XI.

Original Diameter in Inches.	Ultimate Tensile Resistance in Pounds per Square Inch of		Contraction of Original Area of Section.
	Original Area.	Fractured Area.	
0.122	94,871	179,032	0.47
0.123	87,395	162,500	0.462
0.124	89,256	145,946	0.388
0.125	88,618	137,974	0.358
0.122	92,308	168,750	0.453
0.124	91,735	156,338	0.413
0.124	90,082	170,313	0.471
0.122	92,308	168,750	0.453
0.124	91,735	173,437	0.471
0.124	86,776	164,063	0.471
0.125	87,805	156,522	0.439
0.124	86,776	152,174	0.43

Table XI is a condensed form of one given in the "Transactions of the Am. Soc. Civ. Engrs.," Vol. III, p. 212, and contains an account of the tests made by Prof. R. H. Thurston on some wires that had been in use 32 years in the cables of the Fairmount Suspension Bridge at Philadelphia. It is both interesting and important to observe that the long service cannot have appreciably injured either the ductility or ultimate resistance of the wire.

Table XII contains the records of tests on other wire at the same time (1875) by Prof. Thurston. The small

TABLE XII.

Original Diameter.	Diameter After Fracture.	Ultimate Resistance in Pounds per Square Inch of Original Area.
0.134	0.133	92,890
0.1205	0.1185	84,442
0.08	0.0795	94,299
0.071	0.068	90,384
0.0535	0.0532	105,871
0.029	0.029	113,546

reduction of diameter at fracture shows the iron to have been not very ductile. It will also be noticed that the smaller diameters give much the highest resistances.

Reviewing the values given, it appears

1. *That wire is the strongest form in which iron can be used to resist tensile stress.*
2. *That, as a rule, the ultimate tensile resistance increases as the diameter of the wire decreases.*

Tensile Resistance of Shape Iron.

The phenomena exhibited in the breaking by tensile stress of wrought-iron shapes naturally depended much on the character of the piles from which they were rolled. The webs of channels and I beams were sometimes rolled from old rails and other scrap of less excellent quality than the metal of which the flanges were made. In such instances the metal of the flanges would evidently give higher ultimate resistance than that of the webs. As a rule, however, when the production of wrought-iron shapes was at its best the ultimate tensile resistance of such shapes as channels, I beams, and angles was but little if any less than that of square or flat and round bars, and it could usually be taken at 44,000 to 48,000 pounds per square inch, while that of bars ranged from 45,000 to 50,000 pounds per square inch, the ductility of the latter pieces being a little greater than that of the former.

Fracture of Wrought Iron.

The characteristic fracture of wrought iron broken in tension, either directly or transversely, is rather coarsely fibrous, not infrequently exhibiting a few bright granular spots which, in rare cases, may possibly be crystalline.

This characteristic (fibrous) fracture is always produced by the *steady* application of an external force, under the influence of which the piece is draw out in jagged points at the place of failure.

The best of fibrous wrought iron, however, will exhibit a granular fracture if broken suddenly. In making tests, therefore, it is of the greatest importance to observe and direct the mode of application of the external forces producing fracture.

When some grades of iron in bars are broken transversely by shocks (such as are produced by falling weights), a phenomenon known as "barking" is produced. A skin of metal from a sixteenth to an eighth of an inch in thickness, on the tension side of the bent piece, tears apart and separates from the core of the bar. At the place of fracture and on each side of it this skin or "bark" remains essentially straight. This kind of fracture shows remarkably well the fibrous character of wrought iron; it is simply the separation of the fibres near the outside of the bar from those within.

Crystallization of Wrought Iron.

The subject of crystallization of wrought iron is one about which there is much dispute. In "Strength of Wrought Iron and Chain Cables," by Beardslee, as abridged by Kent, p. 36, the following is given as the opinion or view of the United States Testing Commission: "The question as to whether crystallization can be produced in iron by stress, or by repetition of stress with alternations of rest, or by vibration, has been much discussed, and very opposite views are entertained by experts.

"We have met with but one unmistakable instance of crystallization which was probably produced by alternations of severe stress, sudden strains, recoils, and rest."

The committee then state the case of a connecting-rod, carefully made of the best quality of wrought-iron scrap, which had been used in a testing machine for forty years, in the Navy Yard at Washington. It was five inches in diameter, but one day while in use it suddenly broke under a stress (total) of less than 200,000 pounds. "The surface of the fractured ends showed well-defined crystallization, the facets being large and bright as mica."

The data at hand at present are not sufficient for a decision of the question, but it may be confidently stated that in many cases granulation has been mistaken for crystallization.

There seems to be little doubt that in some, or possibly many, cases the metal in wrought-iron railroad axles has been crystallized in use by rapid rotation under stresses, probably far above the elastic limit, induced by shocks resulting from unevenness of tracks or other causes. In such cases the metal would be subject to comparatively rapid alternations of stresses of high intensity. It is probable even in such cases that granulation has frequently been mistaken for crystallization.

Elevation of Ultimate Resistance and Elastic Limit.

It was first observed by Prof. R. H. Thurston and Commander L. A. Beardslee, U.S.N., independently, in this country, that if wrought iron be subjected to a stress beyond its elastic limit, but not beyond its ultimate resistance, and then allowed to "rest" for a definite interval of time, *a considerable increase of elastic limit and ultimate resistance may be experienced.* In other words, the application of stress and subsequent "rest" increases the resistance of wrought iron.

This "rest" may be an entire release from stress or a simple holding the test piece at a given intensity.

Prof. Thurston's investigations were on torsion, while those of the United States Commission were on tension, and will be given here.

The Commission prepared twelve specimens and subjected them to an intensity of stress equal to the ultimate resistance of the material without breaking the specimens. These were then allowed to rest, entirely free from stress, from twenty-four to thirty hours, after which period they were again stressed until broken.

The gain in ultimate resistance by the rest was found to vary from 4.4 to 17 per cent.

These tests, remark the committee, seem to indicate that the tough fibrous irons gained the most, while those which broke with a steel-like fracture gained the least.

Before the rest, the stress which produced the first permanent elongation was about 65 per cent. of the ultimate resistance, but after the rest the two were nearly identical.

The committee then took forty-two other specimens and subjected them to precisely the same operations, except that the rest periods varied from one minute to six months.

The gains were as follows:

In less than 1 hour	1.1 per cent., mean of 5 tests.
In less than 8 and over 1 hour	3.8 per cent., mean of 8 tests.
In 3 days	16.2 per cent., mean of 10 tests.
In 8 days	17.8 per cent., mean of 2 tests
Between 8 and 43 days	15.3 per cent., mean of 5 tests.
In 6 months	17.9 per cent., mean of 12 tests.

After seven other experiments involving a rest of twenty-four hours, with an average gain of 15.4 per cent., the committee concluded "that at the end of one day the result is, with very ductile irons, practically accomplished."

The manifestation of this phenomenon in different grades of iron was then investigated.

"Thirteen pieces were prepared, five of which were of soft charcoal bloom boiler iron, five of coarse contract chain iron, and three of a fine-grained bar of . . . very pure iron with high tenacity."

After testing these specimens subsequent to an eighteen hours' rest, the committee state (Kent's abridgment):

"These experiments confirmed the opinion already formed, and indicate that a bridge, cable, or other structure composed of iron of either of the latter two varieties will receive comparatively slight benefit from the operation of this law; while ductile fibrous metal . . . gains . . . to a great extent by the effect of strains already withstood." The gain in these specimens varied from about 3 per cent. (for the coarse iron) to about 18 per cent. (for the soft iron).

Again, two sets of specimens were prepared: one from the two portions of fractured bars after having been pulled asunder, the other from the bars in their normal condition. After a rest of several days the first set showed a gain over the second in ultimate resistance varying from about 8 to 39 per cent., the higher values belonging to the more ductile irons.

Bauschinger's Experiments on the Change of Elastic Limit and Coefficient of Elasticity.

In "Der Civilingenieur," Heft 5, for 1881, are contained the results of the experiments of Prof. Bauschinger, of Mūnich. The observations in these experiments were made by the aid of a piece of apparatus which gave the elongations (all experiments were tensile) in ten-millionths of a metre, or approximately in $\frac{1}{250000}$ of an inch. An extraordinarily high degree of accuracy was therefore attained.

Prof. Bauschinger's elastic limit was strictly a proportionality limit between stresses and strains. He also observed what may be called the "stretch-limit" (Ger. Streckgrenze), at which point the stretching or elongation suddenly increases and continues to increase for more than a minute after the application of the stress. In ordinary experimenting this point has probably frequently been considered the elastic limit.

The test pieces were subjected to loads which gradually increased from zero by an increment a little less than 3000 pounds per square inch, each load having been allowed to act one minute before adding the succeeding increment. At intervals of the loading, separated by about 11,500 or 12,000 pounds per square inch, each piece was entirely unloaded and allowed to remain so for fifteen or twenty minutes. After the "stretch-limit" was found the piece was subjected to a final load somewhat greater than the "stretch-limit" and then entirely unloaded.

In some cases the piece was immediately put through the same process of testing either once or a number of times, and the results of such tests will be found in the columns of the following tables, indicated by the contraction "*Im'y.*"

In the remaining cases intervals of time, shown at the tops of the columns, were allowed to elapse between any one test and the succeeding one.

The tables, Nos. 1 to 7, inclusive, give the results of the experiments on seven specimens of a grade of iron called "Schweisseisen" (weld iron). These specimens were a very little less than 1 inch (25 millimetres) in diameter. Nos. 1 and 2 were about 32 inches long and the others about 16 inches long.

Tables Nos. 8 to 13, inclusive, give the results obtained with Krupp's "Flusseisen." These specimens were about 1 inch in diameter and 16 inches long.

The tables have been condensed from those given by Bauschinger and reduced to English measures.

The following is the notation:

E. L. =elastic limit in pounds per square inch.
S.-L. =stretch limit in pounds per square inch.
F. L. =final load in pounds per square inch.
E. =coefficient of elasticity in pounds per sq. in.

WELD IRON.

No. 1.	In Original Condition.	Im'y.	Im'y.	Im'y.
E. L.	20,110	14,370	14,900	15,500
S.-L.	27,300	31,600	41,700	49,500
F. L.	31,600	40,200	47,700	——
E.	39,293,000	27,928,000	27,672,000	27,544,000

WELD IRON.

No. 2.	In Original Condition.	After 19 Hrs.	After 27 Hrs.	After 24 Hrs.
E. L.	20,110	28,970	35,500	39,100
S.-L.	28,700	34,750	44,300	45,100
F. L.	31,600	40,540	47,300	——
E.	29,037,000	28,923,000	28,198,000	28,241,000

WELD IRON.

No. 3.	In Original Condition.	After 51 Hrs.	After 41 Hrs.	After 45 Hrs.
E. L.	23,164	28,440	39,080	45,580
S.-L.	29,000	34,750	45,090	——
F. L.	31,900	40,540	48,100	——
E.	29,208,000	28,397,000	28,483,000	28,170,000

WELD IRON.

No. 4.	In Original Condition.	After 80 Hrs.	After 68 Hrs.	After 64 Hrs.
E. L.	22,890	31,900	35,340	43,800
S.-L.	30,050	34,750	44,170	——
F. L.	31,470	40,540	47,110	——
E.	29,293,000	28,810,000	28,227,000	28,696,000

WELD IRON.

No. 5.	In Original Condition.	Im'y.	After 63 Hrs.	Im'y.
E. L.	21,070	14,720	42,090	15,260
S.-L.	30,670	35,320	48,110	51,860
F. L.	34,611	42,700	51,110	——
E.	29,293,000	28,312,000	28,056,000	26,705,000

WELD IRON.

No. 6.	In Original Condition.	After 48.5 Hrs.	After 44.5 Hrs.	After 49 Hrs.
E. L.	27,730	26,720	33,350	24,940
S.-L.	32,120	37,110	45,480	——
F. L.	35,040	43,040	51,550	——
E.	29,720,000	28,639,000	28,483,000	28,881,000

WELD IRON.

No. 7.	In Original Condition.	After 47 Hrs.	After 50.5 Hrs.	After 42.5 Hrs.
E. L.	20,110	26,720	27,170	18,540
S.-L.	30,160	38,590	45,290	——
F. L.	34,470	43,040	51,320	——
E.	28,668,000	28,611,000	28,568,000	29,592,000

MELTED WROUGHT IRON.

No. 8.	In Original Condition.	Im'y.	Im'y.	Im'y.
E. L.	35,340	——	8,990	9,230
S.-L.	36,750	46,910	53,890	58,490
F. L.	45,230	52,770	59,880	——
E.	31,256,000	——	31,483,000	30,488,000

MELTED WROUGHT IRON.

No. 9.	In Original Condition.	Im'y.	Im'y.	Im'y.
E. L.	37,870	5,770	14,720	15,160
S.-L.	42,080	46,140	53,000	60,630
F. L.	44,880	51,920	58,880	——
E.	32,379,000	31,796,000	29,947,000	28,213,000

MELTED WROUGHT IRON.

No. 10.	In Original Condition.	After 3 Hrs.	After 15 Hrs.	After 7 Hrs.
E. L.	33,790	11,490	17,730	15,260
S.-L.	36,600	43,090	53,210	61,020
F. L.	42,230	51,704	59,130	——
E.	31,881,000	31,953,000	31,895,000	32,393,000

MELTED WROUGHT IRON.

No. 11.	In Original Condition.	After 2.5 Hrs.	After 15.5 Hrs.	After 5.5 Hrs.
E. L.	33,930	11,590	11,774	12,070
S.-L.	39,570	43,442	53,000	60,380
F. L.	42,404	52,130	58,880	——
E.	32,536,000	32,237,000	32,222,000	31,085,000

MELTED WROUGHT IRON.

No. 12.	In Original Condition.	After 51 Hrs.	After 47 Hrs.	After 46 Hrs.
E. L.	36,900	43,800	39,820	40,950
S.-L.	39,731	49,640	55,940	60,630
F. L.	42,630	52,560	58,880	——
E.	32,479,000	31,754,000	31,696,000	31,568,000

MELTED WROUGHT IRON.

No. 13.	In Original Condition.	After 43.5 Hrs.	After 54 Hrs.	After 44.5 Hrs.	After 45.5 Hrs.	After 10 Days.
E. L.	——	35,340	38,930	41,740	42,720	61,560
S.-L.	——	36,745	47,600	56,640	——	——
F. L.	——	42,400	51,920	59,630	——	——
E.	31,853,000	32,165,000	31,298,000	31,454,000	31,440,000	32,364,000

During the progress of the various tests, the bars Nos. 6, 7, 9, 11, and 12 were subjected to shocks in addition to the static tests. These shocks were produced by striking the test piece on its end by a hammer. It does not appear that these blows of the hammer perceptibly influenced the results.

The ultimate resistance of the weld iron was found to vary from 55,300 to 58,870 pounds per square inch. That of the melted wrought iron was about 65,000 pounds per square inch.

Although there are some irregularities, the following general conclusions may be drawn from the tables:

By "immediate" testing the elastic limit of weld iron is very much decreased.

With a rest (entirely free from load) between the tests, the elastic limit of weld iron is very much increased.

The greatest proportional gain, except in the case of previous immediate testing, seems to be acquired after a rest no greater than twenty hours.

Bar No. 6 is seen to give anomalous results.

In all cases of the weld iron the stretch limit is considerably raised by repeated testing.

In no case is the coefficient of elasticity, after once testing, equal to its original value; as a rule, a steady decrease is seen to take place by repeated testing, but there are some exceptions.

The elastic limit of "Flusseisen," after repeated testing, is found to be much less than its original value until the length of rest becomes about fifty hours.

The stretch-limit of the same metal is invariably raised by repeated testing, either with or without "rests."

In nearly all the cases of Nos. 8 to 13 the coefficient of elasticity is found to be slightly decreased by repeated testing.

For a very clear and detailed account of these experiments reference must be made to the "Civilingenieur."

Resistance of Iron to Suddenly Applied Stress.

The effect of suddenly applied tensile stress to a test piece of wrought iron or other ductile material has been more or less obscured by the attendant conditions whose influence on the results are not easily recognizable. It is extremely difficult to make a sudden application of loading without comparatively rapid movement of heavy parts of the testing machine whose accumulated energy is usually expended in stretching the test piece. The true influence of sudden loading can only be observed where such accumulated energy is absent, and that is rarely if ever the case. Some of the early investigators, like Kirkaldy ("Experiments on Wrought Iron and Steel"), found that with suddenly applied loading the ultimate resistance was apparently considerably decreased, as well as the ductility of the piece tested. Kirkaldy's results show an apparent decrease of ultimate resistance of about 18 per cent. as an average and nearly 20 per cent. decrease in the final stretch. As a matter of fact, the accumulated energy in the weights on the scale beam of his machine probably fully compensated for the less amount of static load when suddenly applied.

The movements of the molecules exhibited in the final contraction and final stretch of a test specimen undoubtedly require time, and when a sufficient time is not afforded, doubtless there will be less deformation, but it may be and is a serious question whether in the case of wrought iron and steel that effect becomes sensible within the rates of application of loading ordinarily available. These effects will vary with different classes of material. The investigations of Bauschinger, Fischer, Martens, and others show that they may have considerable value in connection with such metals as zinc and tin. On the other hand, it is probably safe to state that there are no investigations whose results are available at the present time which are sufficient to prove that sudden applications of test loads to iron and steel will sensibly affect the ultimate resistance, although it is possible that the final stretch and final contraction may be somewhat affected.

These observations must not be taken to mean that the conditions under which tests are made need not be carefully prescribed and controlled if the results are to be either intelligently interpreted or compared. In order to attain either of those ends it is necessary that the conditions of testing be completely controlled or made as nearly identical as possible.

In the consideration of such matters as those of suddenly applied loading or making tests by shock or impact, it is imperative that precautions be taken to apply the loading in such a manner as to make the distribution of stress uniform throughout the section of the test piece. The contradictory or inconsistent results which have in many cases been found in experimental work connected with suddenly applied loads are undoubtedly largely due to the confusing attendant conditions of the tests which have been either disregarded or unobserved.

Reduction of Resistance Between the Ultimate and Breaking Point.

It has already been observed that the ultimate tensile resistance of wrought iron is the *greatest* tensile resistance which it offers to being pulled asunder, and that a test specimen finally parts at much less than the ultimate resistance. This is due to the ductility of the iron, which allows it to "pull out" or stretch, thus decreasing the cross-section as well as the actual resisting capacity of the metal.

The ultimate resistance, therefore, is not exerted on the final section of fracture, but on a section somewhat greater; referring it (the ultimate resistance) to the section of fracture, then, may mean little or nothing.

The United States Commission made six tests, for the purpose of determining this reduction, on some specimens which had previously been stressed with a subsequent rest. The highest, lowest, and mean losses were as follows:

Highest	14.5 per cent.
Mean	13.8 per cent.
Lowest	12.9 per cent.

It was observed from a number of specimens, by the same Commission, that the reduction of area at the instant of ultimate resistance (or greatest resistance) was about one half, and the elongation or strain a little over three quarters, of the corresponding quantities at the instant of fracture, supposing failure to be produced by a steady strain.

Some further observations seemed to show that if failure were produced by shock, the final contraction would be nearly the same as at the instant of greatest resistance in the case of a steady failure.

Kirkaldy's Conclusions.

The following conclusions were deduced by Mr. Kirkaldy from the results of his experiments. As will be seen, they belong both to wrought iron and steel in tension, and are taken from his "Experiments on Wrought Iron and Steel," 1861.

1. The breaking strain does *not* indicate the quality, as hitherto assumed.
2. A *high* breaking strain may be due to the iron being of superior quality, dense, fine, and moderately soft, or simply to its being very hard and unyielding.
3. A *low* breaking strain may be due to looseness and coarseness in the texture, or to extreme softness, although very close and fine in quality.
4. The contraction of area at fracture, previously overlooked, forms an essential element in estimating the quality of specimens.
5. The respective merits of various specimens can be correctly ascertained by comparing the breaking strain *jointly* with the contraction of area.
6. Inferior qualities show a much greater variation in the breaking strain than superior.
7. Greater differences exist between small and large bars in coarse than in fine varieties.
8. The prevailing opinion of a rough bar being stronger than a turned one is erroneous.
9. Rolled bars are slightly hardened by being forged down.
10. The breaking strain and contraction of area of iron plates are greater in the direction in which they are rolled than in a transverse direction.
11. A very slight difference exists between specimens from the centre and specimens from the outside of crank shafts.
12. The breaking strain and contraction of area are greater in those specimens cut lengthways out of crank shafts than in those cut crossways.

13. The breaking strain of steel, when taken alone, gives no clue to the real qualities of various kinds of that metal.

14. The contraction of area at fracture of specimens of steel must be ascertained as well as in those of iron.

15. The breaking strain, *jointly* with the contraction of area, affords the means of comparing the peculiarities in various lots of specimens.

16. Some descriptions of steel are found to be very hard, and, consequently, suitable for some purposes; whilst others are extremely soft and equally suitable for other uses.

17. The breaking strain and contraction of area of *puddled steel* plates, as in iron plates, are greater in the direction in which they are rolled; whereas in *cast* steel they are less.

18. Iron, when fractured suddenly, presents invariably a crystalline appearance; when fractured slowly, its appearance is invariably fibrous.

19. The appearance may be changed from fibrous to crystalline by merely altering the shape of specimen so as to render it more liable to snap.

20. The appearance may be changed by varying the treatment so as to render the iron harder and more liable to snap.

21. The appearance may be changed by applying the strain so suddenly as to render the specimen more liable to snap from having less time to stretch.

22. Iron is less liable to snap the more it is worked and rolled.

23. The "skin" or outer part of the iron is somewhat harder than the inner part, as shown by appearance of fracture in rough and turned bars.

24. The mixed character of the scrap iron used in large forgings is proved by the singularly varied appearance of the fractures of specimens cut out of crank shafts.

25. The texture of various kinds of wrought iron is beautifully developed by immersion in dilute hydrochloric acid, which, acting on the surrounding impurities, exposes the metallic portion alone for examination.

26. In the fibrous fractures the threads are drawn out and are viewed externally, whilst in the crystalline fractures the threads are snapped across in clusters and are viewed internally or sectionally. In the latter cases the fracture of the specimen is always at right angles to the length; in the former it is more or less irregular.

27. Steel invariably presents, when fractured slowly, a silky fibrous appearance; when fractured suddenly the appearance is invariably granular, in which case also the fracture is always at right angles to the length; when the fracture is fibrous the angle diverges always more or less from 90°.

28. The granular appearance presented by steel suddenly fractured is nearly free of lustre and unlike the brilliant crystalline appearance of iron suddenly fractured; the two combined in the same specimen are shown in iron bolts partly converted into steel.

29. Steel which previously broke with a silky fibrous appearance is changed into granular by being hardened.

30. The little additional time required in testing those specimens whose rate of elongation was noted had no injurious effect in lessening the amount of breaking strain, as imagined by some.

31. The rate of elongation varies not only extremely in different qualities but also to a considerable extent in specimens of the same brand.

32. The specimens were generally found to stretch equally throughout their length until close upon rupture, when they more or less suddenly drew out, usually at one part only, sometimes at two, and, in a few exceptional cases, at three different places.

33. The ratio of ultimate elongation may be greater in short than in long bars in some descriptions of iron, whilst in others the ratio is not affected by difference in the length.

34. The lateral dimensions of specimens form an important element in comparing either the rate of or the ultimate elongation, a circumstance which has been hitherto overlooked.

35. Steel is reduced in strength by being hardened in water, while the strength is vastly increased by being hardened in oil.

36. The higher steel is heated (without, of course, running the risk of being burned) the greater is the increase of strength by being plunged into oil.

37. In a highly converted or hard steel the increase in strength and in hardness is greater than in a less converted or soft steel.

38. Heated steel by being plunged into oil instead of water is not only considerably *hardened* but *toughcned* by the treatment.

39. Steel plates hardened in oil and joined together with rivets are fully equal in strength to an unjointed soft plate, or the loss of strength by riveting is more than counterbalanced by the increase in strength by hardening in oil.

40. Steel rivets fully larger in diameter than those used in riveting iron plates of the same thickness being found to be greatly too small for riveting steel plates, the probability is suggested that the proper proportion for iron rivets is not, as generally assumed, a diameter equal to the thickness of the two plates to be joined.

41. The shearing strain of steel rivets is found to be about a fourth less than the tensile strain.

42. Iron bolts, case-hardened, bore a less breaking strain than when wholly iron, owing to the superior tenacity of the small proportion of steel being more than counterbalanced by the greater ductility of the remaining portion of iron.

43. Iron highly heated and suddenly cooled in water is hardened, and the breaking strain, when gradually applied, increased, but at the same time it is rendered more liable to snap.

44. Iron, like steel, is softened, and the breaking strain reduced, by being heated and allowed to cool slowly

45. Iron subject to the cold-rolling process has its breaking strain greatly

increased by being made extremely hard, and not by being "consolidated" as previously supposed.

46. Specimens cut out of crank-shafts are improved by additional hammering.

47. The galvanizing or tinning of iron plates produces no sensible effects on plates of the thickness experimented on. The result, however, may be different should the plates be extremely thin.

48. The breaking strain is materially affected by the shape of the specimen. Thus the amount borne was much less when the diameter was uniform for some inches of the length than when confined to a small portion—a peculiarity previously unascertained and not even suspected.

49. It is necessary to know correctly the exact conditions under which any tests are made before we can equitably compare results obtained from different quarters.

50. The startling discrepancy between experiments made at the Royal Arsenal and by the writer is due to the difference in the shape of the respective specimens, and not to the difference in the two testing machines.

51. In screwed bolts the breaking strain is found to be greater when old dies are used in their formation than when the dies are new, owing to the iron becoming harder by the greater pressure required in forming the screw thread when the dies are old and blunt than when new and sharp.

52. The strength of screw-bolts is found to be in proportion to their relative areas, there being only a slight difference in favor of the smaller compared with the larger sizes, instead of the very material difference previously imagined.

53. Screwed bolts are not necessarily injured, although strained nearly to their breaking point.

54. A great variation exists in the strength of iron bars which have been cut and welded; whilst some bear almost as much as the uncut bar, the strength of others is reduced fully a third.

55. The welding of steel bars, owing to their being so easily burned by slightly overheating, is a difficult and uncertain operation.

56. Iron is injured by being brought to a white or welding heat, if not at the same time hammered or rolled.

57. The breaking strain is considerably less when the strain is applied suddenly instead of gradually, though some have imagined that the reverse is the case.

58. The contraction of area is also less when the strain is suddenly applied.

59. The breaking strain is reduced when the iron is frozen; with the strain gradually applied, the difference between a frozen and unfrozen bolt is lessened, as the iron is warmed by the drawing out of the specimen.

60. The amount of heat developed is considerable when the specimen is suddenly stretched, as shown in the formation of vapor from the melting of the layer of ice on one of the specimens, and also by the surface of others

assuming tints of various shades of blue and orange, not only in steel, but also, although in a less marked degree, in iron.

61. The specific gravity is found generally to indicate pretty correctly the quality of specimens.

62. The density of iron is *decreased* by the process of wire drawing, and by the similar process of cold rolling, instead of *increased*, as previously imagined.

63. The density in some descriptions of iron is also decreased by additional hot-rolling in the ordinary way; in others the density is very slightly increased.

64. The density of iron is decreased by being drawn out under a tensile strain, instead of increased, as believed by some.

65. The most highly converted steel does not, as some may suppose, possess the greatest density.

66. In cast steel the density is much greater than in puddled steel, which is even less than in some of the superior descriptions of wrought iron.

Ultimate Resistance and Working Stresses for Wrought Iron.

The ultimate resistance of wrought iron in such bars as were commonly used for structural purposes was generally taken from 45,000 to 50,000 pounds per square inch, the smaller value for large bars and the larger for small sections. The working stresses ranged from 12,500 down to 8000 pounds per square inch, the former value for highway bridges and the latter for the net sections of the tension flanges of stringers and floor beams in railroad bridges. The usual value for the main tension members of railroad bridges was 10,000 pounds per square inch.

Art. 42.—Cast Iron.

Coefficient of Elasticity and Elastic Limit.

Cast iron is a material of much less value to the engineer than wrought iron or steel, and consequently has been a subject of much less experimental investigation. Its quality depends upon the character of the ore from which the pig iron is made, the mixtures of different qualities of pig or of pig and scrap or other metal which is introduced into the melting furnace to produce the final casting, upon the manner of cooling the casting, and upon various other influences, including chemical composition, condition of the elements entering the cast product, and manipulation in the foundry. As it is a result of fusion without subsequent working, such as forging or rolling, it is a metal of exceedingly irregular texture. Bubbles of gases not escaping from the molten mass will leave voids or "blowholes" in the final product and the carbon exists both in the graphitic and combined condition, but in varying proportions. The mode of production and the practically unavoidable irregularities in cooling induce both variable conditions of crystallization and internal stresses that are sometimes high enough even to create a rupture of the completed casting. For all these and other reasons the physical characteristics, both elastic and otherwise, are exceedingly variable. There are some grades of cast iron, like those used for car wheels and ordnance purposes, which give a high ultimate resistance and comparatively high coefficients of elasticity and which exhibit an approximation at least to what may be termed an elastic limit, although the latter point is never well defined as it is in wrought iron and steel. The ordinary soft grade castings used in engineering practice for water pipes, machine frames, and other similar purposes disclose in testing such erratic properties that they cannot be said to have either a well-defined coefficient of elasticity or any real elastic limit whatever. The irregular elastic behavior of cast iron under stress is shown by the strain diagrams in Fig. 1, in which the vertical ordinates are intensities of stress, while the horizontal ordinates or abscissæ are the strains or elongations per linear inch. The amounts written at the extreme upper ends of the curves indicate the loads or stresses per square inch at which the test specimens failed. The two curves *Of* and *Oe* were constructed from data given on pages 597 and 605 of the "U. S. Report of Tests of Metals and Other Materials" for 1899. These two cast-iron test specimens were of metal of superior or special grades, proposed to be used for ordnance purposes, as is indicated by the high ultimate resistances, 22,300 and 35,280 pounds per square inch.

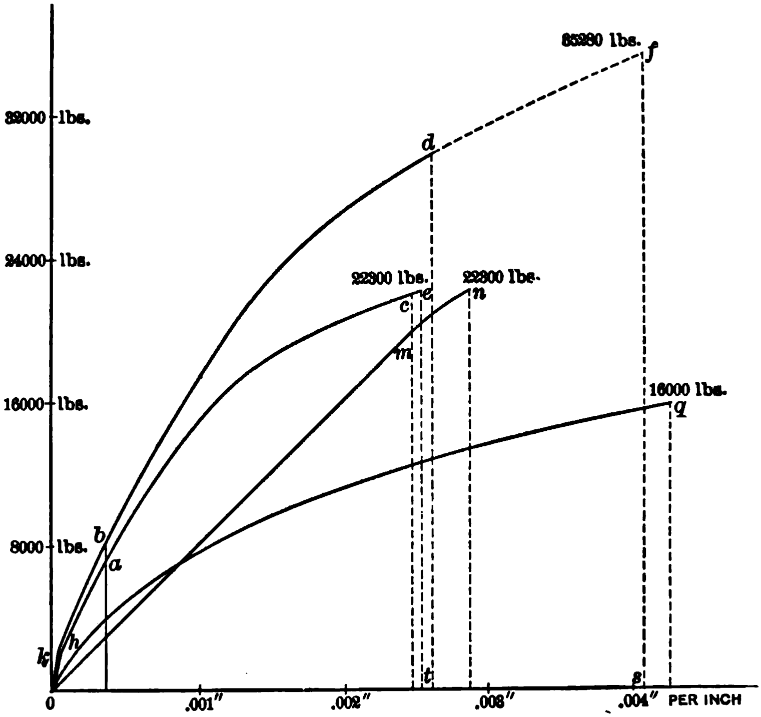


FIG. 1.

The two curves *On* and *Oq* are taken from a paper on "Cast Iron," by Prof. J. B. Johnson, in the "Trans. Am.

Soc. C. E." for February, 1890. The specimen to which the curve *On* belonged failed at 22,300 pounds per square inch, while the metal represented by *Oq* failed at 16,000 pounds per square inch.

There is seen to be the greatest diversity in the inclination and general character of the four strain curves. The curve *Oe* has a fairly straight portion *ha*, the point *a* representing an intensity of stress of 7000 pounds, while the point *h* represents an intensity of 2000 pounds per square inch. The cross-sectional area of this test specimen was 1 square inch. The difference in strains at the two points *a* and *h*, or for a range in intensity of 5000 pounds, was .0003 inch. Hence the coefficient of elasticity for these data would be

$$E = \frac{5000}{.0003} = 16{,}667{,}000 \text{ pounds.}$$

In the same manner the increase of strain per linear inch of test specimen resulting from increasing the stress of 2000 pounds per square inch at *k* to 8000 pounds per square inch at *b* was .00032. Hence with these data the coefficient of elasticity would be

$$E = \frac{6000}{.00032} = 18{,}750{,}000 \text{ pounds.}$$

The strain curve *On* is an extraordinary one for cast iron, as it is straight for nearly its entire length. For the intensity of stress of 16,200 pounds the strain or stretch is seen to be .002 inch; hence the coefficient of elasticity would be

$$E = \frac{16{,}200}{.002} = 8{,}100{,}000 \text{ pounds.}$$

The metal represented by the strain curve *Oq* cannot be said to have any coefficient of elasticity at all, as no part of the curve is straight. These instances selected from a large number of tests are representative of what may be expected in elastic behavior of cast iron. As a rule, the grades possessing the higher ultimate resistances exhibit a more nearly normal elastic character and possess what may be termed not very well-defined coefficients of elasticity running from about 14,000,000 to perhaps 18,000,000 pounds per square inch, while the usual grades or quantities employed in engineering castings may have no coefficient of elasticity at all or as low as 8,000,000 or 10,000,000 pounds per square inch. In view of all experimental data available at the present time it is probably about as near correct as practicable to take the tensile coefficient of cast iron for ordinary engineering purposes, as

$$E = 12{,}000{,}000 \quad \text{to} \quad 14{,}000{,}000 \text{ pounds},$$

or one half that of wrought iron. For the special grades of stronger cast iron, such as are used for ordnance and car-wheel purposes, a coefficient or modulus of 16,000,000 pounds to 18,000,000 pounds per square inch may be used.

As is usually the case in cast iron, the elastic limits of the curves in Fig. 1 are so ill-defined that they cannot be placed with certainty even on the curves *Of* and *Oe*, or scarcely on *On*, and not at all on curve *Oq*. If the points are approximately located on the first three of these curves they may perhaps be placed at *b* (8000 pounds per square inch), at *a* (7000 pounds per square inch), and at *m* (19,000 pounds per square inch). In none of these cases, however, can the metal be said to have either a well-defined limit of elasticity or a true yield point, and that observation is in general true of all cast iron.

Resilience, or Work Performed in Straining Cast Iron.

As the scale of the original of Fig. 1 was 8000 pounds to each inch of vertical ordinate and .001 inch to each inch or horizontal ordinate or abscissa, and as the strains shown in Fig. 1 belong to a test piece 1 inch square in section and 1 inch long, each square inch of area on the original diagram between any one of the strain curves and the axis of abscissæ drawn through O will represent $8000 \times .001 = 8$ inch-pounds of work performed in stretching that test piece. The strain at the point b on the curve Of is .00036 inch, as shown in the figure, while the mean intensity of stress in producing that strain is 4400 pounds. Hence if b represents the elastic limit the resilience or work performed in stretching the metal up to the elastic limit of 8000 pounds per square inch is

$$4400 \times .00036 = 1.58 \textit{ inch-pounds per cubic inch.}$$

Similarly, if a is the elastic limit in the strain curve Oc, the total strain for each inch in length of the test specimen is .00038 inch and the mean intensity of stress is 3750 pounds, all as shown in Fig. 1. Hence the resilience or work performed was

$$3750 \times .00038 = 1.43 \textit{ inch-pounds per cubic inch.}$$

A similar computation may be made for the straight portion of the strain curve On, but the preceding operations sufficiently illustrate the procedure.

The total work performed in breaking each specimen may readily be found in precisely the same manner. In the case of the curve Of the strains or elongations of the specimen were actually observed only up to the point d, although failure actually took place at f or at the intensity, 35,280 pounds per square inch. The part *df* of the curve is drawn approximately as a continuation of the observed curve and therefore is shown as a broken line. The area included between the curve *Of* and the horizontal ordinate *Os*, i.e. the area of the figure *Ofs*, is 11.97 square inches. Hence the work performed in rupturing the test piece was

$$11.97 \times 8000 \times .001 = 95.76 \textit{ inch-pounds per cubic inch.}$$

Again, in the case of the strain curve *Oe* the area of the figure *Oet* is 4.69 square inches. The total work expended, therefore, in rupturing the specimen was

$$4.69 \times 8000 \times .001 = 37.5 \textit{ inch-pounds per cubic inch.}$$

In the latter case the short portion *ce* of the strain curve is drawn approximately, as the strain observations ceased at *c*. It is to be remembered, as is indicated in each of these cases, that when the data apply to each linear inch of test piece and each square inch of sectional area, the work computed will be for 1 cubic inch of material. It is only necessary to multiply by the number of cubic inches in the test piece in order to obtain the work performed in the entire piece.

Ultimate Resistance.

The ultimate tensile resistance of cast iron is an exceedingly variable quantity; it may range from not more than 8000 or 10,000 pounds in castings of indifferent quality to values of nearly 50,000 pounds per square inch in such special grades of metal as those which have been used for car wheels and ordnance. Cast iron has passed completely out of use for the manufacture of heavy guns, but there are other ordnance purposes for which it is still used. The castings usually employed by civil engineers are generally of soft-grade iron; they are such as water pipes, frames, beds of machines, and other similar purposes which do not require special grades produced by special mixtures of raw material or special processes of manufacture. The ultimate resistances will, therefore, be considerably less than those belonging to ordnance and car-wheel irons, or for specially strong grades of metal. As with all material, the character of cast iron affects to a great extent its resistance, i.e., whether it is fine or coarse grained, as does also the character of the ore from which it is produced.

Three specimens turned down to a diameter of about .625 inch taken from iron used in the Boston water pipes and broken at the Warren Foundry, Phillipsburg, New Jersey, gave the following ultimate resistances in pounds per square inch:

18,300, 15,470, 13,070.

Although these tests were made nearly twenty years ago, they represent fairly the ultimate resistance of castings ordinarily used at the present time in civil engineering practice.

It has sometimes been urged that the outer portions or "skin" of castings possesses a higher degree of ultimate resistance than the interior of the same casting, but this is not confirmed by the results of actual tests. It is not improbable that the impression may have been gained from the tests of pieces in which internal strains have been created by irregular or other injudicious cooling of the casting. While there are many irregularities exhibited by cast iron, it is probable, if not practically, certain that when other things are equal the exterior metal of a casting has about the same ultimate resistance as the interior portions. This point has been investigated with care by Prof. J. B. Johnson, the results of whose tests are given in a paper on "Cast Iron" in the "Transactions of the American Society of Civil Engineers" for February, 1890, in which he says that the ultimate resistances of five cast-iron test specimens turned down to $\frac{3}{4}$ inch in diameter gave ultimate resistances running from 14,545 pounds to 24,597 pounds per square inch, with a mean of 21,100 pounds, while the ultimate resistances of test specimens, as cast, ran from 12,470 pounds to 20,387 pounds per square inch with a mean of 16,660 pounds per square inch. These tests seem to indicate that the interior portion of the metal was stronger than the skin, but that conclusion probably would be discredited. It is more probable that the difference is due to cooling strains in the original specimens, which were relieved by turning off the outside or skin of the material.

In the same paper by Prof. Johnson he concludes, from his tests of cast iron used in the vicinity of St. Louis, that the ultimate resistance of cast iron of ordinary castings may be taken at 20,000 to 25,000 pounds per square inch. While such values may readily be obtained by special mixtures, they are too high for the castings commonly employed by civil engineers.

Mr. George A. Just, C.E., in the discussion of the paper of Prof. Johnson, to which reference is made in the preceding paragraph, cites a number of tensile tests of castings made for the Jackson Architectural Iron Works of New York City. The metal of these test specimens was such as is ordinarily used in cast-iron columns or other castings requiring a similar grade of metal. The test specimens were 1 inch square, and the maximum, mean, and minimum ultimate resistance in pounds per square inch of six test pieces were as follows:

18,981, 17,081, 15,693.

These values are fair and reasonable for the general run of castings employed in engineering practice.

A good grade of cast iron tested by Mr. Just in four samples averaging an ultimate tensile resistance of 17,160 pounds per square inch exhibited the following partial chemical analysis:

Phosphorus	.733
Silicon	1.646
Sulphur	.017
Manganese	.227

Another, Southern iron, untested gave

Phosphorus	.672
Silicon	4.080
Sulphur	.025
Manganese	.239
Combined carbon	.271
Graphitic carbon	3.164
Iron	91.549

The "Report of the Tests of Metals and Other Materials" at the United States Arsenal, Watertown, Mass., for 1900, contains a mass of tensile tests of pig irons and ordnance castings of a great variety of grades and qualities, from which the following tabular statement of greatest and least values have been taken. There are also given the results of two tests of gear teeth taken from the same source.

TENSILE TESTS OF CAST IRON.

Iron.	Ultimate Resistance. Lbs. per Sq. In.	
	Greatest.	Least.
Pig	31,890 Fine granular, gray.	11,820 Coarse granular, dark gray.
Ordnance Castings	33,500 " " "	14,900 " " " "
Gear teeth.	12,200 Fine or medium granular, gray.	12,080 Fine or medium granular, gray.

As a recapitulation there may be written:

For ordinary castings:

Coefficient of elasticity $\begin{cases} \text{12,000,000 lbs. per sq. in.} \\ \text{to} \\ \text{14,000,000 lbs. per sq. in.} \end{cases}$

Ultimate tensile resistance, 15,000 to 18,000 lbs. per sq. in.

For specially excellent grades:

Coefficient of elasticity $\begin{cases} \text{16,000,000 lbs. per sq. in.} \\ \text{to} \\ \text{18,000,000 lbs. per sq. in.} \end{cases}$

Ultimate tensile resistance, 20,000 to 35,000 lbs. per. sq. in.

Tensile working resistances in pounds per square inch may be taken as follows:

For water pipes and other similar purposes:

3000 to 3500 lbs. per sq. in.

With higher grades of cast iron for special purposes:

4000 to 7000 lbs. per sq. in.

Effects of Re-melting and Continued Fusion.

The preceding physical qualities of cast iron are supposed to belong to the metal as actually used in castings, which ordinarily means "second-fusion" metal. In other words, the crude pigs are first-fusion iron, formed from the molten metal as it flows from the blast-furnace. These pigs are ordinarily melted in a cupola furnace and run into suitable moulds. If the iron were twice re-melted before being used to make the finished castings, it would be considered third-fusion metal. The effect of this repeated fusion is an improvement in quality. As far as experience indicates it appears that the softer grades of iron can be re-melted a greater number of times to advantage than the harder or higher grades. Evidently re-melting in–

volves a material additional cost, and hence it is seldom employed. Indeed, there have been cases in which the metal has been run from the blast-furnace directly into moulds for the finished castings. Such castings must be of inferior quality and they are seldom or never produced other than in a clandestine manner. It cannot be definitely stated how many re-meltings will continue to improve a given quality of cast iron.

Continued fusion, like repeated re-melting, has the general result of increasing tensile resistance. In some cases this improvement has been observed to continue up to three and three fourths hours in fusion, but there is not sufficient information available to predict the entire period during which improvement would continue and at the end of which no further improvement could be expected. Fusion continued for three and three fourths hours has been shown in one instance to increase the ultimate resistance from 17,840 pounds to 25,770 pounds per square inch.

Effect of Repetition of Stress.

When Captain Rodman, U.S.A., was investigating the qualities of cast iron suitable for ordnance, he subjected a number of test specimens of different grades of cast iron to repeated stresses with the following results:

Specimen.										
Ao	broke	at	2301st	repetition	of	22,000	pounds	per	square	inch.
Ao	"	"	282d	"	"	26,000	"	"	"	"
Bo	"	"	252d	"	"	20,000	"	"	"	"
Bo	"	"	150th	"	"	20,000	"	"	"	"
Cc	"	"	651st	"	"	22,500	"	"	"	"
Co	"	"	457th	"	"	23,500	"	"	"	"
Do	"	"	172d	"	"	21,600	"	"	"	"

The tests for ultimate resistance of the same grade of iron under a single application of breaking load gave the following results, which show considerable fatigue by the repetition of applied loading substantially below the ultimate resistance:

Specimen.	Density.	Ult. Ext.	Ult. Resist.
Ao	7.267	.00303	30,117
Bo	7.178	.00291	23,617
Co	7.255	.00287	28,220
Do	7.221	.00424	25,627

On pages 166 and 167 of his "Reports" he also gives some very interesting results of intermittent repetitions of stresses. He subjected a cylinder of cast iron 1.382 inches in diameter and 35 inches long to intermittent repetitions of 15,000 pounds per square inch (about three quarters of its ultimate resistance) as follows: 250 repetitions, then a rest of 40 hours; next, 375 additional repetitions, then a rest of 30 days; next, 155 additional repetitions, then a rest of 29 days; next, 1020 additional repetitions, then a rest of 26 days; finally 156 additional repetitions, followed by breakage at the 1956th repetition. In every case "rest" signifies entire freedom from load. Captain Rodman's table gives a detailed account of these experiments. He remarks upon them as follows: "The most interesting point . . . is the fact that at every interval of rest of any considerable time the permanent set and the extension due to the last previous application of the force diminished. And in some instances it required some fifty repetitions to bring up the extension and set to the same points where they had been at the beginning of the period of rest, thus indicating clearly that the specimen was partially restored by the interval of rest from the injury which it had received, and that it endured a greater number of repetitions, owing to the intervals of rest, than it would have done had the repetitions succeeded each other continuously and at short intervals of time."

These experiments show the "fatigue" of cast iron and the *increase* of the ratio of stress over strain produced by "rest"—so far as tensile stress is concerned.

An examination of the tables also shows that in any series of repetitions, between any two consecutive rests, both the extension and set *were constantly increasing*, consequently that the ratio of stress over strain was constantly *decreasing*.

It is interesting to observe that, in experimenting upon cast-iron cannon, Major Wade ("Reports," pages 77 and 78) found that water was forced through the "pores" of the metal of one cannon at a pressure of 7000 pounds per square inch, and through those of another with thicker metal (thickness equal to radius of bore) at a pressure of 9000 pounds per square inch.

Effect of High Temperatures.

The effect of high temperatures upon the resisting capacity of cast iron is not in general different from that found in connection with steel and wrought iron. Little if any softening is observed until a temperature of 500° F. is approached, but beyond that limit it is liable to begin to lose capacity of resistance to a material extent, if not rapidly. Some experimental data on this point will be found in Table XIV, Art. 44.

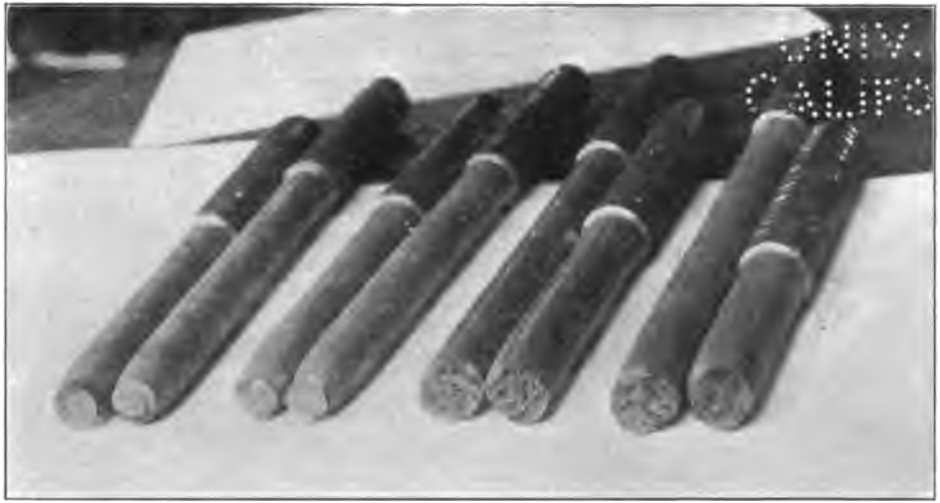
Art. 43.—Steel.

Coefficient of Elasticity.

The great number of varieties and grades of steel renders possible the existence of a correspondingly great number of physical quantities and coefficients used in its consideration in connection with the "Resistance of Materials."

That class of steel generally to be considered here is included under the term "Structural Steel," which may be divided into low, medium, and high steel. These three grades of structural steel are mainly based upon the amounts of carbon which they contain. While each class shades insensibly into another without well-defined limits, it may be approximately stated at least that low or soft steel will have carbon ranging from about .1 to .2 per cent., and that the carbon in medium steel will run from about .2 to .3 per cent., while high steel will show about .3 to .45 per cent. of carbon. The ultimate resistance of low steel may run from 52,000 to 60,000 pounds per square inch, medium steel from 60,000 to 68,000 pounds per square inch, and high steel from 68,000 to about 76,000 pounds per square inch, or possibly higher. Experimental investigations have shown that the coefficient of elasticity is essentially the same for all grades of steel used in construction. This observation holds true also for nickel steel, which has within the past few years come into use for special structural purposes. A considerable number of tests of nickel-steel specimens, in some cases containing 3.375 per cent. of nickel with .3 per cent. of carbon and .73 per cent. of manganese, given in the U. S. Report of Tests of Metals for 1898 and 1899, show that the coefficient of elasticity for this metal may be taken at values ranging from 28,700,000 pounds to 30,385,000 pounds per square inch. In other words, the coefficient of elasticity of this nickel steel may be taken between the usual limits for ordinary structural steel of 28,000,000 and 30,000,000 pounds per square inch.

In Table I are contained the coefficients of elasticity of the hardened and tempered steel wire (see Table XII), supplied by the different makers named, in response to the call for bids for the steel cable wire for the New York and



UNIV. CALIF.

Tensile-test specimens of steel and wrought iron broken in the testing laboratory of the Department of Civil Engineering of Columbia University. The four steel specimens on the left show the characteristic "cupping" fracture. The four specimens on the right are of wrought iron and show the characteristic fibrous fracture of that material.

(*To face page 272.*)

Brooklyn suspension bridge. (Washington A. Roebling's "Report," January 1, 1877.)

In the same "Report," page 72, the specifications state: "The elastic limit must be no less than $\frac{47}{100}$ of the breaking strength. . . . Within this limit of elasticity it must stretch at a uniform rate corresponding to a modulus of elasticity of not less than 27,000,000 nor exceeding 29,000,000."

TABLE I.

Producer.	Coefficients of Elasticity.		Number of Tests.
	Greatest *E*.	Least *E*.	
J. Lloyd Haigh	29,817,067	28,815,797	12
Cleveland Rolling Mills	30,142,026	28,917,715	6
Washburne & Moen	29,757,300	28,887,006	6
Sulzbacher, Hymen, Wolff & Co.	30,389,946	29,103,238	6
Jno. A. Roebling's Sons Co.	30,231,929	28,788,619	13
Carey & Moen	31,261,041	29,418,025	12

Table I gives the greatest and least results of these tests, in pounds per square inch, in the columns headed "*E*," together with the number of tests of the product of each maker. All the wire was No. 8 Birmingham gauge; i.e., 0.165 inch in diameter.

It is not evident from the "Report" whether these values were obtained for some particular intensity of stress or whether they are mean values for the entire range below the elastic limit.

Table II gives a condensed statement of the results of a very elaborate investigation in the "constants" of steel by Prof. P. C. Ricketts at the mechanical laboratory of the Rensselaer Polytechnic Institute during the year 1886. The paper containing the detailed account of this series of important tests was given to the Am. Soc. of C. E. in February, 1887. Although this table contains

TABLE II.

	Mark.	Per Cent. Carbon.	TENSION.					
			Specimen.			Pounds per Square Inch.		
			Diam. Inches.	Per Cent. Reduc. of Area	Per Cent. Elong in 8 Ins.	Elastic Limit.	Ultimate Resistance.	Coefficient of Elas.
Rivet steel*...	1_1	.09	0.756	61.7	30.5	39.600	63,600	30,039,000
" ...	1_2	"	0.758	61.7	30.5	38,800	63,300	30,010,000
" ...	1_3	"	0.757	60.8	28.9	37,800	63,000	31,160,000
" ...	4_1	"	0.757	65.3	29.6	37,800	62,000	31,063,000
" ...	4_2	"	0.758	65.1	29.4	38,600	63,200	30,471,000
" ...	4_3	"	0.758	62.3	29.9	39,400	62,800	29,965,000
" ...	6_1	"	0.760	61.6	30.1	37,400	60,600	30,456,000
" ...	6_2	"	0.760	60.6	29.6	36,900	61,300	30,885.000
" ...	6_3	"	0.760	61.8	32.2	39,100	61,900	27.335,000
" ...	8_1	"	0.760	57.9	29.2	38,100	62,500	30,618,000
" ...	8_2	"	0.759	62.4	28.4	37,100	62,300	30,172,000
" ...	8_3	"	0.758	61.0	28.2	36.600	61,400	30,424,000
" ...	10_1	"	0.756	65.7	28.6	35 600	61,700	29 696,000
" ...	10_2	"	0.755	64.7	29.0	36,800	61,600	30,075,000
" ...	10_3	"	0.754	64.3	29.1	36,900	62,100	30,371,000
" ...	3_1	"	0.757	63.4	27.9	36,700	61,200	30,918,000
" ...	3_2	"	0.758	64.0	30.4	37,700	61,900	30,801,000
" ...	3_3	"	0.758	64.3	29.2	37,100	61,800	31,091,000
" ...	5_1	"	0.757	51.7	30.1	37,800	62,900	30,032.000
" ...	5_2	"	0.755	49.4	29.2	38,500	63,600	31,646,000
" ...	5_3	"	0.757	51.2	28.1	37,800	61,300	30,031,000
" ...	7_1	"	0.750	62.1	30.9	36,200	61,200	30,160,000
" ...	7_2	"	0.749	60.5	29.6	36,800	62,400	30,415,000
" ...	7_3	"	0.751	61.3	31.7	37,800	62,000	30,232,000
" ...	11_1	"	0.752	64.3	29.4	36,400	62,400	30,030,000
" ...	11_2	"	0.754	63.0	29.4	36,400	61,700	30,556,000
" ...	11_3	"	0.749	62.3	29.2	36,700	62,200	30,011,000
" ...	2_1	"	0.752	55.1	29.9	37,200	61,600	30,210,000
" ...	2_2	"	0.757	53.7	31.0	36,700	60,100	32,965,000
" ...	2_3	"	0.753	53.2	32.0	39,300	61,000	30,097,000
Bessemer † ...	N_1	.11	0.748	60.3	28.4	41,500	66,600	28,950,000
" ...	N_2	"	0.754	58.3	28.2	41,400	65,200	29,391,000
" ...	N_3	"	0.750	57.0	28.2	43,400	67,000	29,899,000
" ...	O_1	.12	0.751	59.7	27.4	41,500	65,300	29,186,000
" ...	O_2	"	0.750	59.2	28.5	41,100	65,100	29,252,000
" ...	O_3	"	0.750	57.4	27.0	41,400	65,700	20,464,000
" ...	T_1	"	0.747	57.3	30.6	42,000	66,100	29,907,000
" ...	T_2	"	0.750		30.1	41,900	65.400	29.809.000
" ...	T_3	"	0.751	57.1	28.7	41,300	65,400	29.270,000
" ...	S_1	.13	0.763	58.1	26.8	48,100	69,400	20,706,000
" ...	S_2	"	0.760	59.5	27.0	47,400	69,300	29,500,000
" ...	S_3	"	0.760	56.4	27.1	47,100	70,100	29,238,000
" ...	U_1	"	0.763	59.1	28.2	42,200	65,300	29,436,000
" ...	U_2	"	0.760	56.6	27.6	42,300	65,600	29,678,000
" ...	U_3	"	0.756	58.3	27.0	42,300	66,400	29,390,000
" ...	R_1	.16	0.747	54.8	28.9	42,000	68,300	30,083,000
" ...	R_2	"	0.745	55.7	27.6	41,700	68,500	30,266,000
" ...	R_3	"	0.745	55.0	27.4	41,000	68,600	29,442,000
" ...	P_1	.17	0.746	56.3	27.1	42,100	70,400	20,375,000
" ...	P_2	"	0.744	57.2	27.4	42,700	70,500	30,158,000
" ...	P_3	"	0.749	55.8	27.1	41,500	79,600	30,784,000
" ...	V_1	.36	0.761	40.7	20.5	60,900	97,500	29,045 000
" ...	V_2	"	0.756	38.5	19.1	60,400	99,600	30,236,000
" ...	V_3	"	0.759	39.5	19.4	69,700	99,100	20,989,000
" ...	W_1	.39	0.763	39.0	20.0	69,500	95,800	30,025.000
" ...	W_2	"	0.762	36.8	19.2	69,600	96,200	30,044,000
" ...	W_3	"	0.765	36.7	19.0	69,100	95,200	20,291,000

* Open hearth from Steelton, Pa. † From Troy, N. Y.

TABLE II.—*Continued.*

COMPRESSION.		SHEAR.				Double Shear Ultimate Over Single Shear Ultimate.
Pounds per Square Inch.		Pounds per Square Inch.				
		Single Shear.		Double Shear.		
Elastic Limit.	Coefficient of Elasticity.	Elastic Limit.	Ultimate Resist.	Elastic Limit.	Ultimate Resist.	
39,000	29,897,000					
39,500	27,113,000	39,600	45,440	43,600	46,460	1.022
39,000	28,444,000					
41,100	29,110,000					
41,100	29,025,000	34,600	45,260	38,200	47,450	1.048
41,000	29,045,000					
40,200	30,045,000					
40,200	28,853,000	31,500	46,020	33,800	47,590	1.034
40,400	29,411,000					
41,600	30,192,000					
41,600	29,302,000	31,700	46,910	33,500	48,300	1.032
41,600	29,216,000					
38,600	29,013,000					
38,600	29,963,000	31,100	44,780	34,000	46,590	1.040
38,600	29,478,000					
38,300	29,090,000					
38,300	29,807,000	35,900	44,600	38,500	47,350	1.062
38,300	28,961,000					
41,700	29,630,000					
41,700	28,941,000	33,800	46,440	39,400	48,890	1.053
41,700	29,696,000					
39,900	29,437,000					
40,000	30,009,000	33,700	45,190	35,700	47,210	1.045
40,000	28,730,000					
39,500	29,005,000					
39,700	29,740,000					
39,900	29,963,000					
40,000	31,433,000					
40,000	29,782,000	35,800	46,100	40,700	47,210	1.024
39,700	29,391,000					
41,800	28,567,000					
41,700	29,144,000	30,500	49,210	38,600	51,000	1.036
41,700	28,747,000					
41,100	28,503,000					
41,400	29,531,000	34,400	51,470	39,500	51,470	1.000
41,200	28,730,000					
42,600	29,162,000					
42,400	29,210,000	37,000	49,740	40,300	50,940	1.024
41,900	28,635,000					
44,400	28,070,000					
44,800	28,729,000					
45,000	29,025,000					
44,100	29,281,000					
44,300	29,830,000	36,600	51,000	40,800	51,510	1.010
44,200	29,324,000					
41,100	28,812,000					
41,400	29,342,000	36,700	51,280	43,800	52,550	1.025
41,000	28,666,000					
41,400	28,860,000					
41,600	29,241,000	41,500	53,260	46,000	53,390	1.002
41,800	29,802,000					
55,200	29,162,000					
54,400	29,454,000	52,500	70,190			
54,400	29,281,000					
59,500	28,602,000					
59,200	28,981,000	51,900	67,760			
59,500	29,281,000					

other values than those immediately desired, the opportunity of directly comparing different physical constants from the same quality of steel is a sufficient reason for inserting the entire table at this place. All the test pieces were uniformly about three quarters of an inch in diameter, and the stretch was in all cases measured on eight inches. The elongations given are per cents of the original length of eight inches.

The reductions of area are the per cents of original sections of the test pieces which indicate the differences between the original and fractured areas.

As indicated, the first half of the table belongs to specimens of open-hearth rivet steel from Steelton, Pa., while the second half contains results drawn from tests on a comparatively wide range of metal from the Bessemer process of the Troy Steel and Iron Co., of Troy, N. Y. The open-hearth rivet steel is all seen to contain only .09 per cent. of carbon, while the Bessemer metal had carbon varying from 0.11 per cent. to 0.39 per cent., with a wide gap between 0.17 and 0.36 per cent.

The specimens 1_1, 1_2, and 1_3 were cut from the two ends and centre of bar 1, and those subjected to tension were located adjacent to specimens of the same name subjected to compression. Similar observations apply to other sets of specimens affected by the same figure or same letter. Hence there is shown in this table the relation of different physical quantities belonging to as nearly identically the same material as the possibilities of the case admit.

The coefficients of tensile elasticity exhibit unusual uniformity. Those for the open-hearth steel show no variation with the small variation in carbon. Although the tensile coefficients for the Bessemer steel are slightly lower for the lowest per cents of carbon than for the highest, yet some of the lowest coefficients are found for the highest carbons, and it is difficult to determine any essential variation with varying proportions of that element.

While the average of the tensile coefficients is a very little more for the open hearth than for the Bessemer steel, there is really no sensible difference between them. The average tensile coefficient may be taken at 30,000,000 pounds per square inch.

Too much importance should not be attached to the percentage of carbon alone in these specimens, as the presence of other elements not given, such as manganese, phosphorus, etc., exert marked influences on the physical characteristics of steel.

The "Report of the Naval Advisory Board," prepared by Asst. Naval Constructor R. Gatewood, U.S.N., contains on pages 71 to 75 a large number of tensile tests.

The least coefficient of tensile elasticity given by Lieut. Gatewood is 24,360,000 pounds per square inch, while the greatest value is 30,890,000 pounds per square inch and the mean 27,720,000 pounds. These values belong to 42 tests of accepted material and were distributed about continuously over the range covered by the limiting values. The carbon varied from 0.11 per cent. to 0.24 per cent., and these extreme values belonged to about average values of the coefficient of elasticity.

Table III contains a most valuable set of results obtained from the tests of full-size steel bars for the Blair Crossing bridge by Geo. S. Morison, chief engineer, and it is taken from his report on that structure.

"E. L." is the elastic limit.

"Ult." is the ultimate tensile resistance.

"E." is the coefficient of elasticity in pounds per square inch.

"C." is carbon and "Mn." manganese.

The bars were broken in the government testing machine at Watertown, Mass., but the chemical analyses were made at the Pittsburg testing laboratory.

The coefficients of elasticity are seen to vary from 28,089,000 to 30,627,000 and thus sensibly coincide with the coefficients for test specimens found by Prof. Ricketts and Lieut. Gatewood, with much lower percentages of carbon. The phosphorus was determined for these bars, although it is not given in Table III; it varied from about 0.06 to 0.2 of one per cent. with an average of about 0.09. It is a significant fact that two of the bars which broke in the head contained the highest phosphorus.

Table VII contains other coefficients of tensile elasticity for full-size bars.

TABLE III.

FULL-SIZE STEEL BARS.

Bar. Inches.	Gauged Length. Inches.	Lbs. per Sq. In.		Per Cent. Final		E.	Per Cent. of	
		E. L.	Ult.	Stretch.	Cont'n.		C.	Mn.
6.5×1.02	120	34,890	69,250	12.2	44.2	30,456,000	0.27	0.63
6.5×1.38	120	33,670	66,700	22.0	45.0	28,846,000	0.25	0.885
6.5×1.54	120	34,470	69,920	16.6	42.1	29,702,000	0.295	0.68
6.5×1.14	200	32,570	64,740	20.0	46.6	29,498,000	0.24	0.78
6.0×1.02		24,550	52,390		18.7	28,089,000	broke	in head.
7.0×1.49	200	24,280	45,270	7.9	17.5	28,776,000		
6.5×1.03		36,270	broke	in	head	29,940,000	0.30	0.745
6.5×1.26		37,250	broke	in	head	30,120,000	0.25	0.82
6.5×1.26	200	34,680	64,750	16.5	45.9	30,627,000	0.295	0.75
6.5×1.40	200	35,230	64,020	20.7	43.9	30,030,000	0.23	0.76
6.5×1.39	200	37,440	70,470	13.1	37.9	29,585,000		
6.5×1.52		32,690				30,030,000	bar	not broken.
*6.5×1.02	200	37,250	67,630	15.0	43.1	28,001,000		
6.5×0.98	160	37,480	67,800	15.8	33.7	29,880,000	0.30	0.665
*6.5×0.98	160	36,650	72,050	13.8	38.4	30,270,000	0.28	0.655
6.5×0.98	200	37,600	68,720	12.3	34.1	29,630,000	0.28	0.66
6.5×0.98	200	35,810	65,850	12.0	39.2	29,960,000	0.30	0.65
6.5×0.97	200	33,230	64,410	16.4	49.5	29,670,000	0.29	0.645
6.5×0.97	200	37,640	68,290	13.9	42.4	29,960,000	0.28	0.68

* Broke in head and retested.

Elastic Limit, Resilience, and Ultimate Resistance.

In scrutinizing the results of tests of specimens and full-size members in this section, it is to be observed that the elastic limit is almost invariably the "stretch-limit," or, as it is commonly called, "the yield-point," and not the true "elastic limit," below which the ratio between intensity of stress and rate of strain is essentially constant. It has already been shown and stated that the true elastic limit is from 2000 to 3000 or 4000 pounds per square inch below the stretch-limit or yield-point. The stretch-limit is so readily observed without delaying the ordinary routine of testing that it has come to be called, although erroneously, the elastic limit, in spite of the fact that it is a little above the intensity of stress to which that term should be applied.

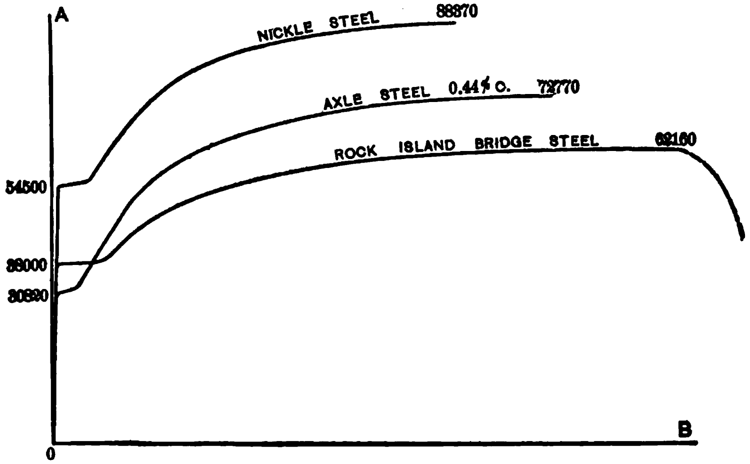


FIG. 1.

The elastic properties of three grades of steel are exhibited graphically in Fig. 1. The curved lines represent the tensile strains of the steel specimens at the intensities of stresses shown. The vertical ordinates are intensities of stress and the horizontal ordinates the rates of stretch, i.e., the stretches per unit of length, the latter being drawn 20 times their actual amounts. The Rock Island Steel belongs to a specimen of steel used for the combined railroad and highway structure across the Mississippi River at Rock Island, Ill., the data being taken from the U. S. Report of Tests of Metals for 1896. The lines for axle steel and nickel steel are the graphical representations of data taken from the "U. S. Report of Tests of Metals" for 1899. As in the previous case, the horizontal ordinates are the stretches per lineal inch shown at 20 times their actual values. The figures at the right-hand extremities of the curves are the ultimate resistances per square inch. The elastic limits and stretch-limits or yield-points are shown with clear definition. The remarkably high elastic limit of the nickel steel is well indicated.

By taking areas first between the horizontal axis *OB* and the inclined straight portion of each line, and then between the same horizontal axis and the entire line in each case, the following values of the elastic and ultimate resilience per cubic inch of each specimen will be found:

	Rock Island Steel.	Axle-steel.	Nickel-steel.
Elastic Resilience	24 in. lbs.	15.7 in. lbs.	49.6 in. lbs.
Ultimate "	10,500 in. lbs.	10,860 in. lbs.	11,040 in. lbs.

The three stress-strain lines or curves of Fig. 1 illustrate completely the physical characteristics of the various grades of steel indicated under all degrees of stress up to actual failure, except that the lines are carried only to the maximum intensities of stress sustained. If those lines were prolonged to the actual parting of the metal, they would show rapidly descending portions like the broken portion of the Rock Island Bridge steel line. That portion of the curve, however, has little practical value, although considerable scientific interest.

Before considering in detail the physical quantities belonging to structural steel, it is advisable to examine

Tables IV and IV*a*, Table IV being condensed from Prof. Woodward's history of the St. Louis Arch. The last four results in that table are from the experiments of Chief Engineer Shock, U.S.N., while the "N. Y. Chrome Steel Co." result is from Kirkaldy's tests.

TABLE IV.

Makers.	Mean of.	Ultimate Resistance, Pounds per Square Inch.			Remarks.
		Greatest.	Mean.	Least.	
Coleman, Rahm & Co., Pittsburg	3	118,400	91,200	74,000	"Very poor steel."
Am. Tool Steel Co., Brooklyn ..	1	——	106,500	——	For lathe tools.
Butcher & Co., Philadelphia	7	144,300	112,100	93,500	
Park Bros., Pittsburg	2	118,000	118,000	118,000	
Steel Works, New York	1	——	85,400	——	
Jessup, Sheffield, Eng.	4	86,000	78,500	74,000	2 Annealed. 2 Unannealed.
Anderson & Woods, Pittsburg ..	2	100,000	100,000	100,000	
Coleman, Rahm & Co., Pittsburg.	1	——	68,000	——	
Miller, Barr & Parkin, Pittsburg.	1	——	90,000	——	Annealed.
Miller, Barr & Parkin, Pittsburg.	2	103,200	102,200	101,200	Unannealed.
Hussey, Wells & Co., Pittsburg .	3	128,000	126,500	125,000	Steel Plate.
Brown & Co., Pittsburg	7	81,300	75,450	45,000	Cast "machinery steel."
Thos. Firth, Sheffield, Eng.	3	113,900	112,600	112,000	"Gun metal."
Butcher & Co., Philadelphia	4	110,100	103,500	99,200	Chrome steel stave.
" " "	4	107,500	106,000	103,500	Chrome steel stave.
" " "	3	151,000	148,700	147,000	Chrome steel ingot.
" " "	5	129,000	99,900	69,800	Carbon rivet steel.
" " "	6	128,300	98,300	65,300	Carbon steel.
" " "	5	142,000	120,100	100,000	Carbon rivet steel.
" " "	5	143,600	139,300	136,300	Chrome steel stave.
" " "	4	135,400	119,200	111,700	Chrome steel stave.
N. Y. Chrome Steel Co.	16	190,260	146,400	115,800	Chrome steel bar.
Park Bros., Pittsburg "Black Dia."	3	131,864	119,500	109,500	Steel normal untemp.
" " "	3	227,500	217,400	201,341	Temp. in oil at 82° F.
" " "	3	176,100	165,500	152,500	Temp. in water at 70° F.
" " "	3	150,500	141,900	132,700	Temp. in water at 79° F.

The diameters of the (circular) specimens varied from 0.357 inch to 1.00 inch, and their lengths from 3 to 12 inches. The elastic limit varied from 45 to 55 (nearly) per cent. of the ultimate resistance.

Table IV*a* contains the results of the experiments of Sir Wm. Fairbairn on the different varieties of English steel given in the left-hand column. The specimens were one inch square, and had previously been subjected to a transverse load. The per cents of strain or elongation are for a length of eight inches, which, it is presumed, included the section of fracture.

TABLE IVa.

BAR SPECIMENS.

Producers.	Ultimate Resistance per Sq. Inch.	Final Strain or Elongation.
C. Cammel & Co.	Pounds.	Per cent.
Cast steel, termed " Diamond-steel"	110,055	1.53
Cast steel, termed "Tool-steel"	109,072	1.50
Cast steel, termed "Chisel-steel"	120,398	2.50
Cast steel, termed " Double-shear Steel" ..	96,665	2.37
Messrs. Naylor, Vickers & Co.		
Cast steel, called " Axle-steel"	88,665	16.25
Cast steel, called "Tire-steel"	91,520	9.00
"Vickers Cast Steel, Special"	134,145	1.00
" Naylor & Vickers' Cast Steel"	118,066	1.75
H. Bessemer.		
Hard Bessemer steel	103,085	1.87
Milder Bessemer steel	88,175	20.00
Soft Bessemer steel	78,606	19.12
Messrs. Turton & Sons.		
Steel for cups	100,155	2.75
" " drills	87,552	1.06
" " cutters	95,372	1.37
" " turning tools	80,273	0.12
" " machinery	102,915	1.43
" " punches	102,567	1.62
" " mint dies	106,237	2.87
" " dies	87,471	0.87
" " taps	97,994	1.87
Double-shear steel	73,266	0.81

Although these tables exhibit results obtained a considerable number of years ago, they are fairly representative of the physical quantities belonging to steel of the same grades at the present time. The grades of metal shown are much higher steel than those employed for structural purposes; they are mostly adapted to special purposes, such as tools, and other similar uses. Chrome steel is not manufactured at the present time (1903).

Table II contains a synopsis of the valuable series of tests of specimens by Prof. P. C. Ricketts. This table has already been explained on page 273. The tension tests show remarkably uniform results in elastic limit and ultimate resistance, and characterize a most excellent material. With the exception of the two Bessemer specimens containing 0.36 and 0.39 per cent. carbon, all specimens were of mild steel.

Table V exhibits the results of late tests of a large number of full-size steel bars of various grades, in connection with the results of tests of specimens in their natural condition, as cut from the rolled bars from which the bars were forged, and after being annealed. It was customary for a number of years to test ¾-inch round specimens, rolled from small 4-inch ingots cast from the same melt which produced the finished bars, but this testing lately has been abandoned as being not sufficiently relevant to the material actually used. Specimens are now cut from the full-size rolled bars and tested, and Table V shows the results of such tests in connection with those of the finished and annealed bars. These specimens are not usually annealed, although the finished bars are always subjected to that process, but the table gives the results of testing some annealed specimens, and thus exhibits the relation existing between the physical properties of specimens, annealed and unannealed, and those of the completed (annealed) bars of the same melt, as closely as it can be shown.

The process of annealing softens the metal and reduces both the elastic limit and ultimate resistance, but it increases the ductility and largely eliminates the internal stresses. The table shows that the reductions of elastic limit and ultimate resistance are greater for the bars than for the specimens. Aside from the influence of the possi-

TABLE V.

ALL FULL-SIZE BARS BROKE IN THE BODY.

No.	Size of Bar, Inches.	Elastic Limit, Lbs. per Sq. In.			Ult. Resist., Lbs. per Sq. In.		
		Natural Specimen.	Annealed Specimen.	Bar.	Specimen.		Bar.
					Natural.	Annealed.	
1	4×13/16	39,280	....	40,540	66,090	....	71,580
2	6×1 1/8	40,490	....	29,690	69,700	....	56,700
3	7×1	40,790	....	31,380	68,730	....	59,850
4	7×1 3/16	33,400	....	30,270	57,170	....	51,500
5	7×1 3/16	32,480	....	27,870	58,050	....	53,520
6	7×1 5/16	34,190	....	28,980	58,460	....	52,010
7	8×1 13/16	40,480	....	28,940	66,880	....	58,010
8	9×1 1/4	38,110	....	32,590	60,920	....	57,410
9	10×1	40,230	....	33,990	69,720	....	63,220
10	10×1 5/16	40,400	....	32,520	70,490	....	60,710
11	10×1 3/8	41,580	....	35,100	73,050	....	67,490
12	10×1 1/2	41,580	....	33,140	73,050	....	65,060
13	10×1 5/8	40,910	....	30,500	70,360	....	58,870
14	10×1 11/16	43,750	....	32,860	75,000	....	65,600
15	10×1 3/4	42,650	....	37,680	75,620	....	70,160
16	10×1 3/4	40,280	....	39,700	70,280	....	65,500
17	10×1 13/16	40,530	....	32,290	72,240	....	62,270
18	10×1 7/8	40,200	....	31,970	71,860	....	53,860
19	10×2 1/16	40,410	....	33,360	69,900	....	73,550
20	10×2 3/16	38,090	....	29,330	71,300	....	63,100
21	10×2 3/16	38,320	....	28,080	70,220	....	55,160
22	3×1 1/16	40,730	39,730	39,640	68,480	68,680	67,770
23	4×3/4	36,660	....	39,030	66,600	....	64,340
24	4×1	39,870	....	36,850	70,650	....	64,620
25	4×1 1/8	41,670	38,880	38,780	68,750	66,850	66,610
26	5×3/4	38,340	....	34,790	65,680	....	61,040
27	5×13/16	37,690	....	38,730	68,560	....	65,210
28	5×7/8	37,840	....	43,600(?)	67,030	....	66,110
29	5×1	39,870	....	39,820	63,480	....	62,980
30	5×1	39,030	....	36,640	59,810	....	58,330
31	5×1 3/16	39,650	39,280	37,330	70,080	69,040	68,100
32	5×1 1/4	40,650	39,760	36,880	69,650	68,580	65,260
33	5×1 1/4	40,570	40,340	38,180	70,000	69,900	66,250
34	5×1 3/8	37,030	....	38,080	61,390	....	62,000
35	5×1 7/16	39,280	38,900	37,170	67,740	67,220	64,630
36	5×1 9/16	42,100	41,490	38,830	71,580	71,420	68,800
37	5×1 13/16	37,950	37,690	35,170	66,770	66,310	63,020
38	5×1 1/8	33,560	....	34,690	62,780	....	63 130
39	6×1	37,370	....	38,480	60,080	....	58.330
40	6×1 1/4	38,690	....	35,880	64,500	....	59,880
41	6×1 1/4	36,530	....	34,840	64,210	....	61,110
42	6×1 3/8	32,310	....	34,440	64,520	....	58,800
43	6×1 13/32	36,530	....	32,420	64,840	....	58,680
44	6×1 13/16	37,490	....	36,080	64,830	....	60,770
45	7×1	35,240	....	35,520	68,850	....	61,200
46	8×1 1/16	38 690	....	40,330	62,770	....	61,840
47	8×1 1/16	36,450	....	37,550	59,100	....	58,120
48	8×1 5/16	39,400	....	36,060	72,400	....	62,210
40	8×1 5/16	38,590	....	37,040	66,820	....	62,460
50	8×1 1/4	34,590	....	34,710	58,600	....	56,210
51	8×2	41,740	....	31,310	65,450	....	53,670
52	8×2	40,400	....	35,640	68,550	....	60,730
53	10×2 5/8	....	....	33,250	....	....	61,720

TABLE V.—*Continued.*

ALL FULL-SIZE BARS BROKE IN THE BODY.

Stretch and Contraction.				Per Cent. of Phos.	Remarks.
Nat. Spec.		Bar.			
Str.	Cont.	Str.	Cont.		
31.3	58.5	10.6	42.6	.084	Acid open hearth steel.
28.1	45.4	13.4	56.9	.026	Basic open hearth steel.
28.1	47.0	17.8	40.9	.03	" " " "
28.8	51.5	21.5	52.5	.014	" " " " Soft steel.
31.9	52.7	17.	45.8	.027	" " " " Foreign matter in section of [fracture.
31.3	34.2	22.5	48.4	.039	" " " " Soft steel.
23.	46.5	15.5	42.2	.03	" " " "
28.7	53.8	23.3	50.3	.014	" " " " Soft steel.
28.1	52.2	13.7	34:6	.025	" " " "
27.	43.6	15.7	44.6	.023	" " " "
27.5	47.5	16.8	30.6	.027	" " " "
27.5	47.5	17.3	38.5	.027	" " " "
25.	40.3	19.3	48.6	.014	" " " "
20.	44.5	13.5	40.	.021	" " " "
24.4	52.6	8.2	39.7	.015	" " " " High steel.
28.8	47.9	11.8	44.4	.062	" " " " " "
21.3	42.8	13.1	23.5	.056	" " " "
27.	40.4	2.5	4.5	.017	" " " "
25.5	51.5	12.3	10.3	.063	" " " " Piping in fracture.
28.8	48.3	13.5	32.6	.017	" " " "
24.2	43.2	6.9	7.3	.023	" " " " Foreign matter in section of [fracture.
27.5	50.0	12.5	49.8	......	The ultimate stretches for these full-size bars belong to lengths varying from about 12 feet to about 35 feet.
30.7	52.8	20.1	53.1	......	
27.5	47.4	14.	46.8	......	
23.7	48.7	14.3	46.7	......	
27.	55.3	13.6	40.3	......	
26.5	47.9	12.5	45.9	......	
23.2	46.7	10.9	51.5	......	
26.2	52.	15.9	51.2	......	
29.5	59.3	13.8	49.7	......	
22.5	52.1	12.3	44.7	......	
22.	44.5	11.4	41.7	......	
22.	44.5	12.	48.	......	
24.5	49.2	16.3	48.	......	
25.5	47.4	11.	50.1	......	
23.7	48.8	13.	40.4	......	
26.	45.1	12.6	52.9	......	
25.5	47.7	19.1	44.3	......	
31.	62.5	17.4	52.7	......	
21.2	54.2	17.5	51.3	......	
27.5	47.5	15.	46.	......	
25.7	46.4	18.	51.7	......	
27.	40.7	14.2	47.9	......	
26.7	50.8	18.1	50.7	......	
27.5	47.2	15.7	55.9	......	
26.	52.7	13.1	46.	......	
31.	62.5	15.7	46.3	......	
23.5	42.	14.3	47.7	......	
27.5	48.6	15.1	48.3	......	
33.	64.3	17.8	49.1	......	
28.	50.4	15.	51.8	......	
25.	44.7	15.	49.1	......	
....	......	20.47	50.4	......	Stretch is for 47 feet length.

ble differences in the annealing temperatures for the two cases, the general average of the bar results should be the lowest, for the reason that the whole mass of the bar includes incidental, but inevitable, small variations of conditions in the metal, which are not found in the small mass of the specimen, and which necessarily lead to still wider variations in the full-size annealed member.

The differences due to the process of annealing will naturally be materially greater in high steel than in low steel.

The wide range in grade of the metal tested gives special value to the results shown in the table.

Nos. 1 to 21, inclusive, were taken from "The Continuous Superstructure of the Memphis Bridge," by Mr. George S. Morison, M. Am. Soc. C. E., in Trans. Am. Soc. C. E. for Sept., 1893; and the remainder, except the last, were taken from a paper in the same Transactions, for Oct., 1892, by Mr. F. H. Lewis, on "The Results obtained from the Tests of Full-sized Bars." The last results in the table belong to a bar manufactured and tested by the Phœnix Iron Co., of Phœnixville, Penn.

The gauged lengths of Nos. 1 to 21, inclusive, in Table V, on which the final stretches were measured, varied from 8 feet to 36 feet.

Table VI shows the results obtained by testing to failure fifteen 10-inch bars among those forged for the 420 feet draw-span of the Central Bridge across the Harlem River, in New York City, Mr. Alfred P. Boller, consulting engineer. They were manufactured and tested under his supervision. Nearly all the bars are of the ordinary open-hearth steel, although No. 7 was of basic open-hearth material. The gauged length on which the final stretch was measured was 26 feet for Nos. 2 and 7; 20 feet for Nos. 1, 3, 4, 10, and 15; 18 feet for No. 8; 17 feet for Nos.

5, 6, 11, 12, and 14; 15 feet for No. 13; and 12.7 feet for No. 9. The final stretch for the 12 inches which included the section of fracture varied from 15 to 44.6 per cent., although very nearly all those values were included within the limits of 29 and 44.6 per cent.

Upon a general view of Tables V and VI there cannot be discovered much difference in uniformity of results between the higher and milder steels, although the latter have what little advantage there is.

TABLE VI.

No.	Size of Bar, Inches.	Pounds per Square Inch at		Per Cent. of Final		Fracture.
		Elastic Limit.	Ultimate Resistance.	Str.	Reduc.	
1	10×1	34,000	63,220	13.7	34.55	30% silky centre, 70% fine grain.
2	10×1	29,340	57,450	15.1	41.2	60% " 40% " "
3	$10\times1\frac{1}{8}$	33,240	59,660	12.1	50.5	100% "
4	$10\times1\frac{1}{8}$	34,510	60,900	12.2	48.9	100% "
5	$10\times1\frac{1}{4}$	30,540	63,770	11.27	36.3	40% " 60% grain.
6	$10\times1\frac{1}{4}$	32,700	65,400	11.76	31.84	100% fine grain.
7	$10\times1\frac{1}{4}$	32,460	61,060	11.2	46.3	70% " 30% grain.
8	$10\times1\frac{5}{16}$	35,160	63,320	12.5	41.2	75% " 25% "
9	$10\times1\frac{5}{16}$	34,100	63.220	12.7	44.5	80% " 20% "
10	$10\times1\frac{11}{16}$	32,860	65,600	13.54	40.	40% " 60% fine grain.
11	$10\times1\frac{3}{4}$	36,890	65,490	15.2	39.4	85% " 15% " "
12	$10\times1\frac{3}{4}$	36,200	65,840	12.3	43.1	70% " 30% " "
13	$10\times1\frac{13}{16}$	32,290	62,280	13.05	41.84	50% " 50% " "
14	$10\times1\frac{13}{16}$	29,330	61,130	14.14	38.13	25% " 75% " "
15	$10\times1\frac{15}{16}$	31,100	61,060	15.33	39.4	40% " 60% " "

Table III contains the results of Mr. Morison's tests, to which reference has already been made on pages 277 and 278. These bars are seen to be of high steel for tension members, and as they were annealed, it is important to observe that their elastic limits average about half their ultimates, i.e., relatively much lower than the unannealed test specimens. Lower steel would be affected less.

Table VII gives the results of tests of some full-size eyebars of Pernot open-hearth steel. The latter was made by the Cambria Iron Co., but the bars were formed by the

Keystone Bridge Co. and tested in the government machine at Watertown, Mass. The tests are reported on pages 194 to 207 of Ex. Doc. No. 5, 48th Congress, 1st Session. The carbon in these bars varied from 0.27 to 0.3 per cent., while the manganese ran from 0.58 to 0.71 per cent., and the phosphorus from 0.074 to 0.098 per cent.

TABLE VII.

6½-INCH × 1-INCH FULL-SIZE BARS.

Length of Bar, Inches.	Pounds per Square Inch.				Per Cent. of	
	Elastic Limit.	Ultimate Resistance.	Coefficient of Elasticity.	Bearing on Pin-hole.	Final Stretch	Final Contraction.
222	37,480	67,800	29,880,000	86,100	15.9	33.7
222	36,650	64,000	30,270,000	79,430		
Retest	Retest	71,560				36.5
Retest	Retest	72,050				38.4
263	37,600	68,720	29,630,000	83,650	12.6	34.1
262	35,810	65,850	29,960,000	80,060	11.5	39.2
262	33,230	64,410	29,670,000	79,060	15.1	49.5
262	37,640	68,290	29,960,000	83,960	13.5	42.4

No bar broke in the head. Tests 3 and 4 were the retests of the two parts of the second bar. The parts of the bar rested five days before being retested. The ultimate resistance is markedly increased in these two experiments.

Some of these bars showed considerable dishing of the head as failure took place, although no head failed. This shows that a bar whose width is 6½ times its thickness has reached about the *limit of thinness*, i.e., that a thinner bar of the same width would be liable to tear in detail through the eye without developing the full resistance of the bar.

Table VIII exhibits results of tests of a number of unusually large eye-bars 12 inches wide with other 8-inch and 7-inch bars used in the Pennsylvania Railroad bridge across the Delaware River a short distance above Philadelphia and completed in 1896. There will also be found in the table tests of specimens taken from the same bars, together with the chemical composition. This table is interesting as disclosing the ultimate tensile resistance of large bars of mild steel having the chemical composition shown. The decrease in ultimate tensile resistance and elastic limit between the original bar and the finished eye-bar, due to the process of manufacture of the latter, is also evident at a glance. Although the steel in the original bars shows ultimate resistances revealed by the tests of specimens running from 58,300 pounds to 69,500 pounds per square inch, no ultimate resistance of the completed bars exceeds 59,500 pounds per square inch, while as small a value as 52,300 pounds per square inch is found. This table is taken from the description of the Delaware River bridge by Mr. F. C. Kunz, Assistant to Vice-President of the American Bridge Company, Engineering Department, published at Vienna, 1901.

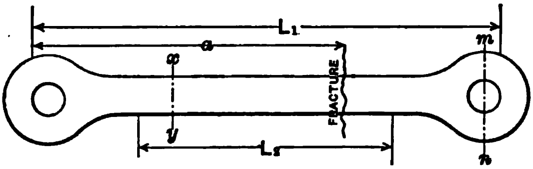
Table VIII*a* gives the results of testing a remarkable series of large steel eye-bars. The table exhibits not only the physical results of the tests but the chemical composition of the metal and the relative results for annealed and unannealed bars. The table was supplied by Mr. Henry W. Hodge, C.E., of the firm of consulting engineers, Boller & Hodge, of New York City. The variation in chemical composition is accounted for by the fact that the bars tested were not specimens of any actual lot, but were forged and broken for the purpose of an investigation to determine specifications under which 12 and 14 inch eye-bars for the Monongahela River cantilever bridge should be manufactured. The machine in which the eye-bar heads were formed was not of sufficient capacity to give satisfactory results, and hence it will be observed that most of the bars broke in the head or

TABLE VIII.*

RESULTS OF TESTS OF EYE-BARS AND OF TEST SPECIMENS

Full-size Eye-bars.								Location of Fracture "a" in Inches.	Pounds per Square Inch.	
No.	Cross-sections xy in Inches.	Pin-hole in Inches.	Percentage of Excess of Section mn over Section xy.	Length L_1 in Inches.		Length L_2 in Inches.			Ultimate Strength.	Elastic Limit.
				Original.	Final.	Original.	Final.			
1	$12\times2\frac{1}{2}$	9	37	410	472	360	414	348	55,000	29,200
2	$12\times2\frac{1}{8}$	9	41	407	478	360	423	434	53,500	25,900
3	$12\times2\frac{3}{16}$	9	41	408	471	360	416	433	53,750	27,800
4	$12\times2\frac{1}{16}$	9	42	422	487	360	418	280	58,300	30,600
5	$12\times2\frac{1}{16}$	9	42	420	485	360	417	284	57,600	30,000
6	$12\times2\frac{1}{16}$	9	42	407	483	360	430	296	56,300	30,090
7	$12\times1\frac{7}{8}$	9	41	403	485	348	423	356	53,000	29,200
8	$12\times1\frac{3}{4}$	9	41	400	470	348	411	332	58,000	31,400
9	$12\times1\frac{11}{16}$	9	38	407	471	360	419	337	54,800	29,800
10	$12\times1\frac{5}{8}$	9	44	407	482	360	430	256	54,900	29,300
11	$12\times1\frac{9}{16}$	9	38	400	447	348	387	414	52,600	29,200
12	$12\times1\frac{9}{16}$	9	41	415	490	360	427	266	52,300	29,900
13	$12\times1\frac{9}{16}$	9	42	411	489	360	430	293	58,500	30,600
14	$12\times1\frac{9}{16}$	—	—	—	—	—	—	—	—	—
15	$8\times1\frac{5}{8}$	$8\frac{1}{2}$	47	428	483	384	434	446	53,500	29,900
16	$8\times1\frac{13}{16}$	$7\frac{1}{2}$	34	328	394	276	334	351	53,300	29,300
17	$8\times1\frac{13}{16}$	$6\frac{1}{2}$	46	328	390	276	332	199	53,000	29,300
18	$8\times1\frac{9}{16}$	$7\frac{1}{2}$	40	328	388	276	328	324	58,500	34,000
19	8×1	$6\frac{1}{2}$	38	470	545	432	501	306	53,300	32,900
20	$7\times1\frac{1}{2}$	—	—	—	—	—	—	—	—	—
21	$7\times1\frac{1}{2}$	$8\frac{1}{2}$	44	501	575	456	525	495	59,500	32,700

* From page 5, "The Delaware River Bridge, Built by Pencoyd Bridge Co.," by F. C. Kunz, "Allgem. Bauzeitung," Heft 1, 1901.

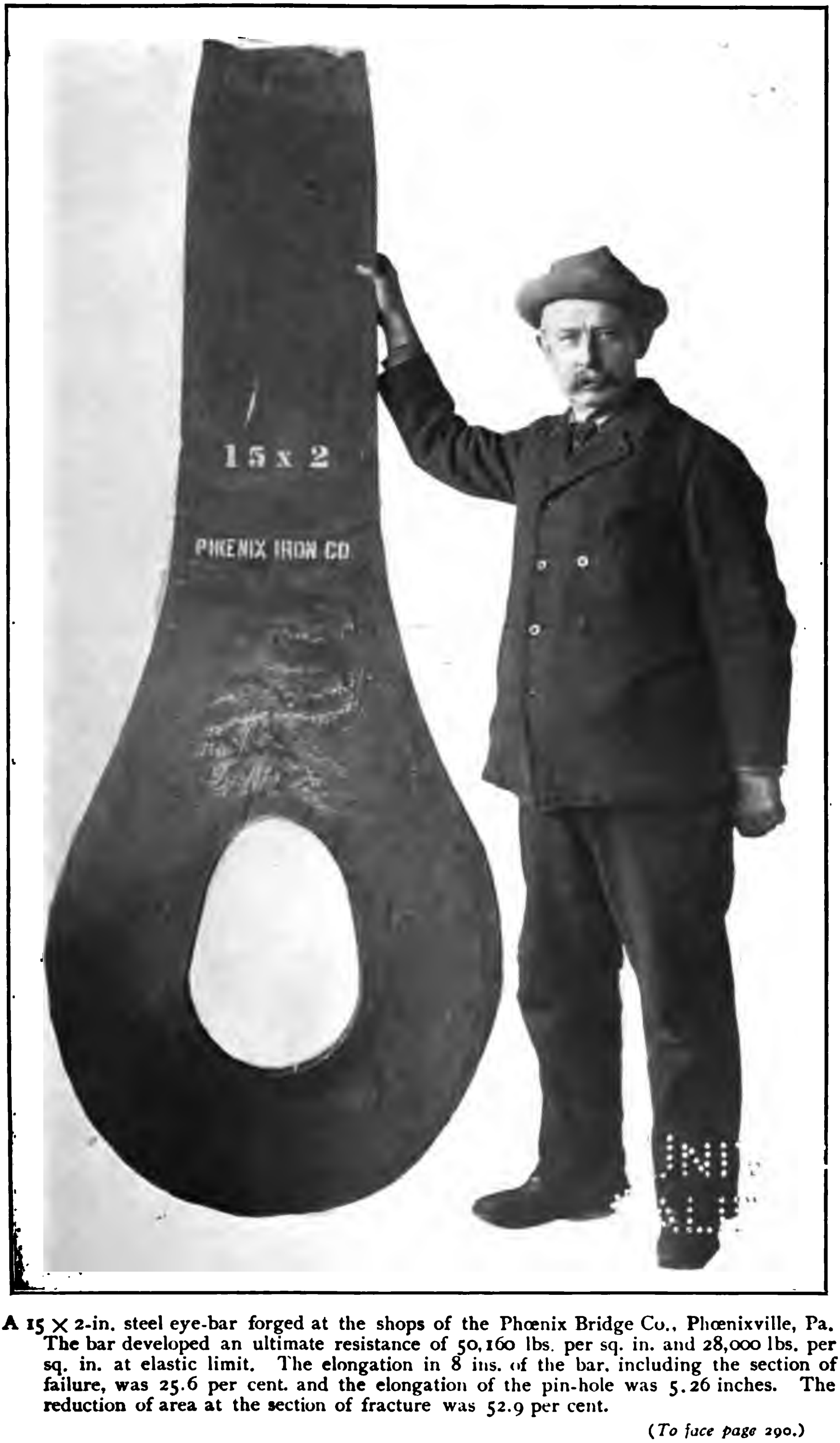


neck. The actual bars for the structure were to be forged in a new machine of greater capacity.

The results are highly interesting as indicating what excellent results may be obtained even for the largest bars under satisfactory conditions of manufacture.

Shape Steel and Plates.

Table IX shows a few of the large number of tests of some of the material used in the construction of the



A 15 × 2-in. steel eye-bar forged at the shops of the Phœnix Bridge Co., Phœnixville, Pa. The bar developed an ultimate resistance of 50,160 lbs. per sq. in. and 28,000 lbs. per sq. in. at elastic limit. The elongation in 8 ins. of the bar, including the section of failure, was 25.6 per cent. and the elongation of the pin-hole was 5.26 inches. The reduction of area at the section of fracture was 52.9 per cent.

(To face page 290.)

TABLE VIII.—*Continued.*

TAKEN FROM THE SAME EYE-BARS, DELAWARE RIVER BRIDGE.

Percentage of			Test Specimen 8″ Long, Approx. 1″ Square.								Process of Manufacture.	No.
Stretch in 12 Inches, including Fractured Section.	Stretch for Length L_2.	Reduction of Area at Fractured Section.	Chemical Composition in Per Cent.				Pounds per Square Inch.		Per Cent.			
			C.	S.	P.	Mn.	Ultimate St'gth.	Elastic Limit.	Str'ch.	Reduction.		
48.0	15.0	49.1	0.18	0.06	0.04	0.49	65,600	33,200	26.0	45.4	basic	1
36.0	17.7	44.7	—	—	—	—	—	—	—	—	″	2
41.0	15.7	38.8	—	—	—	—	—	—	—	—	″	3
43.0	16.1	43.2	—	—	—	—	—	—	—	—	″	4
40.0	15.9	45.3	—	—	—	—	—	—	—	—	″	5
36.0	19.5	38.4	0.24	0.05	0.03	0.62	68,000	40,300	26.75	51.7	″	6
44.0	21.5	41.4	—	—	—	—	—	—	—	—	″	7
49.0	18.3	49.3	—	—	—	—	—	—	—	—	″	8
44.0	16.5	41.1	0.24	0.08	0.06	0.51	69,500	36,200	25.25	42.9	acid	9
42.0	19.4	41.2	0.23	0.04	0.07	0.55	66,700	36,300	27.25	55.2	″	10
30.0	11.2	37.4	0.21	0.07	0.06	0.51	63,800	35,800	27.00	39.5	″	11
45.0	18.8	54.3	—	—	—	—	—	—	—	—	basic	12
47.0	19.5	51.2	0.24	0.05	0.05	0.65	66,500	39,000	28.50	47.9	″	13
—	—	—	0.17	0.07	0.06	0.58	65,000	36,800	26.00	57.3	acid	14
39.0	13.1	44.9	0.21	0.04	0.02	0.45	59,500	33,700	33.50	45.6	basic	15
40.0	21.2	41.9	0.20	0.05	0.03	0.56	60,300	32,000	28.00	58.9		16
33.0	20.5	29.7	0.20	0.05	0.03	0.56	58,300	30,800	29.25	52.7		17
42.0	19.7	48.6	—	—	—	—	—	—	—	—		18
42.0	16.1	49.3	—	—	—	—	—	—	—	—		19
—	—	—	0.22	0.08	0.06	0.59	62,400	35,600	30.25	58.0	acid	20
42.0	15.4	45.8	—	—	—	—	—	—	—	—	basic	21

Chesapeake and Ohio bridge, across the Ohio River, at Cincinnati, Ohio. They are taken from a paper by the author, in the "Trans. of the Am. Soc. C. E.," for August, 1890. All the material tested was steel.

The first seventeen tests were made on specimens cut from steel plates varying from 9 inches by ⅜ inch to 30 inches by ⅞ inch, and all these plates were universal rolled. While the results are by no means regular, the thinner plates in the aggregate yield somewhat higher results both in elastic limit and ultimate resistance than the thicker ones.

A similar result is much more marked in the case of the steel angles, of which there are only two sizes, but with a wide range of weights per yard for the 6 inches by 4 inches. These results are due to the fact that the light

TABLE VIIIa.

RESULTS OF FULL-SIZE EYE-BAR TESTS ON TRIAL STEEL. MONONGAHELA RIVER CANTILEVER. BOLLER & HODGE, CONSULTING ENGINEERS.

The steel was basic open-hearth metal manufactured and rolled by the Carnegie Steel Company, 1902. All bars were about 30 feet long.

"A" means annealed and "N" not annealed.

Bar.	Chemical Composition.				Specimen.			
	Car.	Phos.	Mang.	Sulph.	Elas.	Ult.	Elongation, Per cent in 8".	Reduct.
					Pounds per Sq. In.			
12" × 1¾"	.30	.021	.62	.026	A 37380	67600	27.0	46.5
					N 44330	71080	21.5	40.0
12" × 1¾"	.28	.020	.54	.022	A 32050	60230	26.5	54.1
					N 37230	68000	25.7	39.8
12" × 1⅝"	.26	.03	.52	.02	A 38610	69700	28.7	43.9
					N 39700	72400	27.2	46.2
12" × 1⅝"	.36	.019	.57	.035	A 31250	70720	28.5	45.6
					N 37740	76980	18.7	26.6
12" × 1⅝"	.32	.03	.51	.03	A 29140	67120	28.5	45.5
					N 35600	72120	25.0	38.5
12" × 1⅜"	.28	.03	.46	.04	A 37050	69820	30.5	45.0
					N 40450	73340	27.5	44.7

Bar.	Full-size Bar.				Remarks.
	Elas.	Ult.	Elongation, Per cent. in 20 Feet.	Reduct., Per cent.	
	Pounds per Sq. In.				
12" × 1¾"	38190	64140	15.05	55.54	Broke in body.
12" × 1¾"	37480	58670	5.7	....	" " neck.
12" × 1⅝"	33330	61090	11.55	....	" " head.
12" × 1⅝"	34320	63030	7.05	....	" " "
12" × 1⅝"	34210	59170	6.0	5.96	" " body.
12" × 1⅜"	35930	65520	9.56	....	" " head.

sections are reduced materially more between the rolls than the heavier sections, and thus more work is impressed upon the material of the former. Indeed, it is a matter of serious doubt whether, with the same percentage of reduction from the ingot to the finished piece, the quality of the metal in the larger section would equal that in

TABLE IX.—STEEL.

Material.	Test Specimen, Inches.	Pounds per Sq. In. at		Per Cent. of Final		Fracture.	Bending Test.	Accepted or Rejected.
		Elastic Limit.	Ultimate Resistance.	Stretch in 8 Inches.	Reduction.			
9× 3/8" pl.	1.015× .402	40,690	68,630	30.00	38.48	Silky cup	180° on self	Accepted
10× 3/4" pl	1.015× .760	37,850	62,740	26.87	55.26	" "	" " "	"
12× 1/2" pl.	1.020× .510	49,980	76,320	24.00	35.36	Ang. silky	" " "	"
12× 1/2" pl.	1.018× .505	42,610	66,150	25.00	56.10	" "	" " "	"
16× 3/4" pl.	1.000× .753	43,750	66,060	27.50	51.44	" "	" " "	"
20× 7/16" pl.	1.030× .430	41,100	63,230	25.00	54.74	Silky cup	" " "	"
20× 9/16" pl.	1.02 ×0.56	43,830	67,140	25.00	51.23	" "	" " "	"
20× 9/16" pl	1.01 ×0.57	45,140	67,270	25.00	53.87	Part cup and silky	" " "	"
20× 11/16" pl.	1.01 ×0.68	46,450	68,580	25.62	52.29	" " " "	" " "	"
24× 1/2" pl.	1.025× .501	41,680	68,370	24.75	47.50	Silky cup	" " "	"
30× 1/2" pl.	1.03 ×0.48	52,300	64,920	22.50	52.83	" "	" " "	"
30× 9/16" pl.	1.015× .548	44,950	62,930	23.75	44.50	" "	" " "	"
30× 9/16" pl.	1.02 ×0.56	44,790	62,980	25.00	55.21	" "	" " "	"
30× 5/8" pl.	1.01 ×0.61	39,900	68,410	23.12	45.13	Ang. and silky	" " "	"
30× 7/8" pl.	1.05 ×0.86	41,750	68,280	24.37	42.01	Silky cup	" " "	"
30× 7/8" pl.	1.03 ×0.86	37,390	68,200	27.50	47.04	" "	" " "	"
30× 7/8" pl.	1.03 ×0.86	39,440	69,140	26.87	49.05	" "	" " "	"
Rivet	3/4 round	41,990	61,800	28.75	50.9	" "	" " "	
"	"	40,130	65,590	28.	42.8	Silky irregular	" " "	
"	"	39,170	64,510	30.	61.2	Silky cup	" " "	
"	"	37,840	64,670	29.25	51.3	" "	" " "	
"	15/16 "	37,890	63,620	30.5	56.3	" "	" " "	
"		38,090	62,500	31.	56.5	" "	" " "	
ANGLES.								
5×3" 25 pounds	1×0.32	39,430	71,290	21.25	52.9	Ang. silky		
" 41 "	1×0.55	39,110	71,580	27.25	55.5	Silky cup		
" 49 "	1×0.625	38,650	69,650	26.00	46.4	" "		
6×4" 42 "	1×0.4	39,440	71,105	23.75	54.3	Silky irregular		
" 44 "	1×0.43	39,010	69,860	23.75	52.7	Ang. silky		
" 72 "	1×0.74	35,610	73,970	28.25	51.9	" "		
" 72 "	1×0.73	36,930	72,910	27.75	53.4	" "		
" 84 "	1×0.85	33,290	65,870	26.25	41.0	" "		
" 84 "	1×0.84	34,170	63,060	31.25	60.8	" "		
60"×48" ×2½" pls.	2.5×0.75	29,300	60,350	30.00	30.9	Irregular		Rejected
	1 inch round	31,720	66,240	13.00	13.0	Granular		"
	1 "	30,570	68,150	21.25	27.7	"		"
	1 "	34,650	66,880	23.5	34.4	Irregular		Accepted
	1 "	31,660	61,006	28.5	51.1	Silky irregular		"
	1 "	31,540	61,290	29.75	51.9	Cup		"
	1 "	31,840	64,700	21.25	23.4	Ang silky		"

the smaller; the metal in the interior of the larger mass appearing to feel the pressure of the rolls less than that in the interior of the smaller section.

The last seven results of the Table belong to specimens cut from some unusually heavy plates that were employed as massive links, and which were at that time excessively difficult to obtain of the desired quality. These plates were of irregular shape, and varied from 2½ to 2¾ inches thick; such material had not then generally been rolled under the specifications which governed this work. Repeated attempts secured only plates which proved to be porous, but the most porous portions could be worked under a steam hammer to perfectly satisfactory material, which showed that the reduction between the rolls had been insufficient. These difficulties were finally overcome.

The results given in Table X belong to steel angles and deck beams rolled by the Phœnix Iron Co. for cruisers built by the U. S. Gov't in 1887. The tests were made at the works of the Phœnix Iron Co., and show excellent metal; the carbon varied from about 0.15 to 0.20 per cent.

A number of specimen tests were made, during 1899–1901, of the open-hearth acid and basic steel shapes and plates for the construction of the City Island bridge and the 145th Street bridge across the Harlem River, both in the city of New York, under the direction and supervision of the author. Table XI contains the results of a portion of such tests for the quality of material used. It will be seen that the specimens were taken from a wide range of shapes and plates, and that a large portion of the material was produced by the basic open-hearth process. The table is of special value in consequence of the wide range of sections covered by it, as well as for the chemical data which it contains, showing the percent-

TABLE X.

STEEL ANGLES AND DECK BEAMS.

Original Angle or Deck Beam, Inches.	Test Specimen.		Pounds per Square Inch of Original Section.		Per cent. of Final	
	Section, Inches.	Length, Inches.	Elastic Limit.	Ultimate Resistance.	Contraction.	Stretch in 8 Inches.
2½ × 2½	1.23 × .40	8	39,500	64,154	52.7	28.5
3 × 2½	1.25 × .33	8		62,622	51.5	25.6
3 × 2½	1.25 × .33	8		64,595	49.4	25.4
3 × 2½	1.25 × .33	8		62,980	53.6	27.3
3 × 2½	1.25 × .32	8		62,733	54.1	25.6
3 × 3	1.25 × .38	8		57,990	61.1	26.9
3 × 3	1.25 × .41	8	39,209	61,991	51.8	25.31
3 × 3	1.25 × .43	8	38,791	62,943	51.1	27.9
3 × 3	1.25 × .46	8	39,414	66,538	51.3	26.3
4 × 3 } *a*	1.26 × .38	8		67,356	53.8	26.6
4 × 3	1.27 × .38	8		66,307	51.7	25.5
4 × 3	1.27 × .38	8		66,307	52.3	25.7
4 × 3	1.26 × .38	8		67,878	50.8	23.1
4 × 3	1.24 × .41	8		62,390	52.2	26.4
4 × 3	1.25 × .38	8		64,842	40.0	24.4
4 × 3	1.25 × .40	8		63,256	51.8	24.6
4 × 3	1.25 × .40	8		63,485	47.7	28.6
5 × 3	1.24 × .40	8	37,820	61,407	53.1	25.6
5 × 3	1.25 × .42	8	38,920	62,809	50.9	22.8
5 × 3	1.26 × .40	8	37,685	62,675	51.4	25.7
5 × 3	1.26 × .40	8	37,303	63,078	47.0	25.5
6 × 3.25 } *b*	1.24 × .38	8	37,000	63,803	47.5	23.3
6 × 3.25	1.24 × .34	8	41,321	66,777	48.2	27.0
6 × 3.25	1.25 × .33	8	41,206	63,671	50.3	28.2
6 × 3.25	1.24 × .38	8	30,178	64,632	51.3	29.0
6 × 3.25	1.25 × .39	8	38,777	62,513	51.6	30.3
2 × 2	1.24 × .26	8	42,342	65,237	47.0	27.5
2 × 2	1.24 × .26	8	41,315	63,984	48.5	27.8
2 × 2	1.25 × .26	8	43,600	66,873	45.0	26.0
2 × 2	1.24 × .26	8	42,012	60,310	43.7	25.0
2½ × 2½	1.50 × .28	8	40,596	63,720	48.6	27.8
2½ × 2½	1.50 × .29	8	40,900	62,770	44.6	27.8
2½ × 2½	1.50 × .29	8	41,616	62,933	48.1	26.5
2½ × 2½	1.50 × .26	8	42,460	63,300	47.6	23.8
3½ × 3	1.25 × .41	8	40,850	66,730	50.4	25.0
3½ × 3	1.25 × .42	8	40,380	65,190	57.9	25.8
3½ × 3	1.25 × .48	8	39,230	62,440	64.6	26.5
3½ × 3	1.25 × .42	8	39,570	63,130	56.2	28.3
8 × 5 } *b*	1.50 × .47	8	42,709	65,177	44.8	....
8 × 5	1.50 × .47	8	39,534	63,375	43.9	26.5
8 × 5	1.50 × .55	8	40,272	66,677	44.3	26.8
8 × 5	1.50 × .50	8	37,007	61,924	50.2	31.3

The angles marked "*a*" are of Bessemer steel; all other members represented in the table are of open-hearth steel. The members marked "*b*" are deck beams.

ages of carbon, manganese, phosphorus, and sulphur contained by the steel. Both the chemical analyses and the physical results indicate that many of the shapes are of mild steel, while the remaining portion is of low steel.

TABLE XI.

SPECIMEN TENSILE TESTS OF OPEN-HEARTH STEEL SHAPES AND PLATES. 1899 to 1901.

Specimen from	Loads in Pounds per Square Inch.		Elongation, Per Cent. in 8 Inches.	Final Contraction, Per Cent.	Chemical Analysis.				Character of Fracture.	Remarks.
	Elastic Limit.	Ultimate.			C.	Mn.	P.	S.		
8" × 6" × [illegible]" angle	37,880	63,260	30.5	58.7	.15	.48	.04	.04	Silky ½ cup	Basic open hearth
8 × 6 × [illegible] "	38,650	62,840	29.8	59.1	.18	.46	.03	.04	" " "	" " "
7 × 3½ × [illegible] "	41,720	63,120	28.7	56	.16	.41	.025	.027	" cup	" " "
6 × 6 × [illegible] "	42,030	65,330	26.5	46.9	.25	.48	.023	.027	" ½ "	" " "
6 × 6 × [illegible] "	40,560	62,290	26.8	55.5	.28	.42	.022	.029	" cup	" " "
6 × 4 × [illegible] "	38,350	62,680	31.4	53.9	.15	.42	.02	.05	" "	" " "
6 × 4 × [illegible] "	40,000	62,570	27.5	53.6	.15	.42	.04	.05	" "	" " "
5 × 5 × [illegible] "	40,850	62,150	26.2	51.5	.24	.38	.02	.029	" ½ cup	" " "
5 × 5 × [illegible] "	40,260	64,710	26.2	55.3	.21	.43	.038	.036	" ang.	" " "
4 × 4 × [illegible] "	40,070	62,500	27.5	54.6	.16	.43	.014	.030	" cup	" " "
4 × 4 × [illegible] "	38,070	62,610	29	55.7	.21	.46	.020	.05	" ang.	" " "
3½ × 3½ × [illegible] "	40,040	61,930	28	57.6	.16	.50	.016	.03	" cup	" " "
3 × 2½ × [illegible] "	40,240	61,490	28.8	55.7	.12	.53	.03	.05	" ang.	" " "
20" 80 lb. I	38,300	56,400	30.4	52.6	.16	.47	.03	.05	" "	" " "
20 65 " "	36,810	55,900	30.5	50.9	.17	.50	.03	.05	" "	" " "
15 55 " "	38,030	62,690	29.4	55.5	.18	.44	.03	.05	" cup	" " "
15 42 " "	40,440	63,230	29.6	58.5	.15	.42	.04	.05	" "	" " "
10 35 " "	40,390	70,250	15.9	17.1	.24	.44	.018	.03	Granular	" " "
10 30 " "	40,850	64,890	28.1	54.3	.23	.45	.037	.04	Silky cup	" " "
12 30 " "	40,812	61,810	28.7	55	.26	.38	.024	.035	" ang.	" " "
10 30 " "	37,450	63,000	29.8	57.2	.16	.42	.03	.05	" cup	" " "
9 20 " "	41,070	63,920	30.4	57.4	.15	.46	.03	.04	" ½ cup	" " "
7 20 " "	45,240	67,380	30.4	57.4	.18	.39	.02	.05	" ang.	" " "
6 8 " "	39,640	67,820	28.8	59	.17	.43	.02	.05	" cup	" " "
3" × [illegible]" Z	38,300	65,040	27	47.6	.17	.60	.035	.039	" ang.	" " "
97½" × [illegible]" plate	37,390	63,400	30	59.3	.24	.50	.017	.025	" "	" " "
90½ × ½ "	38,530	68,910	25	52.3	.23	.47	.064	.05	" ½ cup	Acid " "
78½ × [illegible] "	40,340	68,035	24.8	53	.19	.36	.072	.05	" " "	" " "
60½ × [illegible] "	36,270	60,080	28.7	56.9	.21	.54	.028	.024	" "	Basic " "
48 × [illegible] "	38,870	62,860	27.5	57.9	.22	.53	.015	.03	" "	" " "
36 × [illegible] "	42,220	63,730	26	53.1	.17	.39	.077	.025	" ang.	Acid " "
26½ × [illegible] "	37,420	63,380	27.5	58.2	.23	.60	.028	.029	" ½ cup	Basic " "
18 × [illegible] "	39,150	63,420	27.5	48.9	.24	.52	.02	.024	" ang.	" " "
18 × [illegible] "	39,750	64,100	27.5	55	.23	.45	.013	.022	" "	" " "
13 × [illegible] "	39,120	60,115	28.2	59.6	.15	.37	.016	.04	" "	Acid " "
11½ × [illegible] "	40,640	67,580	26.2	52.4	.24	.50	.018	.022	" cup	Basic " "
7 × [illegible] "	40,500	63,070	27.5	57.2	.19	.40	.024	.027	" "	" " "

The quality of metal either in steel shapes or plates depends largely upon the amount of reduction reached in the passage of the blooms through the rolls before the final area of section is attained. In the early days of rolling steel sufficient work between the rolls was not always done, and the quality of the metal suffered correspondingly. This defect is seldom or never found at the present time and the corresponding variations in certain physical qualities are avoided. In the case of wide and thin plates, in which the temperature of the metal may be lower than in thicker plates at the last pass through the rolls, increased hardness may sometimes be found, but as a rule there will be little, if any, difference, as the preceding tables show, in the physical results for the thick and thin sections ordinarily used in engineering construction.

Steel Wire.

The process of production of steel wire is one of cold drawing which, like all cold work, increases ultimate resistance and elastic limit but decreases ductility.

The physical properties of wire will depend to a considerable extent upon its diameter; the smaller the latter, as a rule, the greater will be the ultimate resistance. The wire used in the cables of the new East River bridge (1902) across the East River at New York City having the diameter of 0.192 inch (No. 6 Roebling gauge), is required by the specifications to show an ultimate tensile resistance of at least 200,000 pounds per square inch, with a final stretch of at least 2.5 per cent. in 5 feet and 5 per cent. in 8 inches. The steel from which this wire was made was required to be acid open-hearth metal with elements not to exceed:—phosphorus .04 per cent. and sulphur .03 per cent., manganese .5 per cent. and silicon .1 per cent., copper .02 per cent.

In the U. S. Report of Tests of Metals for 1897 will be found the results obtained by testing piano wire of different diameters, as follows:

Wire.	Ultimate Resistance, Pounds per Square Inch.	Final Contraction.
No. 14 (.033 in. Diam.)	357,440 to 389,470	28 to 37.9%
No. 23 (.0513 in. Diam.)	325,110 to 337,200	42.2 to 45.1%

Table XII contains the results of testing, to ultimate resistance, the wire for which the coefficients of elasticity were given in Table I, together with some belonging to the Chrome Steel Co.'s wire, also tested by the engineers of the New York and Brooklyn bridge. The diameter of this wire was about 0.165 inch (No. 8 Birmingham gauge). As will presently be shown, some of the material was cast steel and other Bessemer steel, all having been hardened and tempered.

TABLE XII.

STEEL WIRE.

Producer.	No. of Tests.	Ultimate Resistance in Pounds per Square Inch.			Per Cent. Final Stretch.	Diameter Fracture, Inch.
		Greatest.	Mean.	Least.		
J. Lloyd Haigh........(1)	12	182,450	175,340	166,169	4.9 1.7	0.161 0.147
Cleveland Rolling Mills (2)	6	182,576	178,400	172,984	4.2 2.1	0.161 0.138
Washburn & Moen.....(3)	6	184,019	176,457	169,706	4.5 0.8	0.147 0.133
Sulzbacher, Hymen, Wolff & Co............(4)	6	179,833	175,291	167,807	4.4 1.8	0.162 0.139
John A. Roebling's Sons Co...............(5)	13	179,019	162,244	125,321	4.8 0.4	0.167 0.130
Johnson & Nephew....(6)	9	206,170	177,706	163,027	6.9 3.1	0.148 0.129
Carey & Moen.........(7)	12	194,227	167,880	126,814	4.2 0.5	0.160 0.125
Chrome Steel Co.......(8)	6	170,150	160,544	150,657	3.4 1.0	

The column "*Per cent. final stretch*" gives the highest values for the 5-feet lengths tested, and the lowest for the 100-feet lengths; these were the greatest and least found.

The column "*Dia. fracture*" gives the greatest and least values of the diameter of the fractured section in decimals of an inch. There seemed to be no definite relation between the ultimate resistance and contraction of section of rupture.

Col. W. A. Roebling states that the character of the above steel was believed to be as follows:

(1) English crucible cast steel.
(2) Open-hearth steel.
(3) English crucible cast steel.
(4) Krupp's Bessemer and cast steel.
(5) Crucible cast steel and American Bessemer steel
(6) English crucible cast steel.
(7) English crucible cast steel.
(8) Crucible cast steel.

It is therefore seen that steel drawn into wire possesses an excess of resistance over that in larger masses, as bars; it thus exhibits the same general phenomenon as wrought iron under similar circumstances.

Steel Castings.

The results exhibited in Table XIII belong to the steel castings used in the turntable of the draw-span of the 145th Street bridge across the Harlem River in New York City. The tests were made in 1901. The left-hand column of the table shows the particular (cast) members of the turntable from which the specimens were taken. They also show that, in steel castings, a sensibly higher grade (in the sense of containing more carbon and manganese) of steel is used than in rolled shapes. As indicated in the heading of the table, the material was acid open-hearth steel. The ultimate tensile resistance runs from about 67,000 pounds to nearly 76,000 pounds per square inch. The elastic limit is also observed to be high, in consequence of the rather large percentage of manganese. The quality of metal exhibited by the physical results of the table is fairly representative of that ordinarily used in steel castings. Obviously the ductility exhibited is less than that found in connection with rolled shapes.

TABLE XIII.

TENSILE TESTS OF ACID OPEN-HEARTH STEEL CASTINGS, 1901.

Specimen from	Loads in Pounds per Sq. In.		Elongation, % in 8 Ins.	Final Contraction, %.	Chemical Analysis.					Character of Fracture.
	Elastic Limit.	Ultimate.			C.	Mn.	P.	S.	Si.	
Turntable wheel.	47,510	72,300	31.2	39.8	.23	.70	.052	.004	.27	Silky cup.
Track segments..	49,500	67,200	30.4	29.	.27	.65	.045	.018	.26	" "
" " ..	47,500	67,900	34.3	46.5	.27	.60	.047	.007	.28	" ang.
" " ..	47,500	71,100	32.0	40.8	.30	.65	.040	.004	.28	Irregular.
Rack segments..	49,775	72,115	31.2	39.8	.23	.70	.052	.004	.27	Silky ang.
Track segments .	47,500	68,100	32.6	44.5	.27	.60	.047	.008	.28	" "
" " ..	46,270	71,920	29.6	38.5	.23	.60	.052	.004	.27	" cup.
" " ..	48,900	74,700	30.4	42.5	.30	.65	.041	.009	.26	Irregular.
Turntable wheel.	46,130	71,860	28.1	37.6	.29	.70	.046	.006	.25	Silky cup.
" "	48,640	71,600	31.4	39.4	.29	.70	.046	.006	.25	" "
Rack segments...	49,775	71,335	21.9	37.1	.29	.70	.046	.006	.25	" "
Track segments..	47,600	76,020	31.6	31.7	.30	.65	.037	.004	.27	" ang.
Shoes...........	45,200	68,000	29.6	37.7	.25	.50	.043	.02	.20	Irregular.
"	47,500	68,700	31.2	40.	.29	.65	.05	.024	.27	"
"	49,800	75,700	29.6	45.5	.31	.60	.036	.003	.26	"

Rail Steel.

The grade of steel adapted to railroad rails is much higher in the hardeners carbon and manganese, and correspondingly higher in physical quantities than structural steel, at the same time it is a quite different metal from that adapted to the finer purposes of tools; it is manufactured by the Bessemer process. The great increase in the immediate past in the weight and speed of railroad locomotives and trains has subjected rails to intensely severe duties which can be performed without deterioration of metal only by steel of the highest powers of endurance, which means a steel of high ultimate resistance, elastic limit, and corresponding ductility. The grades of steel used for rail purposes at the present time are well illustrated by the following tabular statement, showing the chemical composition of the metal and the results of its tests by the West End Street Railway Company of Boston. The tests were made in 1898 and are reported in the *Engineering News* of November 22d of that year.

	Soft.	Hard.	West End.
Carbon	.28	.59	.57
Silicon	.026	.056	.234
Phosphorus	.106	.097	.05
Sulphur	.066	.059	.078
Manganese	.79	.83	.98
Specific gravity	7.855	7.841	7.825

	Ultimate Tensile Resistance.	Elastic Limit.
Soft rail	75,860	45,730 (?)
Hard rail	118,100	62,500 (?)
West End rail	120,380	53,160 (?)

All results are in pounds per square inch.

Under actual traffic the West End rail disclosed the least wear on the head and the soft rail the most, with the hard rail in intermediate position.

The following tabular statement shows the chemical composition of the rails of various weights and sizes used by the N. Y. C. & H. R. R. R. Co., the pounds at the head of the columns indicating the weight of rail per yard. The metal of the lightest or 65-pound rail corresponds to an ultimate resistance of 85,000 to 90,000 pounds per square inch, with an elastic limit of .5 to .7 of that value. The highest or 100-pound rail corresponds to metal having an ultimate tensile resistance of probably

110,000 to 120,000 pounds per square inch, with an elastic limit of .6 to .7 of those amounts. In these chemical compositions it is pertinent to observe the high carbon and manganese, and the low phosphorus and sulphur.

NEW YORK CENTRAL & HUDSON RIVER R. R. SPECIFICATIONS.

	65-Lb.	70-Lb.	75-Lb.	80-Lb.	100-Lb.
Carbon	0.45 to 0.55	0.47 to 0.57	0.50 to 0.60	0.55 to 0.60	0.65 to 0.70
Silicon	0.15 to 0.20	0.15 to 0.20	0.15 to 0.20	0.15 to 0.20	0.15 to 0.20
Manganese	1.05 to 1.25	1.05 to 1.25	1.10 to 1.30	1.10 to 1.30	1.20 to 1.40
Sulphur not to exceed	0.069	0.069	0.069	0.069	0.069
Phosphorus not to exceed	0.06	0.06	0.06	0.06	0.06
Rails having carbon below will be rejected	0.43	0.45	0.48	0.53	0.60
Rails having carbon above will be rejected	0.57	0.59	0.62	0.65	0.70

The numbers represent the per cents of the various elements.

Rivet Steel.

The grade of steel ordinarily used for rivets is the softest, or lowest in hardeners, employed in engineering construction; it should thus be correspondingly low in phosphorus and carbon. In Table II of this article there will be found the measures of ductility and other physical properties of a number of specimens of rivet steel, which are fairly representative of that metal, except that the ultimate resistance is frequently much lower than is shown there. In much of the rivet metal used at the present time the ultimate tensile resistance may run from 52,000 to 60,000 pounds per square inch. In such steel the carbon may run down to .06 or .08 per cent. with sulphur between .02 and .03 per cent., and phosphorus even lower. The treatment to which rivet metal must be subjected in the heading of rivets makes it imperative that the metal possess qualities of ductility and toughness to an unusual degree and that the variations of temperature in the rivet shall not reduce its resisting capacity. In other words, rivet steel must possess physical properties enabling it to resist torturing treatment to the highest practicable degree.

Nickel Steel.

The alloy nickel steel, to which allusion has already been made in connection with the subject of the coefficient of elasticity of steel, possesses marked characteristics of high ultimate resistance and elastic limit, the latter usually running from six tenths to three fourths of the

TABLE XIV.

Shape of Member.	Kind of Steel.	Ultimate Strength, Pounds per Square Inch.	Elastic Limit, Pounds per Square Inch.	Elastic Ratio, Percentage.	Elongation in 8 Inches, Percentage.	Elongation in 2 Inches, Percentage.	Reduction of Area, Percentage.
Rounds.........	Nickel......	86,015	63,575	73.9	20.19	34.00	46.3
	Hard forging	87,663	58,055	66.2	16.70	24.44	30.3
	Forging.....	78,066	51,793	66.3	23.94	....	52.0
Angles..........	Nickel......	86,960	58,553	67.3	21.75	39.67	50.5
	Hard forging	87,820	54,153	61.7	19.25	34.83	43.3
	Forging.....	76,970	49,544	64.4	....	....	49.6
Universal plates, longitudinal....	Nickel......	85,773	58,410	68.1	21.08	39.25	52.0
	Hard forging	82,773	50,163	60.6	20.50	37.67	47.0
	Forging.....	78,996	46,654	59.1	26.78	....	52.1
Universal plates, transverse.....	Nickel......	86,417	58,203	67.4	16.50	28.92	36.1
	Hard forging	85,173	(50,000)*	(58.7)*	18.83	23.17	27.4
	Forging.						
Sheared plates, longitudinal....	Nickel......	85,337	58,169	68.1	19.00	35.50	48.1
	Hard forging	85,012	(50,000)*	(58.8)*	22.10	39.40	48.4
	Forging.....	78,918	49,128	62.3	22.03	....	50.8
Sheared plates, transverse.....	Nickel......	84,377	57,260	67.9	17.13	32.50	43.4
	Hard forging	84,327	(50,000)*	(59.3)*	21.71	37.00	41.3
	Forging.						

* Approximate; could not determine accurately.

former. The amount of nickel in the alloy is usually about 3 per cent., although it sometimes runs from 3.25 to 3.375 per cent. The amount of carbon in connection with the nickel usually runs from about .25 to .30 per cent.

Table XIV exhibits comparative tests of specimens of nickel and two grades of carbon steel, taken from the shapes and plates indicated. This table and the succeeding one are taken from a paper in the "Transactions of the American Society of Civil Engineers," for 1895, by Mr. H. H. Campbell, of the Penna. Steel Co. The results are averages of at least 3 in each case, and frequently more, but the greatest and least values do not generally differ more than about 2 to 3 per cent. from the average. The high ratio of the elastic limit over the ultimate resistance is at once noticeable and indicates one of the most valuable qualities of nickel steel.

The composition of these acid open-hearth steels is given in the following tabular statement.

COMPOSITION OF ACID OPEN-HEARTH STEELS.

	Carbon.	Man.	Sulphur.	Phos.	Nickel.	Copper.
Nickel steel	.24	.78	.027	.032	3.25	
Hard forging steel	.30 to .35	.60 to 1.00	.03 to .05	.03 to .05		
Forging steel	.25 to .30	.60 to .80	.03 to .07	.03 to .06	....	.20 to .40

The relative ductility under various conditions of these three grades of metal are further illustrated by the comparative miscellaneous tests of specimens shown in Table XV.

TABLE XV.

COMPARATIVE MISCELLANEOUS TESTS ON THE STEELS IN TABLE XIV.

Shape.	Kind of Steel.	Nature of Tests.
Rounds	Nickel	Bent cold, flat upon itself.
	Hard forging	Bent cold, flat upon itself.
	Forging	Bent cold, flat upon itself.
Rounds	Nickel	When nicked, six bars bent an average of 57° before cracking.
	Hard forging	When nicked, five bars bent an average of 54° before cracking.
	Forging	When nicked, six bars bent an average of 35° before cracking.
Rounds	Nickel	When quenched, bent flat upon itself.
	Hard forging	When quenched, bent flat upon itself.
	Forging	When quenched, bent flat upon itself.
Angles	Nickel	Bent cold, flat upon itself.
	Hard forging	Bent cold, flat upon itself.
	Forging	Bent cold, flat upon itself.
Angles	Nickel	When quenched, bent flat upon itself.
	Hard forging	When quenched, bent flat upon itself.
	Forging.	
Angles	Nickel	Opened out flat, without fracture.
	Hard forging	Opened out flat, without fracture.
	Forging	Opened out flat, without fracture.
Angles	Nickel	Closed about 45°, before fracture.
	Hard forging	Closed about 60°, before fracture.
	Forging	Closed shut, without fracture.
Sheared plates, longitudinal	Nickel	Bent cold, flat upon itself.
	Hard forging	Bent cold, flat upon itself.
	Forging	Bent cold, flat upon itself.
Sheared plates, longitudinal	Nickel	When quenched, broke at a very small angle of bend.
	Hard forging	When quenched, broke at a very small angle of bend.
	Forging.	

It should be observed from the results of both tables that the ductility of the nickel steel is in general in excess of that of the other two accompanying grades.

In the U. S. Report of Tests of Metals for 1898 there will be found the results of the tests of two specimens of nickel steel, one giving an ultimate resistance of 107,950 pounds per square inch and the other an ultimate resistance of 111,350 pounds per square inch. Again, in the corresponding report for 1899 there will be found the results of tests of nickel-steel specimens given in Table XVI and having the following chemical composition:

Carbon............	.30 per cent.	Phosphorus.....	.027 per cent.
Manganese.......	.73 " "	Copper..........	.035 " "
Silicon............	.203 " "	Nickel.........	3.375 " "
Sulphur..........	.035 " "		

TABLE XVI.

TENSILE TESTS OF NICKEL-STEEL SPECIMENS.

Diameter of Piece, Inches.	Pounds per Square Inch.		Percentage Final		Remarks.
	Elastic Limit.	Ultimate.	Stretch.	Contraction.	
.8	44,500	88,370	20.9	54.4	Annealed.
.8	56,000	88,900	21.	54.4	"
.79	73,000	97,290	18.6	56.9	Tempered and annealed.
.79	70,000	96,000	18.2	55.6	" " "
.21	57,310	91,680	26.5	56.2	Annealed.
.21	55,500	90,480	26.5	56.5	"
.21	74,210	100,770	23.	58.9	Tempered and annealed.
.21	73,630	101,410	24.	59.4	" " "

As would be expected, the annealed specimens gave less ultimate resistance and elastic limit than the tempered and annealed. In all cases the ductility is excellent, although the final stretch is materially less for the tempered and annealed specimens than for the annealed, as would be anticipated.

It has been stated that nickel steel carrying about 3 per cent. of nickel and .25 per cent. of carbon will yield a metal of ultimate resistance and elastic limit equivalent to a carbon steel of .45 per cent. carbon, but having the ductility of low carbon steel, and the preceding results with those that follow indicate that that statement is well within the actual results of tests.

The following tabular statements give the physical qualities of nickel steel adapted to the various purposes indicated. They are taken from results published in the *Railroad Gazette* for August 8th, 1902.

NICKEL-STEEL FORGINGS.

	Tensile Strength, Lbs.	Elastic Limit, Lbs.	Exten., Per Cent.	Cont., Per Cent.
Driving-wheel axles	99,310	64,170	25.00	53.76
Piston-rods	90,140	60,090	25.50	54.08
Main crank-pins	93,570	65,450	24.00	49.37
Front crank-pins	92,180	64,170	24.50	51.00
Connecting-rods and guides	92,040	59,820	26.00	53.01

NICKEL-STEEL CASTINGS.

	Tensile Strength, Lbs.	Elastic Limit, Lbs.	Exten., Per Cent.	Cont., Per Cent.
Crosshead	84,540	53,980	18.50	31.10
Furnace-bearer, bearer-guide	85,050	54,490	18.00	26.04
Annealed:				
Carbon steel	109,500	51,440	19.50	36.31
Nickel steel	100,330	66,720	25.00	54.56
Oil-tempered:				
Carbon steel	129,360	67,230	17.50	38.53
Nickel steel	103,890	76,390	25.00	61.56

SMALL RIFLE BARRELS—NICKEL STEEL.

Tensile Strength, Lbs.	Elastic Limit, Lbs.	Ext. in 2 Inches, Per Cent.	Cont. or Area, Per Cent.
115,100	99,820	23	64.00
114,080	97,780	23	64.95
114,590	99,820	23	65.45
116,620	96,770	22.50	62.05
116,120	97,780	23	64.00
114,590	98,800	24	62.53

Effect of Low and High Temperatures on Steel.

The results of some German experiments and the experience of the Massachusetts Railroad Commissioners with steel rails for one year, have already been given in connection with wrought iron.

Table XVII contains the results of the experiments by Mr. Chas. Huston, as given in the "Journal of the Franklin Institute" for Feb., 1878.

"U. R." is the ultimate resistance in pounds per square inch, while "C." is the per cent. of contraction at the section of fracture.

Each result is a mean of three experiments.

TABLE XVII.

Kind of Material.	"Cold."		572° Fahr.		932° Fahr.	
	U.R.	C.	U.R.	C.	U.R.	C.
Charcoal boiler-plate, piled	55,400	26	63,100	23	65,300	21
Siemens - Martin (exceptionally soft)........................	54,600	47	66,100	38	64,400	34
Crucible steel (ordinarily soft)..	64,000	36	69,300	30	68,600	21
Crucible steel (not quite hard enough to temper)............	78,400	27	82,800	16	77,300	20

The method of producing rupture at the desired place was such as to make the specimens partake, to some extent at least, of the nature of "short" ones, which, however, would not affect the *comparative* results.

It will be observed that the charcoal boiler-plate iron gave the highest resistance at the highest temperature, but that all the steels gave the highest "U. R." at the intermediate temperature 572° Fahr.

It is somewhat remarkable that in every case but the last (the hardest steel) the contraction of fractured section *decreased* with the rise in temperature.

Other results for steel will be found in Table XIV, of Article 44, and it will be seen that they tend to confirm the conclusions just drawn.

In the "Annales des Ponts et Chaussées" for Feb., 1881, page 226, are given the number of breakages of steel rails which occurred in Russia in 1879. The following is the table showing the number of failures for each month of the year.

These results conflict somewhat with those given by the Massachusetts Railroad Commissioners, in Art. 32.

January	699
February	598
March	854
April	235
May	235
June	160
July	247
August	156
September	214
October	328
November	341
December	692

The greatest number is found in the coldest half of the year, but the greatest number for any one month belongs to March, which is not the coldest month. It is probable that this is due to the effect of long wear on the frozen ground of the entire winter in connection with the possible alternate freezing and thawing of the ground in the month of March.

Prof. R. C. Carpenter of Cornell University has investigated the effect of high temperatures on the tensile resistance of wrought iron and steel circular test specimens .5 inch in diameter, and reported the results in the "Trans. of the Am. Soc. of Mechanical Engineers" for 1896. The number of wrought-iron specimens tested was about thirty, and about twenty of steel. Fig. 2 shows graphically the relation between the ultimate resistances and the corresponding temperatures in degrees Fahr.

The ductility represented by the final elongations or stretches in 8 inches at the corresponding temperatures of rupture are exhibited in Fig. 3.

Prof. Carpenter observes "that all the curves have a point of contraflexure at about 70° F., and another at a temperature not far from 500°. The maximum strength is found at temperatures of 400° to 550°. At temperatures higher than this all the materials show a rapidly decreasing strength."

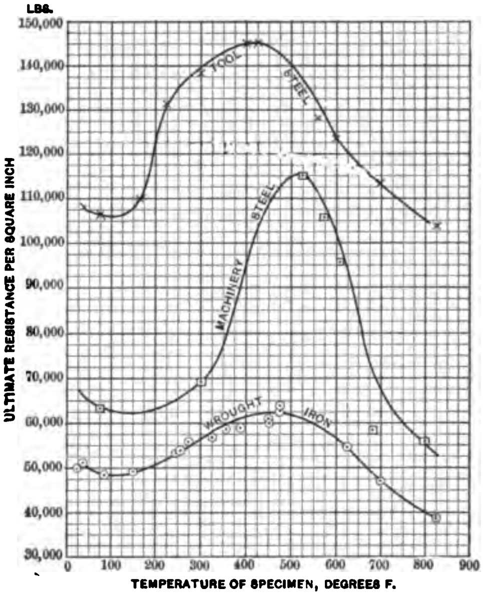


FIG. 2.

As a general result or consensus of all results, including the older and the later, it may be stated that iron and steel lose no sensible portion of their resisting capacity under about 500° Fahr., but that softening is liable to begin when the temperature rises much above that limit. At a temperature of about 800° Fahr. these metals may lose as much as 20 per cent. of ultimate resistance.

Hardening and Tempering.

The processes of hardening and tempering are not usually applied to structural steel, but to those higher grades of metal used for such special purposes as tools or wire. The hardening process consists in heating the steel to such temperature as may be desired to accomplish a given purpose and then quenching in water, brine, oil, molten lead, or other proper bath. The temperature from which the quenching is done may be that indicated

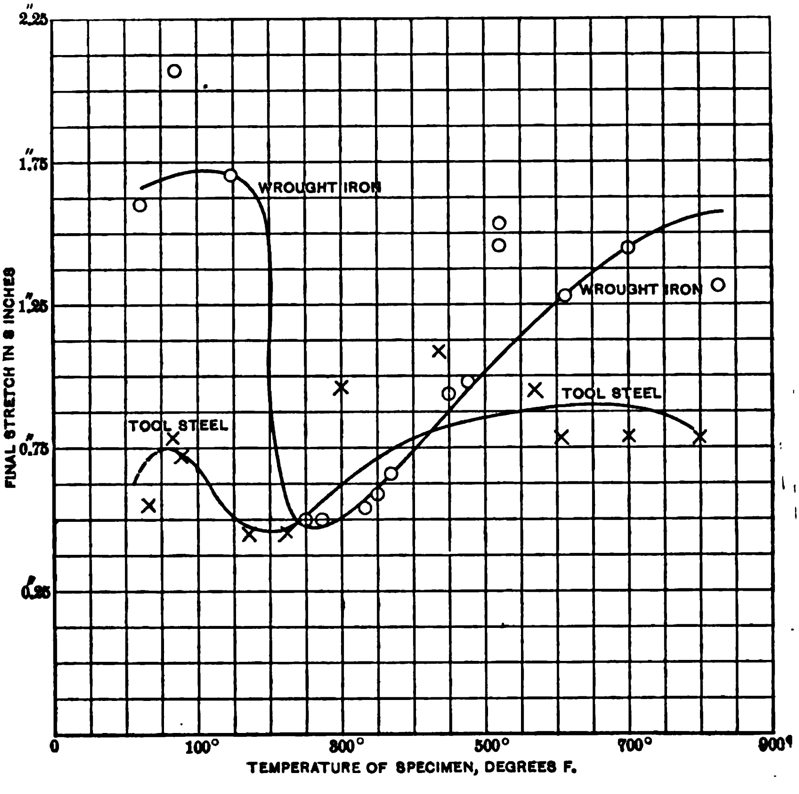


FIG. 3.

by an orange color; it depends upon the size or character of grain of metal desired. In general terms, the higher the content of carbon, the more marked will be the results of the hardening processes. Quenching has a comparatively small effect upon low or medium structural steel.

The process of tempering is, in reality, supplementary to the process of hardening in the manner just described. After a piece of steel has been hardened by quenching so that its temperature is that of the air, if it be again heated it will exhibit different colors as the temperature is increased. The first noticeable color will be a light delicate straw, then deep straw, light brown, dark brown, brownish blue, called "pigeon wing," light bluish, light brilliant blue, dark blue, and black, after which the temper is completely removed. The preceding colors are due to thin films of oxide that form on the exterior surfaces of the pieces as the temperature increases. When this heating is stopped at any color and the steel allowed to cool, the metal is said to be drawn to the temper shown by the corresponding color.

The tempers at different colors for different processes are sometimes stated as follows:

Light straw................	For lathe-tools, files, etc.
Straw....................	" " " " "
Light brown................	" taps, reamers, drills, etc.
Darker brown..............	" " " " "
Pigeon wing................	" axes, hatchets, and some tools.
Light blue..................	" springs.
Dark blue..................	" some springs, occasionally.

Tempering or hardening increases both the elastic limit and ultimate resistance, but decreases the ductility.

Annealing.

The processes of annealing, like those of hardening and tempering, produce more marked results in the higher steels than in the lower. Steel has a sensibly varying density at different temperatures; in other words, a given weight of metal will occupy sensibly different volumes at different temperatures. Hence if a piece of steel be subjected to any operation, such as forging, which gives to different portions concurrently widely varying temperatures, those portions will necessarily be subjected to considerable intensities of internal stresses, and if those stresses are not removed they may reduce greatly both the ultimate resistance and ductility. In the higher grades of steel and in special steels it is, therefore, imperative to anneal members which have been subjected to such operations. These observations are specially pertinent to such high steels as those adapted to the manufacture of tools or other similar purposes. In general it is necessary in structural engineering practice to resort to annealing only in the case of eye-bars, or other members which have been subjected to the operations of forging. The process consists simply in heating the member to be annealed to about a cherry-red temperature until the piece is heated through, and then allowing it to cool gradually to a normal temperature. At the cherry-red heat the metal is sufficiently softened to allow the molecules to readjust their relative positions so as to remove the internal stresses. After the operation of cooling is completed the metal will be at least approximately, if not entirely, in a condition of no internal stresses, i.e., if the annealing has been properly done. The more gradually and uniformly the cooling is accomplished the more excellent will be the results. Sometimes resort is made to such special means to accomplish these ends as covering the members, after bringing them to a proper temperature, with sand, ashes, or other similar material, to insure a slow and uniform cooling.

The preceding tables show what is always found in a comparison of results for the natural and the annealed metal. The process of annealing will diminish the ultimate resistance of structural steel in general from about 4,000 to 6,000 or 8,000 pounds per square inch, and the elastic limit will be reduced correspondingly. These effects will be found more marked as the metal is finished between the rolls at lower temperatures. In general, steel which is hardened by the conditions of manufacture, like that of comparatively low temperature in rolling, will exhibit greater decreases of ultimate resistance and elastic limit under annealing.

The process of annealing increases the ductility of the steel, since it softens the metal. In spite of the reduction in ultimate resistance and elastic limit, therefore, the operation gives a valuable quality to the steel.

Table XVIII exhibits the effects of simply annealing, or of first oil-tempering and subsequent annealing, on specimens of gun steel manufactured by the Midvale Steel Company of Phila., for the Ordnance Dep't U. S. Army, 1884. The results are taken from the Report of Chief of Ordnance for 1884. Oil-tempering, or hardening in oil, may be said to increase almost universally both the elastic limit and ultimate resistance.

Annealing in all cases reduces the ultimate resistance and increases the final stretch and final elongation, i.e., increases the ductility. Oil-tempering, with subsequent annealing, however, is seen in Table XVIII to produce a very irregular effect, although on the whole it slightly reduces the final elongation, except in the case of high

TABLE XVIII.

SPECIMEN TESTS.

Treatment.	Results Before Treatment.			Results After Treatment.		
	Ult. Resist, Pounds per Sq. Inch.	Per Ct. of Final		Ult. Resist, Pounds per Sq. Inch.	Per Ct. of Final	
		Elong.	Con.		Elong.	Con.
Annealed....................	100,500	10.5	15.6	87,700	16.0	20.0
"	80,400	9.0	8.0	71,500	25.5	39.7
Oil-tempered and annealed....	95,068	12.5	13.0	102,900	14.5	22.0
" " " "	88,900	20.5	33.0	106,600	17.5	35.0
" " " "	94,100	19.5	32.0	102,000	20.5	47.0
" " " "	89,500	21.5	31.0	100,200	17.5	35.0
" " " "	97,500	21.0	28.7	107,000	18.0	40.0
" " " "	89,000	18.5	27.0	107,000	19.0	39.0
" " " "	97,000	22.0	28.0	106,000	20.0	41.0
" " " "	86,132	11.5	12.0	97,720	15.0	31.0
" " " "	84,880	24.0	41.0	94,880	20.0	41.0
" " " "	88,400	20.0	37.0	86,920	24.0	48.0
" " " "	103,813	5.5	5.2	97,500	17.5	29.0
" " " "	95,820	13.0	13.0	94,700	19.0	40.0
" " " "	85,780	22.0	34.0	101,160	19.0	34.0

steel, for which the opposite effect is produced. In all cases the combined operations are seen to produce a very material increase in the final contraction.

Effect of Manipulations common to Constructive Processes; also Punched, Drilled, and Reamed Holes.

The shop treatment of steel must in some respects be peculiar to that metal, and different from that which characterizes the manufacture of iron-bridge members. The processes of punching and shearing have been considered very injurious to steel and comparatively uninjurious to wrought iron. More intimate acquaintance with steel, in the various shop processes requisite for the production of finished members, has shown that the idea is certainly erroneous for the milder grades of metal. There is no question that punching and shearing are injurious, but it is found that, for steel with a tensile resistance below 65,000 pounds per square inch, the injury may be even less than for iron. It is chiefly necessary that no crack, even the most minute, should be started in any punched or sheared surface, for the reason that such a minute fissure may, within a short time, extend itself injuriously before it is discovered. If punches and dies are maintained in a sharp condition the metal will be more cleanly severed and there will be less liability to this peculiar injury. Table XIX is taken from a paper by Mr. James Christie, M. Am. Soc. C. E., on the Treatment of Metals for Structural Purposes, in the Transactions of that society for October, 1893. It will be observed that the effect of punching is a great reduction both in the final contraction and in the final stretch. It has been found by these and other experiments that

TABLE XIX.

BARS ROLLED THREE INCHES WIDE, WITH HOLES THROUGH THE MIDDLE OF THE BAR, AND TREATED AS DESCRIBED.

Size of Bar Inches.	Treatment.	Ultimate Resistance, Pounds per Square Inch.	Elastic Limit, Pounds per Square Inch.	Final Stretch, Per Cent.	Final Contraction, Per Cent.	Fracture.
$3\times\frac{3}{8}$	As rolled.	65,900	41,800	29.	54.	Silky.
	Punched 1".	60,800	46,100	11.	13.	"
	" $\frac{7}{8}$" Reamed $\frac{15}{16}$"	52,600	39,500	12.	32.	"
	Drilled 1".	65,500	45,000	15.	29.	"
$3\times\frac{5}{8}$	As rolled.	61,900	36,400	31.	53.	"
	Punched 1"	52,300	51,100	2.3	5.6	Crystalline.
	" $\frac{7}{8}$" Reamed $\frac{15}{16}$"	60,200	46,700	14.	25.	Silky.
$3\times\frac{3}{4}$	As rolled.	60,300	37,300	31.	49.	"
	Punched 1"	46,300	44,200	7.3	3.7	Crystalline.
	" $\frac{7}{8}$" Reamed $\frac{15}{16}$"	55,000	45,700	4.	27.	"
$3\times\frac{7}{8}$	As rolled.	63,100	39,900	32.	55.	Silky.
	Punched 1"	51,000	47,200	2.3	4.	Crystalline.
	" $\frac{7}{8}$" Reamed $\frac{15}{16}$"	65,900	49,400	14.	20.	Silky.
$3\frac{3}{4}\times\frac{7}{8}$	As rolled.	70,500	37,900	26.7	47.	
	Drilled $\frac{15}{16}$"	73,000	39,000	10.		
	Punched $\frac{11}{16}$" Reamed $\frac{15}{16}$"	73,700	39,400	11.		

Four tests of each. Elongations of bars measured on 8-inch length, of perforated specimens on 3-inch length, except last one, on 2-inch length.

the region affected by the punch, or jaw of the shear, extends but a very short distance from the cutting edge of the tool. Within that region, however, the metal is very much hardened, and the loss of ductility and elevation of elastic limit is due to that hardening. The loss of ultimate resistance is probably due to the violent disturbance of the molecules, and the resulting minute fissures in the metal within the same region. The prejudicial effect is therefore removed by reaming from the punched hole a thin ring of metal about one sixteenth of an inch thick. It will be shown further on, by reference to some French tests, that usually a thickness of one twenty-fifth of an inch at most is sufficient for the removal of all injured metal, but as the die is usually about one sixteenth of an inch larger than the punch a little greater thickness must be removed in order to cover the requirements of the die side of the plate. It is probable that the metal on that side is less injured than that on the punch side. In all cases in Table XIX the ultimate resistance is reduced from ten to twenty-three per cent., with a very large elevation of the elastic limit. The effects on the thinner plates are somewhat less than on the thicker ones. Indeed it is possible to take a plate so thin that the prejudicial effects would disappear, and, as a general rule, it may be stated that the injurious effects increase with the thickness of the plate, as they also increase with the carbon, i.e., as the steel becomes higher in grade. It will be seen that reaming to the dimensions indicated removes essentially all the injurious effects of the punch so far as elastic limit and ultimate resistance are concerned, and the observation can be extended to the final stretch and final reduction, although, apparently, the table does not justify it. This is due to the fact that the reamed hole makes the specimen a short one at the section of fracture, and, as usual in such cases, both percentages of stretch and elongation are reduced. The same shortening effect raises. the ultimate resistance of the piece with the reamed hole above that of the original specimen.

Table XX, also from Mr. Christie's paper, shows the result of testing strips of steel plates of each given thickness, with widths varying from one and a half to three inches.

The general effects of the cutting edge of the shear are precisely the same as those of the punch, as the operation in each case is a shear. Hence, if sheared edges are planed off to a depth of one sixteenth to one eighth of an inch, the injured metal will be entirely removed. The hardening effects of both shearing and punching may also be removed by the process of annealing, although less effectually than by reaming and planing. As naturally would be inferred by experience in punching, higher steels and thicker plates are more injuriously affected by shearing than low steels and thinner plates.

TABLE XX.

No.	Thickness of Steel.	Treatment of Edges.	Pounds per Square Inch.		Per Cent, Final	
			Ultimate Resistance.	Elastic Limit.	Stretch.	Contraction.
1	$\frac{5}{16}''$	Sheared..................	64,600	45,100	10.6	28
2	$\frac{5}{16}''$	Planed..................	65,100	41,600	23.2	61.8
3	$\frac{1}{2}''$	Sheared..................	68,800	43,600	16.1	28.3
4	$\frac{1}{2}''$	Planed..................	69,000	43,000	25.7	57.8
5	$\frac{11}{16}''$	Sheared..................	64,300	41,600	16.5	25.
6	$\frac{11}{16}''$	Planed..................	61,600 doubtful	37,200	28.4	47.5
	General average of sheared edges......		65,900	40,600	14.4	27.1
	" " " planed edges......		67,000	40,600	25.8	55.7

Strips 1.5 to 3 inches wide, cut from plates 6 to 10 inches wide, $\frac{1}{16}$ to $\frac{1}{8}$ inch material removed from sheared edges by planing, to make planed edges.

In consequence of the irregular edge of a large sheared plate, bridge specifications frequently require that at least one quarter of an inch of metal shall be removed from the edge of such plates by planing.

It has been found that the deterioration of the riveted joints in steel plates with punched holes is frequently, and perhaps usually, appreciably less than might be expected from such tests with punched plates as have been just cited. This is probably due largely to a possible annealing effect of the hot rivet on the walls of the punched holes. Comparative tests of riveted joints with steel and iron plates would appear almost certainly to indicate that with steel running not higher than 65,000 to 70,000 pounds per square inch in ultimate resistance, and with a thickness not exceeding one half inch, the loss in such joints due to punched holes is less, and sometimes very materially less, for steel than for iron. Nevertheless, as there is a material gain in reaming, it is usual, as it is certainly the best practice, to remove the injured material with the reamer in all cases of steel plates, and if a thickness exceeds about three quarters of an inch it is not unusual to require holes to be drilled. It is also good practice, although not usually required, to remove the sharp corners of reamed and drilled holes so as to form a little fillet. This removes all liability of sharp shearing action on the shaft of the rivet, and it makes the contact of adjacent plates perfectly satisfactory.

Steel seems to be very sensitive to the effects of hammering or working at what is termed a "blue heat." Consequently it is necessary to heat the rivet to such a temperature as will enable the operation of heading to be completed before the rivet cools to the blue stage. A bright red or yellow heat is requisite for good work, and the rivet should be held under a pressure of fifty or sixty tons per square inch of the shaft section until the metal has time to flow throughout the rivet length and thus completely fill the hole, otherwise the upsetting will be complete at and in the vicinity of the rivet-heads only. An additional advantage in holding the rivet under the greatest pressure of the riveter for a short time is the fact that the rivet becomes cool enough to prevent the separation of the plates.

The forging of steel requires unusual skill and experience. When a piece has been heated to a proper temperature it should be kept under work until it has fallen in temperature to a proper point to secure all the advantages of working, but of course not below red heat. The forging should be done with a hammer whose weight is suitably proportionate to the mass to be forged. If the hammer is too light, the result will be a surface effect only, with the interior but little changed. Pressure forging, with appropriate facilities for attaining great pressures, is probably capable of producing the best results.

The operation of annealing, particularly as applied to full-size bars, is one of great importance in the manufacture of structural steelwork. The metal is heated as uniformly as possible, so that undue stresses will not be developed, to a bright cherry-red, corresponding probably to about 1100 or 1200 degrees Fahr., and then allowed to cool gradually. By this means any internal stresses that may have been produced by the process of forging, or any other shop manipulation, are eliminated. The metal is sufficiently softened at the highest temperature to allow the molecules to adjust themselves to a condition of essentially no stress, and if the cooling is gradual the internal stresses will not be re-developed.

TABLE XXI.

Carbon, Per Cent.	Number Bent Without Crack.	Number Cracked.	Number Breaking.	Per Cent.		
				Without Crack.	Cracking.	Breaking.
.13	17	1	0	94.4	5.6	0
.14	32	5	0	86.5	13.6	0
.15	30	6	2	78.9	15.8	5.3
.16	40	10	3	75.5	18.9	5.6
.17	35	18	7	58.3	30	11.7
.18	18	11	7	50	30	19.4
.19	16	5	14	45.7	14.3	40
.20	5	3	12	25	15	60
.21	4	5	6	26.7	33.3	40
.22	4	3	4	36.4	27.3	36.4
.23	1	0	7	11.5	0	87.5

The hardening of steel is accomplished in many different ways, but the common method of testing structural steel in this connection is by quenching in water, usually at about 80 degrees Fahr. In the paper by Mr. Christie, already cited, he gives an account of tests made with a series of three-quarter-inch square bars heated to a bright red, and quenched in water at 80 degrees Fahr. The specimens were then bent around a curve whose radius was not greater than the thickness of the piece.

This bending was accomplished with a light power hammer, but there is less liability of cracking when it is done with pressure. Table XXI shows his complete results, with the percentage of carbon in each piece. No specimen containing less than .13 per cent. of carbon cracked, but all containing more than .23 per cent. of carbon broke. The table shows what may be expected from this quenching test, with ordinary structural steel, as the latter usually will be found considerably within the limits of carbon shown.

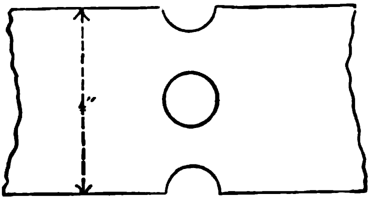


FIG. 4.

The experiments of Prof. Alex. B. W. Kennedy, on the effect of punching and drilling holes, in mild-steel boiler-plate, are well illustrated by Table XXII, which is condensed from one given in London *Engineering*, May 6, 1881. None of these plates was annealed, but all were drilled or punched as received.

Within the limits of these experiments, Prof. Kennedy observes, neither the width of the test piece nor the different diameters of die had any essential influence on the results.

The injurious effect of punching is shown by the fact that the punched specimens gave only 92 to 98 per cent. of the resistance of the drilled ones.

It will be noticed that both the drilled and punched specimens gave higher resistances than the natural plate. This is due to the "shortening" and other influence (i.e., the disturbance of the lateral strains) of the rivet-holes, as before observed, and explained in Art. 41, "*Ultimate Resistance and Elastic Limit.*"

A duplicate set of experiments on 32 specimens of a somewhat softer steel boiler-plate gave essentially the same results (see *Engineering*, May 6, 1881).

By experimenting on mild Fagersta steel plates with the thickness $\frac{1}{8}$, $\frac{1}{4}$, $\frac{3}{8}$, $\frac{1}{2}$, and $\frac{5}{8}$ inch, Mr. Kirkaldy found the ratio of the resistance of drilled specimens over that of punched ones to vary from about 1.1 (for $\frac{1}{8}$, $\frac{1}{4}$, and $\frac{3}{8}$-inch specimens) to 1.5 (for $\frac{5}{8}$ specimen) when unannealed, and to be about 1.1 for all the thicknesses when annealed. All the specimens were 12.5 inches wide, with three rows of 0.77-inch holes, pitched 2.5 inches apart, running across the specimens. The average resistance per square inch of net section was greater than that of the original plate for the drilled holes, but considerably less for the punched ones.

Table XXII.

PUNCHED AND DRILLED HOLES.

Thickness in Inches.	Hole.	Diameter of Hole. Inches.	Tenacity in Tons, Sq. Inches. Net Section.	Tenacity Compared with that of	
				Nat. Plate.	Drilled Plate.
$\frac{1}{2}$	Drilled...........	0.940	38.12	1.105	——
$\frac{1}{2}$	Drilled...........	0.940	38.22	1.108	——
$\frac{1}{2}$	$\frac{7}{8}$ punch, $\frac{31}{32}$ die....	0.912–0.876	35.04	1.000	0.918
$\frac{1}{2}$	$\frac{7}{8}$ punch, $\frac{29}{32}$ die....	0.892–0.871	34.44	1.025	0.902
$\frac{3}{8}$	Drilled...........	0.926	35.39	1.126	——
$\frac{3}{8}$	Drilled...........	0.934	34.90	1.111	——
$\frac{3}{8}$	$\frac{7}{8}$ punch, 1 die.....	0.998–0.890	33.91	1.073	0.965
$\frac{3}{8}$	$\frac{7}{8}$ punch, $\frac{15}{16}$ die....	0.945–0.875	34.38	1.096	0.978

Each result is a mean of four, from plate specimens 2, 4, 6, and 8 inches wide. The pitch of rivet-holes across the middle of specimens was 2 inches, and the width of each specimen was so chosen that each side passed through the centre of the rivet-hole, as shown in Fig. 4. A "ton" is 2,240 pounds. The two diameters of punched holes are for the two sides of the plate.

Mr. Kirkaldy states that "the loss from punching is not constant, but varies with the thickness and also with the hardness of the material." He also concluded that punching hardens the material in the vicinity of the punch, and that the effect of punching is counteracted "to a considerable extent" by annealing.

At the commencement of the use of steel for structural purposes the effects of punching and shearing were regarded with much apprehension, and many tests were made to ascertain the extent of the prejudicial influence. The preceding matter is practically a résumé of American and English experience.

Experiments on French steel plates, produced by the Bessemer and Martin processes (*métal fondu*), confirm this result and form a basis for other conclusions, as follows ("The Use of Steel," by J. Barba, A. L. Holley, translator, p. 40):

"1st. That the effects of punching and shearing are essentially local and spread only over a very restricted region, less than 0.039 inch on the edges of the sheared or punched parts;

"2d. That no cracks exist in this altered region;

"3d. That tempering destroys the effects of shearing and punching by bringing the metal back to the state it would be in if drilling or planing had been substituted for punching or shearing;

"4th. That annealing alone or after tempering destroys, as tempering alone does, the alterations caused by shearing and punching."

These conclusions relate to plates from 0.27 inch to 0.46 inch thick.

In first-class practice, holes in steel plates and shapes are frequently first punched and then reamed to a diameter 0.125 inch greater.

Experiments on some narrow specimens of steel plate seem to indicate that conical punching (the die 0.16 to 0.20 inch greater in diameter than the punch) injures the material less than cylindrical punching (with a clearance of perhaps $\frac{1}{16}$ inch).

In the working of steel plates and shapes, during ordinary constructive processes, all local pressure of great intensity, and hammering while cold or at a low temperature, tend to produce internal strains of great intensity or other changes in molecular condition which cause the finished plate or shape to be liable to great brittleness and unlooked-for failure of a local character.

For these reasons M. Barba gives the following directions in regard to the working of steel:

"1st. Avoid any local pressure of whatever nature it may be; 2d. If local pressures have been produced by blows of a hammer, the action of the punch, etc. (which may, as we have seen, cause ruptures), heat the piece to a cherry-red in a very regular manner and as much as possible in its entirety—the whole of it at once—and let it cool in the open air on a homogeneous surface, which has all over equal conducting power. This simple reheating, which may be considered as annealing for plates and bars, on account of their slight thickness, restores to the worked metal its original qualities, even if it was in a very unstable state of equilibrium."

If a large amount of working (such as bending or curving) of a single kind is to be done to a single piece, it is best, if possible, to heat to a cherry-red and do the work by stages, rather than all at once; and then anneal after the working is completed. If the working is local and the heating irregular, it may be necessary to anneal once or more during the progress of the work.

Local heating in the production of the ordinary steel eye-bar head, for example, frequently gives much trouble, unless resort is had to subsequent annealing.

These difficulties in the working of steel are found more pronounced in the higher grades.

On account of the homogeneous character of the metal, upsetting processes, as in riveting, etc., seem to injure the molecular condition of steel much less than that of iron.

Bauschinger's Experiments on the Change of Elastic Limit and Coefficient of Elasticity.

The details of these experiments are given in "Der Civilingenieur," Part 5, 1881. The manner of application of the tests, and remarks on the quantities, elastic limit, stretch-limit, and final load, will be found by referring to page 246. The following is the notation:

E. L. = elastic limit in pounds per square inch.
S.-L. = stretch-limit in pounds per square inch.
F. L. = final load in pounds per square inch.
E. = coefficient of elasticity in pounds per square inch.

BESSEMER STEEL.

	In Original Condition.	After 69 Hours.	After 0.5 Hour.	After 68 Hours.
E. L.	25,970	43,272	8,760	14,970
S.-L.	40,380	51,920	55,470	71,850
F. L.	46,140	57,690	70,080	———
E.	29,848,000	29,549,000	29,009,000	30,146,000

A small specimen of this Bessemer steel, about an inch in diameter, gave an ultimate resistance of 75,800 pounds per square inch.

The elastic limit rises twice after two long periods of rest and falls in a very marked manner after the short rest of 0.5 hour.

The stretch-limit rises steadily while the coefficient of elasticity falls twice and then rises above its original value.

Prof. Bauschinger was the first to determine, in regard to Bessemer steel, that by stretching the metal beyond its elastic limit its elasticity is elevated, not only during the time of action of the load, but also during a longer period of rest, without load, of one or more days; and that, in this manner, the elastic limit may exceed the load which caused the stretching. (Dingler's Journal, Band 224.)

Fracture of Steel.

The character of steel fractures has been incidentally noticed, in some cases, in the different tables.

If the steel is low, or if some of the higher grades are thoroughly annealed, the fracture is fine and silky, provided the breakage is produced gradually. In other cases the fracture is partly granular and partly silky, or wholly granular.

In any case a sudden breakage may produce a granular fracture.

The Effects of Chemical Elements on the Physical Qualities of Steel.

Anything more than a meagre statement of the influence of the chemical composition of steel on its physical properties is obviously out of place here, but a knowledge, however slight it may be, of the influence of certain elements on those properties is so essential to the engineer in his structural work that attention should at least be called to it.

Although other elements exert highly important influences upon the resisting qualities of steel, carbon is undoubtedly the most prominent hardener. The effect of a given percentage of carbon, at least within certain rather wide limits, is to give greater toughness and resisting qualities to steel with less concurrent brittleness than any other contained element. It is made, therefore, the basis of classification of structural steel, the low steels being low in carbon and the high steels high in carbon.

The metal manganese also gives to steel some advantageous qualities. At the present time it seldom enters steel to an amount less than .5 per cent., nor more than about 1 per cent. Its presence seems to confer the capacity of resisting the effects of high temperatures in shop processes. Metal low in phosphorus and sulphur appears to require less manganese than that which is higher in those impurities. It has been found that the influence of manganese upon steel depends in a rather extraordinary manner upon its amount. If the content reaches 1.5 or 2 per cent. steel becomes practically worthless on account of its brittleness, but when a content of 6 or 7 per cent. of manganese is reached, the metal becomes extremely hard and acquires to a high degree the property of toughness by quenching in water without becoming much harder.

When steel is alloyed with more than about 7 per cent. of manganese, manganese-steel is the product, which, in its natural state, may have an ultimate tensile resistance running from 74,000 to over 116,000 pounds per square inch. When quenched in water the ultimate tensile resistance of the same metal may run from about 90,000 pounds per square inch up to nearly 137,000 pounds per square inch. Before quenching the final stretch ranged from 1 to 4 or 5 per cent., and after quenching from 4 to 44 per cent. The preceding figures belong to a range in manganese from about 7 per cent. to over 19 per cent. concurrently with carbon from about .61 per cent. up to 1.83 per cent. This metal is an interesting alloy, but is never used in structural engineering work.

Opinions vary much as to the influence of silicon on steel, but it seems now to be well established that that influence within the limits ordinarily found is of minor consequence, or at least not prejudicial to either resistance or ductility. In structural steel it usually ranges from less than .03 to .05 per cent., while in rail steel it may run as high as .3 per cent. In some excellent tool-steel it may run even from .2 to .75 per cent.

Sulphur is an impurity carrying with it highly prejudicial effects. It essentially injures metal for rolling, as it makes the steel liable to crack and tear at the usual temperatures found between the rolls. It also diminishes capacity to weld. Its effects may, to some extent, be overcome by the presence of manganese and by proper care in heating. It is, however, highly prejudicial as an element and is usually kept below about .04 per cent.

Of all the objectionable elements found in steel, phosphorus has the position of primacy. Although it is a hardener which may increase the ultimate resistance to some extent, it produces brittleness and diminishes most materially the capacity to resist shock, and it is one of the chief purposes of the best methods of steel production to reduce phosphorus to the lowest practicable limit. Its effects are sometimes erratic, being occasionally found in excess in apparently good material. In structural steel it is seldom permitted to run over .08 per cent., and in the basic processes of manufacture it frequently falls to .03 or .04 per cent.

The presence of .1 to .25 per cent. copper appears to have no deleterious effect upon steel and may even be beneficial. As high as 1 per cent. of copper has been found in steel without serious effects where sulphur was low.

Aluminum steel is an alloy containing at times as high as 5 to 6 per cent. of aluminum. The effect of aluminum on ultimate resistance does not seem to be prejudicial, nor, again, is it of any special advantage; nor does it act seriously upon the ductility until its amount approaches about 2 per cent. or more. On the whole it does not seem to be a valuable element for steel.

There are other special alloys such as tungsten and chromium steel. They are used for the special purposes of tools on account of their hardness, which is so extreme that neither quenching nor tempering is required. They do not, however, enter into structural use.

As an illustration of the effect of some of the more prominent constituents which enter the composition of steel, Table XXIII is taken from a paper presented by

TABLE XXIII.

For Carbon, Hundredths Per Cent.	Increase of Ultimate Resistance per 0.01 Per Cent. P added, Lbs. per Sq. Inch.	Effect of Unit of P to Unit of C as 1 to
9	900	$1\frac{1}{8}$
10	1,000	$1\frac{1}{4}$
11	1,100	$1\frac{3}{8}$
12	1,200	$1\frac{1}{2}$
13	1,300	$1\frac{5}{8}$
14	1,400	$1\frac{3}{4}$
15	1,500	$1\frac{7}{8}$
16	1,500	$1\frac{7}{8}$
17	1,500	$1\frac{7}{8}$

INCREASE IN ULTIMATE RESISTANCE BY SUCCESSIVE INCREMENTS OF MANGANESE

Manganese, Per Cent.		Increase in Ultimate Resistance.	Total Increase in Ultimate Resistance from 0. Manganese.
From	To	Lbs. per Square Inch.	Lbs. per Square Inch.
0.00	0.15	3,600	3,600
0.15	0.20	1,200	4,800
0.20	0.25	1,100	5,900
0.25	0.30	1,000	6,900
0.30	0.35	900	7,800
0.35	0.40	800	8,600
0.40	0.45	700	9,300
0.45	0.50	600	9,900
0.50	0.55	500	10,400
0.55	0.60	500	10,900
0.60	0.65	500	11,400

Mr. Wm. R. Webster, C.E., before the Chicago meeting of the Am. Inst. of Mining Engineers, at the International Engineering Congress, August, 1893. The investigations relate to basic Bessemer steel plates (both sheared and universal rolled), varying in ultimate resistance from about 50,000 to about 80,000 pounds per square inch. While phosphorus (P), sulphur (S), carbon (C), and manganese (Mn) are all "hardeners," the influence of the two former is wholly prejudicial.

According to these investigations of Mr. Webster, the increase of ultimate resistance per square inch in this particular metal for each .01 per cent. increase of hardener depends upon the amount of that and other hardeners.

Art. 44.—Copper, Tin, Aluminum, and Zinc, and their Alloys—Alloys of Aluminum—Phosphor-bronze—Magnesium.

Table I gives the tensile coefficients of elasticity (*E*) of copper and the alloys indicated as determined by Prof. R. H. Thurston.

TABLE I.

Metal.	Authority.	*E*.	Remarks.
Gun-bronze....	Thurston	11,468,000	Copper, 0.90; tin, 0.10 (nearly).
Alloy.........	"	13,514,000	Copper, 0.80; zinc, 0.20.
Alloy.........	"	14,286,000	Copper, 0.625; zinc, 0.375.
Tobin's alloy...	"	4,545,000	Composition, below table.
Copper........	"	9,091,000	Cast metal.

Tobin's alloy is a composition of copper, tin, and zinc, in the proportions (very nearly) of 58.2, 2.3, and 39.5, respectively. The value of *E* for this metal, and those for the two preceding and one following it, are calculated for small stresses and strains given by Prof. Thurston in the "Trans. Am. Soc. Civ. Engrs.," for Sept., 1881.

There will also be found in Tables IX, X, XI, and XII coefficients of elasticity for aluminum-zinc, aluminum-magnesium, and other alloys, and for magnesium, aluminum, and zinc.

TABLE II.

CAST TIN.

p.	E.	p.	E.
1,950	1,147,000	3,200	96,400
2,360	472,000	4,000	41,540
2,580	172,000	Broke at 4,200 lbs.	

TABLE III.

CAST COPPER.

p.	E.	p.	E.
800	10,000,000	12,000	18,750,000
2 000	9,091,000	13,600	8,193,000
4,000	9,091,000	16,000	2,235,000
8,000	14,815,000	22,000	137,000

Broke at 29,200 lbs.

The values of E (stress over strain) for different intensities of stress (pounds per square inch) for cast tin, cast copper, and Tobin's alloy, are given in Tables II, III, and IV.

" p " is the intensity of stress in pounds per square inch, at which the ratio E exists.

Each of these metals is seen to give a very irregular elastic behavior.

Tables II, III, and IV are computed from data given by Prof. Thurston in the United States Report (page 425) and "Trans. Am. Soc. Civ. Engrs.," already cited.

TABLE IV.

TOBIN'S ALLOY.

p.	*E*.	*p*.	*E*.
2,000	4,545,000	18,000	5,455,000
4,000	4,545,000	24,000	5,941,000
6,000	4,688,000	30,000	6,250,000
8,000	4,938,000	40,000	6,390,000
10,000	5,263,000	50,000	4,744,000
14,000	5,110,000	60,000	3,436,000

Broke at 67,600 lbs.

Ultimate Resistance and Elastic Limit.

Table V is abstracted from the results of the experiments of Prof. Thurston as given in the "Report of the U. S. Board Appointed to Test Iron, Steel, and other Metals," and "Trans. Am. Soc. of Civ. Engrs.," Sept., 1881. The composition of the various alloys was as given in the table, which also contains results for pure copper, tin, and zinc. All the specimens were of cast metal.

The mechanical properties of the copper-tin-zinc alloys have been very thoroughly investigated by Prof. Thurston ("Trans. Am. Soc. of Civ. Engrs.," Jan. and Sept., 1881). As results of his work he has found that the ultimate tensile resistance, in pounds per square inch, of "ordinary bronze, composed of copper and tin, ... as cast in the ordinary course of a brassfounder's business," may be well represented by

$$T_c = 30,000 + 1,000t;$$

"where t is the percentage of tin and not above 15 per cent."

TABLE. V.

Percentage of			Pounds Stress per Square Inch at		Per Cent., Final	
Copper.	Tin.	Zinc.	Elastic Limit.	Ultimate Resistance.	Stretch.	Contraction.
100	00	00	11,620	19,872	0.05	10.0
100	00	00	11,000	12,760	0.005	8.0
100	00	00	14,400	27,800	0.065	15.0
90	10	00	15,740	26,860	0.037	13.5
80	20	00	——	32,980	0.004	00.0
70	30	00	5,585	5,585	——	00.0
62	38	00	688	688	——	00.0
52	48	00	2,555	2,555	——	00.0
39	61	00	2,820	2,820	——	00.0
29	71	00	——	1,648	——	00.0
21	79	00	——	4,337	——	00.0
10	90	00	3,500	6,450	0.07	15.0
00	100	00	1,670	3,500	0.36	75.0
	Queensl'd					
00	100	00	——	2,760	——	47.0
	Banca					
00	100	00	2,000	3,500	0.36	86.0
Gun	Bronze					
90	10	00	10,000	31,000	4.6	——
80	00	20	——	33,140	32.4	40.0
62.5	00	37.5	——	48,760	31.0	29.5
58.2	2.3	39.5	——	67,600	4.0	8.0
100	0.0	0.0	——	29,200	7.5	16.0
90.56	0.0	9.42	——	——	——	——
81.91	0.0	17.99	10,000	32,670	31.4	43.0
71.20	0.0	28.54	9,000	30,510	29.2	38.0
60.94	0.0	38.65	16,470	41,065	20.7	28.0
58.49	0.0	41.10	27,240	50,450	10.1	17.0
49.66	0.0	50.14	16,890	30,990	5.0	11.5
41.30	0.0	58.12	3,727	3,727	——	——
32.94	0.0	66.23	1,774	1,774	——	——
20.81	0.0	77.63	9,000	9,000	0.16	0.0
10.30	0.0	88.88	14,450	14,450	0.39	0.0
0.0	0.0	100.00	4,050	5,400	0.69	0.0
70.0	8.75	20.25	18,000 (?)	31,600	0.36	0.0
57.50	21.25	21.25	1,300	1,300	——	——
45.0	23.75	31.25	2,196	2,196	——	——
66.25	23.75	10.00	3,294	3,294	——	——
58.22	2.30	39.48	30,000 (?)	66,500	3.13	7.0
10.00	50.00	40.00	5,000 (?)	9,300	0.7	0.0
60.00	10.00	30.00	21,780 (?)	21,780	0.15	0.0
65.00	20.00	15.00	——	3,765	——	——

TABLE V.—*Continued.*

Percentage of			Pounds Stress per Square Inch at		Per Cent., Final	
Copper.	Tin.	Zinc.	Elastic Limit.	Ultimate Resistance.	Stretch.	Contraction.
70.00	10.00	20.00	24,000(?)	33,140	0.31	—
75.00	5.00	20.00	12,000(?)	34,960	3.2	5.4
80.00	10.00	10.00	12,000(?)	32,830	1.6	4.0
55.00	0.50	44.50	22,000	68,900	9.4	25.0
60.00	2.50	37.50	22,000	57,400	4.9	6.6
72.50	7.50	2.00	11,000	32,700	3.7	11.0
77.50	12.5	10.00	20,000	36,000	0.7	0.0
85.00	12.5	2.50	12,000(?)	34,500	1.3	3.0

The values of the elastic limit in the lower part of the table were not at all well defined.

"For brass (copper and zinc) the tenacity may be taken as:

$$T_z = 30,000 + 500z,$$

where z is the percentage of zinc and not above 50 per cent."

He found that a large portion of the copper-tin-zinc alloys is worthless to the engineer, while the other, or valuable portion, may be considered to possess a tenacity, in pounds per square inch, well represented by combining the above formulæ as follows:

$$T_{tz} = 30,000 + 1,000t + 500z.$$

These formulæ are not intended to be exact, but to give safe results for ordinary use within the limits of the circumstances on which they are based.

Prof. Thurston found the "strongest of the bronzes" to be composed of:

Copper	55.0
Tin	0.5
Zinc	44.5
	100.00

This alloy possessed an ultimate tensile resistance of 68,900 pounds per square inch of original section, an elongation of 47 to 51 per cent. and a final contraction of fractured section of 47 to 52 per cent.

The first and sixth alloys of copper, tin, and zinc, in Table V, are called by Professor Thurston "Tobin's alloy." "This alloy, like the maximum metal, was capable of being forged or rolled at a low red heat or worked cold. Rolled hot, its tenacity rose to 79,000 pounds, and when moderately and carefully rolled, to 104,000 pounds. It could be bent double either hot or cold, and was found to make excellent bolts and nuts."

As just indicated for the particular case of the Tobin alloy, the manner of treating and working these alloys exerts great influence on the tenacity and ductility.

Baudrimont found for a copper wire 0.0177 inch in diameter, an ultimate resistance of about 45,000 pounds per square inch, the wire being unannealed, while for a diameter of 0.064 inch, Kirkaldy found about 63,000 pounds per square inch.

Professor Thurston states: "brass, containing copper 62 to 70, zinc 38 to 30, attains a strength in the wire mill of 90,000 pounds per square inch, and sometimes of 100,000 pounds."

All of Professor Thurston's specimens were what may be called "long" ones, i.e., they were turned down to a diameter of 0.798 inch for a length of five inches, giving an area of cross-section of 0.5 square inch.

The U. S. Report of Tests of Metals for 1897 contains a number of tests of bronze castings and cast copper, a synopsis of which is given in the following two tables:

TENSILE TESTS OF BRONZE DRY SAND CASTINGS (COPPER-TIN-ZINC ALLOYS), THE SECTIONAL AREA BEING .25 SQUARE INCHES AND THE GAUGED LENGTH 2 INCHES.

Per Cent.			Ultimate Resistance, Lbs. per Sq. Inch.		Final Per Cent. of		Number of Tests.
Copper.	Tin.	Zinc.			Stretch.	Contraction.	
57.5	.5	42	Max.	59,220	37	42	30
			Mean	55,000	25.3	31.7	
			Min.	50,760	17.5	21.6	
88.	10.	2	Max.	34,640	20.5	21.6	4
			Mean	30,700	17.4	20	
			Min.	25,280	12.5	15.2	

TENSILE TESTS OF CAST COPPER.

Sect. Area Specimen, Square Inch.	Ultimate Resistance, Lbs. per Square Inch.	Final Per Cent. of		Gauged Length, Inches.	Remarks.
		Stretch.	Contraction.		
.448	28,840	39.8	41.1	4	
.792	18,720	14.6	19.7	10	
.795	22,040	18.2	28.7	5	Cast in chill
.449	23,360	27.2	28.3	4	" " sand

General Results.

Table VI gives general results of various European experimenters. *T* represents the ultimate tensile resistance in pounds per square inch.

Some of these results are from the experiments of early investigators, who attached little importance to the size and form of the test specimen. In all the cases the results would be more valuable if the circumstances of testing were given. Those belonging to the more unusual alloys, however, possess considerable general interest in spite of the uncertainty surrounding their experimental origin. The presence of a little phosphorus in copper is seen to increase its resistance in a marked manner.

Table VI.

Metal.	Experimenter.	*T*.
Copper, wrought	Anderson	33,600
Copper, cast	"	19,000 to 26,100
Copper, bolts, with phosphorus 0.01	"	16,900
Copper, bolts, with phosphorus 0.015	"	38,400
Copper, bolts, with phosphorus 0.02	"	45,400
Copper, bolts, with phosphorus 0.03	"	47,900
Copper, bolts, with phosphorus 0.04	"	50,000
—Proportions.—		
Gun metal, copper 12, tin 1	"	29,000
Gun metal, copper 11, tin 1	"	30,700
Gun metal, copper 10, tin 1	"	33,000
Gun metal, copper 9, tin 1	"	38,100
—Weights in 100.—		
Alloy, copper 84.29, tin 15.71	Mallet.	36,100
Alloy, copper 82.81, tin 17.19	"	34,050
Alloy, copper 81.10, tin 18.90	"	39,650
Alloy, copper 78.97, tin 21.03, brasses	"	30,500
Alloy, copper 34.92, tin 65.08, small bells	"	3,140
Alloy, copper 15.17, tin 84.83, speculum metal	"	6,950
Tin	"	5,600
—Proportion.—		
Aluminum bronze, copper 90, Al. 1	Anderson.	73,000
Aluminum bronze, greatest	"	96,300
Tin, cast	Rennie.	4,740
Zinc, cast	Stoney.	3,000
Brass, yellow	Rennie.	18,000
Brass, yellow, copper 67, zinc 33	Anderson.	28,900
Brass, tube, copper 62, zinc 38	Everitt.	103,000
Brass, tube, copper 70, zinc 30	"	80,600
Brass, wire	Dufour.	91,300
Muntz metal, copper 60, zinc 40	Anderson.	49,300
Sterro-metal, copper 10, iron 10, zinc 80	"	7,100
Sterro-metal, copper 60, iron 3, zinc 39, tin 1.5	"	53,800
Sterro-metal, copper 60, iron 4, zinc 44, tin 2.0		
—cast in sand	"	43,100
—cast in iron, annealed	"	54,300
—cast in iron, forged red hot	"	69,400
Copper 60, iron 2, zinc 37, tin 1	"	76,200
Copper 60, iron 2, zinc 35, tin 2	"	85,100
Copper 55, iron 1.77, zinc 42.36, tin 0.83—cast	"	60,500
—forged red hot	"	76,200
—drawn cold	"	85,100

Alloys of Aluminum.

Prof. R. C. Carpenter, of Cornell University, in the transactions of the Am. Soc. Mech. Engrs., vol. xix, has reported a number of interesting and valuable tests of alloys of aluminum, as well as tests of pure magnesium.

TABLE VII.

ALLOYS OF GREATEST RESISTANCE.

Percentage of			Ultimate Resistance, Lbs. per Square Inch.	Specific Gravity.	Per Cent. of Final Stretch.
Aluminum.	Copper.	Tin.			
85.	7.5	7.5	30,000	3.02	4.
6.25	87.5	6.25	63,000	7.35	3.8
5.	5.	90.	11,000	6.82	10.1

The greater part of the results for the aluminum-tin-copper alloys are given in Table VIII, but the composition of those giving the greatest ultimate resistances are exhibited in Table VII. It will be observed that the highest ultimate resistance belongs to the alloy of greatest density, but the alloy of least resistance has nearly as great density. The ductility of none of these alloys of greatest ultimate resistance is specially marked; indeed, the ductility is very low except in the case of the least ultimate resistance.

The composition and corresponding elastic limits and ultimate resistances of aluminum-tin-copper alloys will be found in Table VIII. Like all the aluminum alloys the specific gravity varies between wide limits, being low where there is much aluminum and high where there is little. The ductility is low in all cases except in that of pure tin or the alloy in which it appears to the extent of 90%. There is in this table the usual wide range of physical qualities belonging to such a series of mixtures.

TABLE VIII.

ALUMINUM ALLOYS.

Composition, Per Cent. by Weight.			Ultimate Resistance, Lbs. per Square Inch.		Elastic Limit, Lbs. per Square Inch.	Specific Gravity.	Final Stretch Per Cent. in 6 Inches.
Al.	Tn.	Cu.	*A.*	*B.*			
......	....	100	27,000	28,330	12,000	6.5	6.5
5	5	90	40,815	42,038	13,832	7.6	4.0
10	10	80	32,209	34,200	24,829	6.5	0.8
20	20	60	1,966	2,225	*	5.7	
30	30	40	849	1,077	*	5.05	
40	40	20	4,800	5,672	*	4.91	
100	....	....	15,000	14,316	6,432	2.67	5.6
90	5	5	15,476	17,070	8,227	2.82	3.
80	10	10	18,580	21,140	13,329	3.09	1.2
60	20	20	4,416	5,950	*	3.53	.3
40	30	30	915	1,123	*	4.4	
20	40	40	2,221	2,622	*	5.21	
......	100	....	3,505	3,933	....	7.3	35.51
5	90	5	11,582	10,418	4,823	6.77	10.15
10	80	10	5,999	5,922	2,988	6.24	1.1
20	60	20	1,198	1,200	*	5.55	
30	40	30	993	961	*	4.96	
40	20	40	3,798	3,997	*	4.48	

A. Results of first melting. *B*. Results of second melting.
Test pieces 6 in. between shoulders, diam. $\frac{1}{2}$ inch.
* Could not be turned in the lathe.

The results in this table were obtained by Messrs. Gebhardt and Ward, at the testing laboratory of Sibley College of Mechanical Engineering, Cornell University, in 1896.

The physical properties of aluminum-zinc alloys, including those metals unalloyed, are equally fully given in Table IX, as well as values of the coefficients of elasticity. There is not as wide variation of results in this table as in Table VIII, although there is a considerable range of ultimate resistance, especially if the results for unalloyed zinc be included. It will be observed that this table also includes the intensity of stress found in

TABLE IX.

ALUMINUM-ZINC ALLOYS.

Percentage.		Specific Gravity.	Ultimate Resistance, Lbs. per Sq. In.	Transverse Tests. Maximum Fibre Stress Lbs. per Sq. In.	Coefficient of Elasticity.	Remarks.
Aluminum.	Zinc.					
100	0	2.67	14,460	14,500	6,535,000	Shrinkage uneven.
100	0	2.67	16,750	14,150	....	" "
90	10	2.77	17,940	18,950	7,710,000	
90	10	2.74				
85	15	2.918	....	28,091	9,260,000	
85	15	2.918	18,100	....	....	Shrinkage uneven.
80	20	2.998	21,850	....	9,110,000	" "
80	20	2.975	....	34,600	....	" "
75	25	3.15	22,940	....	8,210,000	" "
75	25	3.14	....	45,080	....	" "
70	30	3.191	24,400	43,200	8,178,000	Shrinkage even.
70	30	3.24	23,950	41,200		
66⅔	33					
65	35	3.326	....	....	....	Poor specimen.
60	40	3.471				
60	40	3.57	19,770	40,350	8,540,000	
50	50	....	19,300	38,100		
50	50	....	19,060	39,850	8,500,000	
25	75	....	13,175	25,500		
25	75	....	14,150	....	8,670,000	
0	100	7.19	2,522	7,556	6,680,000	Elongation of all the specimens less than 1 per cent.

NOTE.—The experimental results given in Table IX are those of Messrs. Hunt and Andrews, obtained at Sibley College of Mechanical Engineering, Cornell University, in 1894.

TABLE X.

TENSILE TESTS OF MAGNESIUM—CAST METAL.

Number of Test Piece.	Diameter.	Ultimate Resistance, Lbs. per Square Inch.	Elastic Limit, Lbs. per Square Inch.	Final Extension, Per Cent.	Coefficient of Elasticity.
1.........	.433	23,800	8,800	4.2	2,040,000
2.........	.433	22,050	....	....	1,860,000
3.........	.442	20,900	10,780	1.8	2,060,000
4.........	.435	19,500	8,400	2.5	1,830,000
5	.424	24,800	7,090	3.1	1,930,000
6.........	.432	22,500	....	2.3	

TABLE XI.

ALLOYS OF ALUMINUM AND MAGNESIUM

Number of Test Piece.	Percentage of Magnesium.	Specific Gravity.	Ultimate Resistance, Lbs. per Square Inch.	Elastic Limit, Lbs. per Square Inch.	Coefficient of Elasticity.
1.........	0	2.67	13,685	4,900	1,690,000
2.........	2	2.62	15,440	8,700	2,650,000
3.........	5	2.59	17,850	13,090	2,917,000
4.........	10	2.55	19,680	14,600	2,650,000
5.........	30	2.29	5,000		

the extreme fibres of beams subjected to transverse loading. Although these values are not required at this point, it is more convenient to insert them here and refer to them in the article devoted to the flexure of such beams. The sizes of the specimens subjected to transverse loading are not given, but they were small.

TABLE XII.

Character of Alloys.						Resistance, Pounds per Square Inch.			
	Test Piece.	Composition and Remarks.				Tension.		Transverse.	
		Al.	Cu.	Tin.		Elastic.	Ultimate.	Elastic.	Ultimate.
		%	%	%					
Cast.	1	100	....	....	....	4,000	12,055	2,345	......
	2	93	7	....	....	6,250	18,555	9,000	25,250
	3	75.7	3	....	20% zinc, 1.3 man.	....	35,075	....	23,420
Rolled.	4	100	....	....	1 inch bars......	12,500	17,185	17,154	......
	5	100	....	....	¾ " "	....	....	18,870	......
	6	98	2	....	1 " "	9,000	18,647	13,720	......
	7	98	2	....	¾ " "	....	....	18,870	......
	8	96	4	....	1 " "	16,000	23,045	22,300	......
	9	96	4	....	¾ " "	....	....	30,880	......
	10	96.5	2	....	1½% chromium.	19,000	26,310	26,313	......
Cast Tin Alloys.	11	98	....	2	....	2,150	8,622	....	......
	12	96	....	4	....	2,400	9,565	....	......
	13	94	....	6	....	2,250	9,315	....	......
	14	92	....	8	....	2,000	7,270	....	......
	15	90	....	10	....	1,750	7,352	....	......

The experimental results given in Tables X and XI were also established at the testing laboratory of Sibley College of Mechanical Engineering of Cornell University. The tests were made by Messrs. Marks and Barraclough, graduate students in 1893. Table X gives results for pure magnesium, including the coefficients of elasticity and the final stretch, while Table XI exhibits the results for alloys of aluminum and magnesium, the per cent. of magnesium being shown in one of the columns, the remaining per cent. being aluminum. The ultimate resistances given in Table X show that magnesium is a metal of considerable tensile resistance, especially in comparison with its density, its specific gravity being but 1.74, that of aluminum being 2.67.

Table XII exhibits the elastic limits and ultimate

TABLE XII.—*Continued.*

Final Stretch Per Cent. (Tension Pieces).	Final Contraction, Per Cent. (Tension Pieces).	Hardness (Relative).	Specific Gravity of Specimen.		Coefficient of Elasticity, Lbs. per Square Inch.	
			Tension.	Transverse.	Tension.	Transverse.
5.62	10.93	3.61	2.670	2.654	8,385,000	8,440,000
1.00	3.08	12.87	2.830	2.810	11,115,000	8,065,000
.15	1.77	35.56	3.117	3.055	9,685,000	8,060,000
8.49	38.30	7.12	2.710		9,780,000	10,110,000
....	....	6.94	2.715		....	10,000,000
19.49	39.02	6.79	2.725		9,505,000	10,330,000
....	....	12.30	2.756		....	9,600,000
3.62	10.10	12.42	2.774		10,440,000	10,595,000
....	....	13.35	2.773		....	10,070,000
1.31	9.78	14.09	2.759		9,850,000	9,813,000
4.00	8.64	3.71	2.689		5,435,000	
5.38	6.86	3.74	2.739		6,210,000	
5.19	7.97	3.49	2.771		5,035,000	
3.06	5.41	3.33	2.804		5,175,000	
3.87	8.89	3.09	2.856		6,675,000	

resistances of all the different alloys shown in the table, and in the conditions also exhibited by the table, i.e., whether cast or rolled. There are also given coefficients of elasticity for both tension and transverse tests, as well as elastic limits and ultimate stresses (intensities) in the extreme fibres of small beams, to which reference will be made in the article devoted to transverse resistance.

It will be observed that both the elastic limits and the ultimate resistances of Table XII are found within the range exhibited by the results already shown in the preceding tables.

If desired, diagrams can readily be constructed from the results of each table which will show the variations of physical quantities corresponding to the variations of composition of the alloys.

In 1895 the Fairbanks Company tested at their New York office four specimens of Tobin bronze manufactured by the Ansonia Brass and Copper Co., with the following results.

ROLLED TOBIN BRONZE PLATES—SPECIMENS 8 INCHES LONG.

Specimen, Inches.	Resistance in Pounds per Sq. Inch.		Per Cent., Final	
	Elastic.	Ultimate.	Stretch.	Contraction.
1×.185	51,350	78,920	20.5	45.4
1×.185	51,350	78,810	17.5	44.33
1×.25	56,000	79,200	17.5	43.2
1×.25	56,450	79,640	16.25	40.72

Phosphor-bronze, and Brass and Copper Wire.

Table XIII contains the results of the experiments of Mr. Kirkaldy on phosphor-bronze, with two results each for brass and copper wire.

The diameter of the phosphor-bronze wire varied from about 0.06 inch to 0.11 inch; that of the copper wire was 0.064 inch, and that of the brass wire 0.0605 inch.

The final stretch is the per cent. of the original length, and belongs to the annealed wire.

TABLE XIII.

PHOSPHOR-BRONZE.

Metal.	*E. L.*	Ultimate Resistance, Pounds per Square Inch.		Final Stretch.
		Unannealed.	Annealed.	
Phosphor-bronze	55,800	75,000		
"	55,200	74,000		
"	40,500	63,700		
"	26,300	54,100		
"	21,700	50,100		
" wire		102,750	49,400	37·5
" "		121,000	47,800	34·1
" "		121,000	53,400	42.4
" "		139,100	54,200	44·9
" "		159,500	58,900	46.6
" "		151,100	64,600	42.8
Copper wire		63,100	37,000	34·1
Brass wire		81,200	51,500	36.5

The contraction of fractured section for the phosphor-bronze specimens varied from about four to thirty-two per cent. of original area.

The first five results belong to metal of the same composition, but subjected to different treatment.

Some specimens tested by Mr. Kirkaldy gave as low as about 21,700 pounds per square inch.

Variation of Ultimate Resistance and Stretch at High Temperatures.

The results contained in Table XIV were obtained at Portsmouth (England) Dockyard, and were published in the *Engineer* Oct. 5, 1877. "*R*" is the ultimate tensile resistance in pounds per square inch, and "*St.*" is the per

TABLE XIV.

Temperature, Fahr.	Gun Metal Rods 1 Inch in Diameter.											
	Copper......87.75 Tin.......... 9.75 Zinc........ 2.5		............87.75 .... 9.75 .... 2.5		............91.00 .... 7.00 .... 2.00		............85.00 .... 5.00 ....10.00		............83.00 .... 2.00 ....15.00		............94.5 .... 5.0 .... 2.5	
	R.	*St.*	*R.*	*St.*	*R.*	*St.*	*R.*	*St.*	*R.*	*St.*	*R.*	*St.*
Atmos.	34,240	12.5	36,800	8.8	33,600	16.0	33,600	21.0	31,040	26.0	35,840	20.0
100°	32,320	10.0	—	—	33,600	15.5	35,200	18.0	30,720	26.0	—	—
150°	33,600	11.0	—	—	33,600	14.0	33,920	19.5	28,800	25.5	—	—
200°	31,040	10.0	34,240	8.8	29,440	9.0	33,470	19.0	29,440	26.3	28,160	11.0
250°	32,320	10.0	24,640	5.0	16,320	3.0	32,960	16.0	28,160	26.0	23,040	6.0
300°	32,000	10.0	18,880	0.7	16,960	0.0	33,960	18.3	27,840	23.0	16,320	0.7
350°	28,800	8.3	18,880	0.0	—	—	31,680	17.0	27,840	25.0	—	—
400°	15,680	0.8	—	—	16,640	0.0	16,640	2.0	27,840	25.0	—	—
450°	16,960	0.0	—	—	—	—	16,000	2.0	9,728	1.2	—	—
500°	16,000	0.0	—	—	17,600	0.0	14,720	2.0	9,728	0.0	—	—

TABLE XIV.—*Continued.*

Temperature, Fahr.	Phosphor Bronze.		Muntz Metal.		Copper.		Cast Iron.		Wrought Iron.				Landore Steel.	
	Dia. = 1 Inch. Copper....92.5 Tin........ 7.0 Phosphor.. 0.5		Dia. = 0.74 Inch. Copper......62 Zinc.........38		Dia. = 0.72		Dia. = 1 Inch. Welsh..... .25 Scotch..... .25 Scrap..... .50		Remanufactured. Dia. = 0.74 Inch.		Yorkshire. Dia. = 0.70 Inch.		0.74 In.×0.49 In.	
	R.	*St.*	*R.*	*St.*	*R.*	*St.*	*R.*	*St.*	*R.*	*St.*	*R.*	*St.*	*R.*	*St.*
Atmos.	38,980	17.5	81,900	2.5	56,810	2.5	30,720	0.0	58,500	22.0	60,030	25.0	76,920	26.0
100°	39,300	17.0	79,560	2.5	57.430	2.5	27,070	0.0	62,010	18.8	60,030	24.3	——	——
150°	39,040	18.0	84,240	3.9	54,960	4.0	28,220	0.0	62,010	15.0	——	——	——	——
200°	38,720	18.0	80,145	3.9	54.340	5.0	28,160	0.0	62,010	15.0	60,680	17.3	74,150	22.5
250°	37,120	15.0	78,390	5.0	53,110	7.0	30,080	0.0	63,180	15.0	——	——	——	——
300°	36,800	12.0	76,050	2.5	53,110	6.0	31,360	0.0	76,050	15.5	62,640	7.5	76,920	11.3
350°	30,080	7.0	76,050	2.3	53,110	6.0	30,080	0.0	77,220	12.5	——	——	——	——
400°	27.140	5.0	70,200	2.3	51,870	6.0	25,660	0.0	63,180	12.0	60,680	15,0	81,770	10.3
450°	24,320	4.0	71,955	3.8	51,260	6.0	25,600	0.0	70,560	15.0	——	——	——	——
500°	26,880	5.0	72,540	5.0	48,220	6.0	25,600	0.0	79,560	20.0	71,830	13.8	83,200	10.0

cent. of stretch for a length of 10 inches in all except the last (steel) specimen.

At 250° to 350° the gun-metal specimens lose about half their ultimate resistance and nearly all their ductility. Phosphor-bronze loses about one third of its resistance and two thirds of its ductility at 300° to 400°. Muntz metal and copper are not much affected, nor is cast iron. Wrought iron and steel gain in ultimate resistance but lose in ductility. These results would probably be somewhat varied by different processes of, and treatment in, manufacture and construction.

The Muntz metal and copper specimens were rolled.

Bauschinger's Experiments with Copper and Red Brass.

Prof. Bauschinger extended his experiments on the repeated application of stress so as to cover not only wrought iron and steel, the results of which have already been given, but also copper and red brass.

The notation is that already used:

E. L. = elastic limit in pounds per square inch.
S.-L. = stretch-limit in pounds per square inch.
F. L. = final load in pounds per square inch.
E. = coefficient of elasticity in pounds per sq. inch.
Im'y = "immediately."

The copper specimens were of rolled material about 16 inches long with a cross-section about 2.4 inches by 0.64 inch. These specimens gave an ultimate tensile resistance, per square inch, of 28,900 to 32,000 pounds and a final contraction of 27 to 46 per cent.

The red brass specimens were turned to about one inch in diameter and 16 inches long. They gave ultimate tensile resistances, in pounds per square inch, varying from 19,600 to 23,460.

With one exception, in the second case of red brass, the elastic limit and stretch-limit were elevated by repeated application of stress, whether immediately or at the end of following periods of rest.

The effects on the coefficient of elasticity are seen to be somewhat irregular.

COPPER.

	In Original Condition.	Im'y.	Im'y.	Im'y.
E. L.	5,475	8,030	8,790	11,450
S.-L.	——	11,670	14,650	22,880
F. L.	——	14,600	21,970	——
E.	16,651,000	17,249,000	16,154,000	15,770,000

COPPER.

	In Original Condition.	After 18 Hours.	After 23 Hours.	After 24 Hours.
E. L.	2,560	7,320	8,080	11,520
S.-L.	——	——	14,680	23,040
F. L.	——	14,650	22,010	——
E.	16,011,000	16,295,000	15,197,000	15,756,000

COPPER.

	In Original Condition.	After 43 Hours.	After 44.5 Hours.	After 51.5 Hours.
E. L.	5,840	8,030	10,340	15,390
S.-L. .	——	11,670	14,760	23,080
F. L.	——	14,600	22,160	——
E.	16,097,000	16,780,000	16,069,000	15,472,000

RED BRASS.

	In Original Condition.	Im'y.	Im'y.	
E. L.	7,680	9,090	9,260	——
S.-L.	13,960	16 070	19,240	——
F. L.	16,770	19,550	——	——
E.	12,030,000	12,485,000	12,727,000	——

RED BRASS.

	In Original Condition.	After 17.5 Hours.	After 21 Hours.
E. L.	5,600	9,115	8,550
S.-L.	14,020	16,130	19,240
F. L.	16,820	19,640	——
E.	12,322,000	12,314,000	12,485,000

RED BRASS.

	In Original Condition.	After 53 Hours.	
E. L.	3,480	9,090	——
S.-L.	13,910	16,070	——
F. L.	16,690	——	——
E.	13,239,000	12,940,000	——

The explanation of the method of applying these repeated stresses will be found in connection with the results for wrought iron on page 246.

Art. 45.—Various Metals and Glass.

Coefficients of Elasticity.

The following values of the coefficients of elasticity, in pounds per square inch, contained in Table I are taken from Wertheim's "Physique Mécanique," pages 57 and

TABLE I.

Metal.	Experimenter.	Coefficient of Elasticity.	
		Drawn.	Annealed.
Lead	Wertheim	2,564,000	2,457,000
Cadmium	"	7,713,000	7,555,000
Gold	"	11,564,000	7,942,000
Silver	"	10,463,000	10,155,000
Palladium	"	16,721,000	13,920,000
Platinum	"	24,237,000	22,067,000

58. The coefficients are the means of a large number of tensile experiments, with the exception of that for cadmium, which was derived from experiments on transverse vibrations. This last method gave results which differed in most cases from the direct tensile ones not more than the latter did from each other.

Wertheim also gives for the tensile coefficients of elasticity of some different glasses:

Mirror glass	$E=8,792,000$	pounds per square inch.
Goblet (common)	$E=9,559,000$	" " " "
Goblet (fine)	$E=8,589,000$	" " " "
Goblet (violet)	$E=7,110,000$	" " " "
"Crystal"	$E=5,830,000$	" " "

Ultimate Resistance and Elastic Limit.

Wertheim determined the elastic limit of many of the more rare metals, such as those named in Table I, and they are here given in pounds per square inch:

	Annealed.		Drawn.
Lead	284	to	355
Cadmium	142	to	171
Gold	4,266	to	19,200
Silver	4,266	to	16,350
Palladium	7,110	to	25,600
Platinum	20,600	to	37,000

His "limit of elasticity" is that force which will permanently elongate the metal 0.000,05 of its original length, and all his experiments were made on wires of very small diameters.

The following ultimate resistances were found for wires about $\frac{1}{60}$th inch in diameter by Baudrimont ("Annales de Chimie," 1850):

Gold	17,100	to	26,200	pounds per square inch.
Silver	40,300	to	40,550	" " " "
Platinum	32,300	to	32,700	" " " "
Palladium	51,750	to	52,640	" " " "

The ultimate resistances of some other metals are:

Metal.	Experimenter.	Ult. Resist.	
Cast lead	Rennie	1,824	pounds per square inch.
Sheet lead	Navier	1,926	" " " "
Pipe lead	Jardine	2,240	" " " "
Soft solder (⅔ tin, ⅓ lead)	Rankine	7,500	" " " "

Art. 46.—Cement, Cement Mortars, etc.—Brick.

The ultimate tensile resistance of cements and cement mortars depends upon many conditions. The two great divisions of cements, i.e., natural and Portland, possess very different ultimate resistances whether neat or mixed with sand, the latter being much the stronger. With given proportions of sand or neat, the ultimate resistances of cement mortar or cement will vary with the amount of water used in tempering and with the pressure under which the moulds are filled. Again, the character of the sand used will obviously influence largely the tensile resistance of the mortar produced, and not only the degree of cleanliness, but the size of grain and the variety of sizes are elements which must be considered. It has also been maintained by some that silica-sand will give better results, other things being equal, than other sand. Finally, the shape of briquette used will affect the results to some extent. Fig. 1, on page 370, shows the form of briquette recommended by the Committee of the American Society of Civil Engineers, and it is the form generally used in American practice. It is foreign to the purpose of this work to enter into the consideration of all these influences; they are only mentioned to enable the few typical experimental results which follow to be interpreted properly.

As the fineness of grinding is an important quality of a cement, it is usually noted by stating the percentage of weight of the cement which either passes through or is retained upon a sieve having a stated number of meshes per linear inch, which number squared gives the number of meshes per square inch. The sizes of the grains of sand used are graded in the same way. The "No." of a sieve to which reference may be made in what follows indicates, therefore, the number of meshes per linear inch.

Coefficient of Elasticity.

In consequence of the fact that cement, mortars, and concrete begin to exhibit permanent stretch at comparatively low tensile stresses there is a little uncertainty as to the value of the coefficient of elasticity unless distinct statement is made of the intensities of stress at which those values are obtained, and whether the total stretch is used or that total less the permanent set. It is not possible to make such statement in connection with all the values which follow, except that they have been reached at low intensities of stress unless otherwise stated, and with elongations which may be considered wholly elastic. Although cement mortars and concrete do not exhibit a perfectly elastic behavior their stress-strain lines for intensities of stress even exceeding those used in practice are essentially straight and, on the whole, exhibit elastic properties at least equal to those of cast iron.

Comparatively few tests have been made to determine either the tensile or compressive coefficient of elasticity of cement, mortar, and concrete, although that quantity is a most important element in the theory and design of concrete-steel members. Among the more valuable determinations of the tensile coefficient of elasticity yet made is the series of values given by Mr. W. H. Henby in a paper read before the Engineers' Club of St. Louis, June, 1900, and shown in Table A. The cross-section of each test specimen was $2\frac{7}{8} \times 3\frac{1}{2}$ inches for a length of 14 inches, the length for which the extensions were measured being 10 inches. The composition of the concrete is given in the table, as are the ages and conditions before testing. The specimens were kept in air either dry or under a damp cloth, or in water.

The entire table is required to exhibit the relative influences of composition, age, mode of keeping specimens prior to testing, and consistency of the original mixtures. Some of the values ranging from 5,000,000 to 8,000,000 pounds or more seem abnormally high, and they are certainly higher than should be taken in practice. It is probable that these test specimens were better mixed and more densely made than would be the large masses in engineering structures, so that in the latter the tensile coefficient of elasticity should probably not be taken higher than 2,500,000 to 3,000,000 pounds per square inch.

TABLE A.

TENSILE TESTS OF PORTLAND CEMENT CONCRETES GIVEN BY HENBY.

	Age, Days.	Parts of Cement, Sand, and Broken Stone.			Cement.*	Kept in	Wt., Lbs. per Cu. Ft.	Coefficient of Elasticity.	Ult. Resist. Lbs. per Sq. In.	Consistency.
1	7	1	2	4	L	Air dry		2,000,000	130	Dry
2	8	"	"	"	A	Air	157	3,720,000	213	Plastic
3	18	"	"	"	M	Air dry		2,286,000	75	Excess water
4	30	"	"	"	L	" "	142	2,882,000	198	Dry
5	30	"	"	"	L	" "	148	4,727,000	227	"
6	30	"	"	"	L	" "	142	2,269,000	191	Plastic
7	30	"	"	"	A	Air	148	3,306,000	192	"
8	30	"	"	"	A	"	150	3,600,000	241	"
9	30	"	"	"	A	"	149	8,360,000	183	Dry
10	30	"	"	"	A	"	149	7,280,000	214	"
11	30	"	"	"	A	"	153	3,750,000	223	Plastic
12	30	"	"	"	A	Water	158	3,550,000	279	"
13	35	"	"	"	L	Air dry	147	4,543,000	149	"
14	64	"	"	"	A	Air	151	7,744,000	252	Dry
15	65	"	"	"	A	"	144	3,810,000	143	Plastic
16	65	"	"	"	A	"	149	3,724,000	102	"
17	65	"	"	"	A	Water	150	5,440,000	82	"
18	90	"	"	"	M	Air dry	147	3,500,000	125	Excess water
19	96	"	"	"	L	" "	146	3,992,000	190	Plastic
20	96	"	"	"	L	" "	147	4,760,000	226	"
21	100	"	"	"	L	" "	147	5,075,000	209	"
22	120	"	"	"	M	" "	143	3,473,000	136	Excess water
23	7	1	2	5	A	Air	152	3,968,000	147	Plastic
24	14	"	"	"	M	Air dry	139	2,000,000	85	"
25	18	"	"	"	M	" "	—	2,106,000	120	"
26	30	"	"	"	A	Air	145	4,532,000	165	"
27	30	"	"	"	A	"	152	4,425,000	242	"
28	30	"	"	"	A	Water	155	5,000,000	239	"
29	60	"	"	"	A	Air dry	144	1,857,000	111	Very dry
30	60	"	"	"	A	" "	146	3,253,000	128	Dry
31	60	"	"	"	A	" "	146	3,023,000	189	"
32	63	"	"	"	A	Air	146	4,980,000	154	"
33	63	"	"	"	A	"	146	3,744,000	192	"
34	90	"	"	"	A	Air dry	144	3,696,000	129	Very dry
35	90	"	"	"	A	" "	144	3,896,000	93	" "
36	90	"	"	"	M	" "	140	3,776,000	142	Dry
37	7	1	3	6	A	Water	154	3,828,000	115	Plastic
38	30	"	"	"	A	Air	150	3,810,000	119	Dry
39	30	"	"	"	A	"	147	2,427,000	104	Plastic
40	30	"	"	"	A	"	148	2,440,000	128	"
41	30	"	"	"	A	"	144	4,496,000	93	"
42	30	1	4	8	A	"	143	3,553,000	71	"
43	30	"	"	"	A	Water	149	6,108,000	125	Dry
44	90	1	3	–	M	Air dry	136	3,988,000	199	"
45	90	1	3	–	M	" "	139	5,202,000	234	"
46	120	1	3	–	M	" "	136	5,144,000	144	Very dry
47	120	1	3	–	M	" "	136	5,150,000	154	" "
48	95	1	–	–	M	Water	137	6,423,000	645	Plastic

* M, Medusa; L, Lehigh; and A, Atlas.

It is important to observe that the coefficient does not appear to gain much in value beyond the age of thirty days. Nor does it seem to make much difference whether the specimens were kept in air or water.

The broken stone used in the concrete of Table A was limestone, the maximum size of some of which was 1½ inches and 2 inches for the remainder. The voids in the broken stone varied from 43 to 46 per cent.

TABLE B.

TENSILE TESTS OF ATLAS (PORTLAND) CEMENT CINDER CONCRETES GIVEN BY HENBY.

	Age, Days.	Parts of Cement, Sand, and Cinders.			Kept in	Weight, Lbs. per Cu. Ft.	Coefficient of Elasticity, Lbs. per Sq. In.	Ult. Resist., Lbs. per Sq. In.	Ult. Comp. Resist., Lbs. per Sq. In.
1	7	1	2	4	Air	118	1,854,000	46	——
2	7	1	2	4	"	114	1,892,000	——	——
3	7	1	2	4	Water	122	1,826,000	——	——
4	30	1	2	4	Air	113	2,800,000	133	993
5	30	1	2	4	Water	121	2,909,000	77	1,415
6	30	1	2	5	"	126	1,900,000	76	1,166
7	30	1	2	5	Air	109	2,853,000	86	1,039
8	30	1	2	5	"	107	2,329,000	86	1,054
9	30	1	2	5	"	102	1,820,000	78	688
10	30	1	2	5	"	112	1,900,000	129	1,005
11	30	1	3	6	Water	114	1,422,000	41	653
12	60	1	2	5	Air dry	117	2,413,000	97	882
13	60	1	3	6	" "	110	1,274,000	58	699
14	60	1	3	6	" "	110	1,892,000	88	949
15	60	1	3	6	" "	107	2,215,000	62	677
16	60	1	3	6	" "	108	2,922,000	52	——
17	30	1	3.5	7	Air	102	1,034,000	30	409
18	30	1	3.5	7	"	104	937,000	31	510

Unscreened commercial Mississippi River sand, as found in the St. Louis market, was used in all of Mr. Henby's tests.

In both tables A and B the ultimate tensile resistances of the concrete are given, and in Table B the ultimate compressive resistances of the cinder concretes are also shown.

Table B exhibits the same class of data as Table A, but for cinder concrete, which is seen to be much lower in both elastic and ultimate properties than stone concrete. It appears from the values in the table that the tensile coefficient of elasticity should not be taken higher, certainly, than 2,000,000, and 1,500,000 would probably be more advisable for general practice, supposing the concrete to be well made and put in place with care. The dimensions of specimens were the same as for the stone concrete of Table A.

Mr. Henby made a number of tests to ascertain the relative values of cinder concrete kept dry and in water. His results indicate that both the coefficient of elasticity and ultimate resistances are materially greater for the material kept dry. Indeed in testing eight specimens he found the average coefficient for the dry specimens about two and a half times that of the wet specimens.

Professor W. Kendrick Hatt, of Purdue University, in a paper read before the American Section of the International Association for Testing Materials, at its convention, 1902, gave the following values for the tensile coefficient of elasticity and ultimate tensile resistances of Portland cement concrete composed of 1 cement, 2 sand, and 4 broken stone at the ages of 25, 26, 28, and 33 days:

	Coefficient of Elasticity, Lbs. per Sq. In.	Ultimate Tensile Resistance, Lbs. per Sq. In.
Maximum	2,700,000	360
Average	2,100,000	311
Minimum	1,400,000	280

It will be seen in discussing the compressive coefficient of elasticity that practically its full value, like the tensile coefficient, is acquired at the age of one to three months. Neither coefficient adds much to its value after one month.

Ultimate Resistance.

The ultimate resistances of neat Portland cement and mortar made with the same cement have been somewhat increased within the past half dozen years; but, upon the whole, those resistances as exhibited in the following tables are fairly representative of the best grades of cement used at the present time (1902). The conditions of manufacture are now so well controlled that a very high 7-day or 28-day test cement may readily be produced; but that is not always desirable; the main purpose in masonry construction being rather the attainment of an ultimate resistance possibly less high under a short-time test but which continues to increase indefinitely. A cement showing a high ultimate resistance on a short-time test may not continue to increase its ultimate resistance satisfactorily, or that resistance may even recede with the lapse of time.

Professor W. Kendrick Hatt, in the paper to which reference has already been made, determined the following ultimate tensile and compressive resistances of the Portland cement and cement mortar used in his tests of reinforced or concrete-steel beams. The ages of the different briquettes are given in the table, as are also their compositions, i.e., whether neat or with proportions of 1 cement to 3 sand:

ULTIMATE TENSILE RESISTANCE IN POUNDS PER SQUARE INCH.

Age.	24 Hours.	7 Days.	1 Month.	3 Months.
Neat	346	803	...	802
3 : 1	78	412	505	626

ULTIMATE COMPRESSIVE RESISTANCE OF 2-INCH CUBES IN POUNDS PER SQUARE INCH.

Age.	24 Hours.	7 Days.	1 Month.	3 Months.
Neat	2,402	7,337	9,537	9,740
3 : 1	610	3,145	5,860	8,005

The ultimate compressive resistances are given at this point because it is more convenient to record them here. Reference will be made to these later results in the article on the compressive resistance of cements, mortars, and concretes.

Probably the latest series of a great number of tests of cement and mortar briquettes is that of the cement-testing laboratory of the Rapid Transit Railroad Commissioners of New York City, William Barclay Parsons, C.E., Chief Engineer. The numbers of briquettes broken during the years 1900 and 1901 were over 2000 and 16,000 respectively. The average ultimate tensile resistances in pounds per square inch disclosed by that series of tests of both Portland and natural cements, as given in the report of the Chief Engineer, are the following:

	Year.	Neat Cement.			* Sand 2, Cement 1	
		1 Day.	7 Days.	28 Days.	7 Days.	28 Days.
Portland:						
Average result	1900	229	552	714	276	434
Average result	1901	300	645	763	380	525
Spec. requirements	——	150	400	500	200	300
Natural:						
Average result	1900	——	172	249	118	215
Average result	1901	——	215	322	218	350
Spec. requirements	——	——	125	200	100	150

* For natural cement a 1 cement 1 sand mortar was used.

These results represent closely the best high test Portland and natural cements produced in the United States during the execution of the Rapid Transit Subway work.

The following tables disclose not only the ultimate resistances of the different compositions but also the variation of those resistances with age.

Table I exhibits the results of tests of briquettes of twelve different brands of domestic and one of foreign Portland cements as made in the testing laboratory of the Bureau of Surveys of the Department of Public Works of Philadelphia, Pa., for the year 1901, Mr. George S. Webster, Chief Engineer. This table gives the fineness of the cements in the terms of the percentages by weight which passed through sieves with 2500, 10,000, and 40,000 meshes per square inch; it also shows the amount of water used for the different mixtures. The table is useful in showing the gain in resistance both for the neat cement and cement mortar briquettes up to the age of six months.

Table II shows the results of testing both mortar briquettes—one part cement to three parts sand—and those of the neat cement at the ends of the periods named, extending to two years. All cements are Portland, and the tests were made for the Water Commissioners of St. Louis. The figures for Table II were taken from the annual report of the Water Commissioners for the year ending April, 1894.

Table III shows the results of a large number of tests of the "Giant" brand of American Portland cement made in connection with the construction of the works named. These results were compiled by Mr. R. W. Lesley from the official records of those works, and they are averages in each case from 3 to over 4000 tests. The table is particularly interesting in consequence of the long period (five years) which elapsed before the oldest specimens were broken. The proportions "2 to 1" and "3 to 1" mean 2 volumes of sand and 3 volumes of sand to 1 volume of cement.

TABLE I.

AVERAGE RESULTS OF PORTLAND CEMENT TESTS MADE DURING 1901.

Brand.	No. of Tests.	Fineness in Per Cent.			Per Cent. of Water.	
	Broken.	No. 50.	No. 100.	No. 200.	Neat.	Sand.
Alpha............	136	99.8	91.0	76.6	20.1	9.3
Atlas............	820	9.9	90.4	76.5	17.1	8.8
* Castle..........	16	99.2	86.4	74.0	25.0	10.2
Dexter...........	28	100.0	95.1	83.1	20.0	9.3
Giant.	816	99.5	90.3	76.4	19.5	9.2
Krause's..........	72	100.0	91.2	76.1	19.0	9.2
Lehigh...........	112	99.8	90.3	77.1	19.9	9.3
Phœnix..........	36	100.0	93.8	78.2	19.5	9.2
Reading..........	16	100.0	96.6	86.9	20.0	9.3
Saylor's..........	764	100.0	95.5	82.4	20.6	9.4
Star.............	4,012	100.0	91.7	76.1	19.7	9.3
Vulcanite.........	204	99.5	89.9	74.7	20.3	9.4
Whitehall.........	356	100.0	91.0	75.4	19.4	9.2

Brand.	No. of Tests.	Ultimate Tensile Resistance in Pounds per Square Inch.						
		Neat.						
	Broken.	24 Hrs.	7 Days.	28 Dys.	2 Mos.	3 Mos.	4 Mos.	6 Mos.
Alpha.........	136	357	770	834	885	813	785	827
Atlas.........	820	542	728	790	802	761	815	825
* Castle.......	16	235	336	387	443	—	—	—
Dexter........	28	363	826	932	778	—	—	—
Giant.........	816	424	669	719	713	745	776	786
Krause's......	72	418	830	864	775	—	—	—
Lehigh........	112	377	699	747	684	735	774	760
Phœnix.......	36	345	721	723	—	—	—	—
Reading.......	16	460	800	955	775	—	—	—
Saylor's.......	764	295	697	766	756	766	733	745
Star..........	4,012	437	721	746	731	715	727	—
Vulcanite......	204	290	748	767	707	807	710	740
Whitehall.....	356	524	713	765	788	796	775	—

* Belgian.

TABLE I.—*Continued.*

AVERAGE RESULTS OF PORTLAND CEMENT TESTS MADE DURING 1901.—*Continued.*

Brand.	No. of Tests.	Ultimate Tensile Resistance in Pounds per Square Inch.						
		1 to 3 Standard Quartz Sand.						
	Broken.	24 Hrs.	7 Days.	28 Dys.	2 Mos.	3 Mos.	4 Mos.	6 Mos.
Alpha	136	81	252	314	344	312	302	262
Atlas	820	104	204	289	324	321	337	308
* Castle	16	65	121	176	215	—	—	—
Dexter	28	68	298	336	312	—	—	—
Giant	816	87	227	309	328	317	328	329
Krause's	72	74	229	285	270	—	—	—
Lehigh	112	76	233	329	296	310	303	325
Phœnix	36	94	264	343	—	—	—	—
Reading	16	150	263	301	338	—	—	—
Saylor's	764	64	217	296	319	301	311	286
Star	4,012	77	219	298	321	301	286	—
Vulcanite	204	45	226	287	269	298	280	330
Whitehall	356	87	232	313	295	295	343	—

* Belgian.

The average results for a considerable number of tests for periods of time up to four years, made in connection with the construction of the Sodom Dam of the Croton water system (the supply for the city of New York), are given in Table IV.

They are taken from a paper by Mr. W. McCulloch in the "Trans. of the Am. Soc. Civ. Engrs." for March, 1893. The composition of the mortar in the mortar specimens is shown in the extreme left-hand column to have been 2 volumes and 3 volumes of sand to 1 volume of cement.

A considerable number of tests made by Mr. P. K. Yates, resident engineer on the Seventh Avenue bridge across the Harlem River, at New York City, are illustrated by the results given in Table V. The greatest age of any specimen is seen to be two years.

Table VI contains the results of tests by Mr. Geo. W. Rafter on domestic Portland cements as given in the report

TABLE II.

AVERAGE OF TEN BRIQUETTES IN EACH CASE. SAND BETWEEN NO. 20 AND NO. 30 SIEVE, ONE DAY IN AIR, REMAINING TIME IN WATER. TENSILE RESISTANCE, POUNDS PER SQUARE INCH.

TESTS OF ONE PART CEMENT TO THREE PARTS SAND.

Sand Tests 1 to 3.	Columbus.	Globe.	Dufossez and Henry.	Alsen's.	Harris.	Heyn.	Zunz.	Phœnix.	H. H. Meier & Co. Puzzolan.	Dyckerhoff.	Lion.	Empire.	Manheimer.	Germania.	Stettin Star.
Fineness on No. 100 sieve.	70.2	83.2	91.7	86.5	71.2	75.2	81.5	76.5	94.1	88.6	82.7	83.6	87.6	89.5	91.7
1 week.	50	160	232	190	130	182	218	184	163	223	195	229	272	266	280.2
2 weeks.	73	187	189	207	150	200	240	215	197	239	212	241	264	312	284
3 weeks.	84	198	178	226	148	235	230	250	215	251	266	259	265	296	296
4 weeks.	96	194	166	215	170	227	242	261	210	287	258	239	288	290	303
3 months.	135	239	220	238	204	276	294	297	290	308	351	299	317	367	353
6 months.	224	235	207	252	246	291	268	292	309	337	349	358	336	419	421
9 months.	227	236	225	276	284	311	306	303	324	332	352	331	350	412	431
1 year.	231	253	255	285	286	306	307	326	341	347.6	366	347	386	426	452
2 years.	245	287	280	306	304	335	342	345	387	371	406	411	418	456	480

TESTS OF NEAT CEMENT.

	Columbus.	Globe.	Dufossez and Henry.	Alsen's.	Harris.	Heyn.	Zunz.	Phœnix.	H. H. Meier & Co. Puzzolan.	Dyckerhoff.	Lion.	Empire.	Manheimer.	Germania.	Stettin Star.
1 week.	350.4	438.4	586.0	606.2	599.4	692.6	540.0	716.0	376.2	665.2	741.6	691.2	559.8	593.6	675.6
2 weeks.	460.2	554.4	687.2	671.4	638.0	702.4	635.6	791.8	448.8	693.8	782.0	855.0	603.4	671.6	761.6
3 weeks.	545.4	552.8	679.4	664.8	711.8	802.8	659.6	812.4	471.0	709.4	847.6	826.6	594.0	691.4	768.8
4 weeks.	590.6	607.0	682.4	707.0	647.6	719.2	659.4	824.2	487.2	710.4	939.6	871.6	625.2	728.6	707.2
3 months.	644.6	541.2	754.6	755.0	681.8	636.5	748.4	746.8	513.8	690.6	832.2	819.8	578.2	749.9	675.2
6 months.	799.8	723.4	854.4	787.7	752.4	675.6	638.6	828.2	525.0	657.8	874.6	819.8	681.2	757.2	742.2
9 months.	858.8	729.8	848.0	746.4	796.2	790.2	696.7	923.1	562.4	795.2	963.6	825.2	828.2	807.6	849.8
1 year.	807.8	695.2	739.0	742.2	755.2	850.2	649.4	815.0	583.2	715.0	924.0	895.8	811.0	707.6	783.2
2 years.	904.0	753.6	754.6	750.8	855.8	817.8	689.4	928.2	597.4	764.4	930.0	753.8	791.8	802.8	772.8

TABLE III.

AVERAGE TENSILE RESISTANCE IN POUNDS PER SQUARE INCH.

	Mode of Mixing.	Age.									Fineness.
		One Week.	One Month.	Six Months.	One Year.	Eighteen Months.	Two Years.	Three Years.	Four Years.	Five Years.	Residue on No. 100 Sieve, 10,000 Holes Square Inch.
		Average in Pounds.	Average in Pounds.	Average in Pounds.	Average in Pounds.	Average in Pounds.	Average in Pounds.	Average in Pounds.	Average in Pounds.	Average in Pounds.	
Sodom and Bog Brook dams, New York Aqueduct.	Neat	348	422	634	682	672	694	736	771	840	18.0
	2 to 1	166	280	468	490	530	564	680	674	700	——
	3 to 1	140	234	428	420	514	512	572	——	590	——
Titicus Dam, New York Aqueduct.	Neat	380	476	575	642	584	641	668	——	——	11.7
	2 to 1	200	317	491	549	541	536	573	——	——	——
	3 to 1	115	185	347	426	416	419	483	——	——	——
Carmel and Craft's Dam, New York Aqueduct.	Neat	343	443	548	617	604	611	——	——	——	14.7
	2 to 1	182	308	480	585	477	475	——	——	——	——
	2 to 1	164	237	494	367	516	568	——	——	——	——
	3 to 1	100	174	372	413	401	406	——	——	——	——
	3 to 1	112	166	480	541	437	406	——	——	——	——
Reading Terminal Railroad and Station, Philadelphia.	Neat	315	376	549	523	——	——	——	——	——	14.79
	3 to 1	79	129	228	261	——	——	——	——	——	——
	2 to 1	84	144	234	325	——	——	——	——	——	——
Niagara Falls Tunnel, Niagara Power Co.	Neat	321	420	562	661	——	——	——	——	——	14.9
	2 to 1	152	242	454	472	——	——	——	——	——	——
	3 to 1	98	177	301	391	——	——	——	——	——	——

TABLE IV.

AVERAGE TENSILE RESISTANCE IN POUNDS PER SQUARE INCH.

Cement.	Fineness, Average.	Time Set in Water.						
		One Day.	One Week.	One Month.	One Year.	Two Years.	Three Years.	Four Years.
Portland:	10,000 mesh.							
Burham, neat.	80%	167	429	615	798	700	764	782
" 1c. 2s.	——	——	141	258	468	532	632	658
" 1c. 3s.	——	——	169	224	404	520	552	——
Natural:	2500 mesh.							
Union, neat.	96%	160	240	228	510	542	650	654
" 1c. 2s.	——	——	34	94	394	430	514	522

TABLE V.

No. Tests.	Cement.	Age and Ultimate Tensile Resistance in Lbs. per Sq. In.							
		Seven Days.	One Mo.	Three Mos.	Four Mos.	Six Mos.	Nine Mos.	One Year.	Two Years.
20	Alsen's Portland, neat	450	453	606	—	—	701	739	766
10	" " 1c. 1s.	—	—	399	418	428	—	655	—
6	" " 1c. 3s.	—	146	193	—	—	—	—	—
—	Rosendale, neat*....	113	155	297	—	377	—	412	—
—	" 1c. 1s.....	—	115	225	—	269	—	309	—

* At end of one day, eighty pounds per square inch.

TABLE VI.

ULTIMATE TENSILE RESISTANCES IN POUNDS PER SQUARE INCH AT THE AGES SHOWN.

Cement.	Mixture.	One Day.	Seven Days.	Twenty-eight Days.	Ninety Days.	One Hundred and Fifty Days.
Empire Portland...	neat	324	520	625	—	—
Wayland Portland	neat	340	519	610	649	698
	1 s. 1 c.	—	272	386	489	527
	2 s. 1 c.	—	203	270	326	440
	3 s. 1 c.	—	144	246	351	—
	4 s. 1 c.	—	132	200	242	—
	5 s. 1 c.	—	104	136	—	—

of the State Engineer and Surveyor of the State of New York for 1894. Each result is an average of twenty tests of briquettes. The sand was clear and rather fine, more than one-third of it passing a No. 50 sieve.

The values exhibited in Table VII were determined during the year 1901 at the testing laboratory of the Bureau of Surveys in the Department of Public Works of Philadelphia, Pa., and are taken from the same report as was Table I: they belong to six brands of domestic natural cement. This table also shows the fineness of the cements

TABLE VII.

AVERAGE RESULTS OF NATURAL CEMENT TESTS MADE DURING 1901.

Brand.	No. of Tests.	Fineness in Per Cent.			Per Cent. of Water.	
		No. 50.	No. 100.	No. 200.	Neat.	Sand.
Cumberland and Potomac.	152	94.4	80.9	68.9	32.9	13.3
Improved Anchor........	3,260	99.9	92.6	79.4	26.4	11.9
Improved Bonneville.....	4,500	99.3	89.4	75.4	24.8	11.5
Improved Shield.........	16	99.8	90.4	76.0	24.0	11.3
Improved Union.........	2,308	98.7	90.2	78.9	26.7	11.9
Union Rosendale........	16	99.2	94.0	85.0	32.0	13.1

Brand.	No. of Tests.	Ultimate Tensile Resistance in Pounds per Square Inch.						
		Neat.						
		24 Hrs.	7 Days.	28 Days.	2 Mos.	3 Mos.	4 Mos.	6 Mos.
Cumberland and Potomac.	152	149	253	325	389	390	380	342
Improved Anchor........	3,260	123	280	408	453	470	475	477
Improved Bonneville.....	4,500	174	342	445	475	470	518	522
Improved Shield.........	16	155	229	279	334	—	—	—
Improved Union.........	2,308	184	253	342	383	413	434	447
Union Rosendale........	16	86	135	230	—	—	—	—

Brand.	No. of Tests.	Ultimate Tensile Resistance in Pounds per Square Inch.						
		1 to 2 Standard Quartz Sand.						
		24 Hrs.	7 Days.	28 Days.	2 Mos.	3 Mos.	4 Mos.	6 Mos.
Cumberland and Potomac.	152	61	137	231	303	374	303	287
Improved Anchor........	3,260	49	159	274	346	381	383	385
Improved Bonneville.....	4,500	62	183	298	379	397	417	417
Improved Shield.........	16	63	143	231	315	—	—	—
Improved Union.........	2,308	91	162	250	332	369	412	460
Union Rosendale........	16	33	63	147	—	—	—	—

and the percentages of water used in the mixtures of both neat cement and cement mortar. It will be observed that the sand used in the tests of both Tables I and VII was the standard crushed quartz used for such purposes.

The recent cement product, called silica-Portland cement, is manufactured by grinding together certain portions of clean silicious sand and Portland cement. The results given below are taken from the tests of such silica-Portland cement, manufactured by the Silica-Portland Cement Co., of Long Island City, N. Y. One part, by weight, of Aalborg Portland cement was ground together with six parts, by weight, of clean silicious sand to such a degree of fineness that essentially all of the product passed through a 32,000-mesh sieve. This finely ground mixture of 1 cement to 6 sand, by weight, is called "neat" in what follows, while "(1-6) s. c.-2 q." is 1 part, by weight, of the "neat" silica-Portland cement to 2 parts, by weight, of crushed quartz, or "standard" sand, all of which passes a No. 20 sieve and is retained on a No. 30 sieve. The results were obtained in the cement-testing laboratory of

TABLE VIII.

SILICA-PORTLAND CEMENT.

Ultimate Tensile Resistance in Pounds per Square Inch.

Mixture.	Per Cent. of Water.	Age.			
		Seven Days.	Fifteen Days.	Twenty-one Days.	Two Hundred and Nineteen Days.
Neat..........	18–21%	8 { 148 130 121	6 { 172 165 147	8 { 166 149 121	——
(1–6) s. c.–2 q...	11%	23 { 81 69 58	——	8 { 114 98 88	5 { 220 204 194

All specimens one day in air and remainder in water.

the department of civil engineering of Columbia University. The figures on the left of the brackets show the number of tests of which the ultimate resistances are the greatest, mean, and least in each case.

Five seven-day tests of the Aalborg Portland cement used in the manufacture of the silica-Portland cement gave the following greatest, mean, and least ultimate tensile resistances, the specimens having been one day in air and six days in water:

Greatest.	Mean.	Least.
594 lbs. per sq. in.	536 lbs. per sq. in.	441 lbs. per sq. in.

Four specimens of the neat silica-Portland cement (1–6), one day in air and the remainder of the time in water, gave the following results:

		Age.
Neat (1–6)........	308 lbs. per sq. in......	199 days.
	264 " "	190 "
	294 " "	189 "
	260 " "	185 "

One of the earliest authorities on English Portland cement was Mr. John Grant, who published his "Experiments on the Strength of Cements" in 1875. Although his experimental determinations are now displaced by those applicable to cements and mortars of the present time, his work is one of the classics in Portland cement literature. The conclusions which he reached through investigations and practical experience still hold in great measure and are of sufficient value to be given here. They are the following:

1. Portland cement, if it be preserved from moisture, does not, like Roman cement, lose its strength by being kept in casks, or sacks, but rather improves by age; a great advantage in the case of cement which has to be exported.

2. The longer it is in setting, the more its strength increases.

3. Cement mixed with an equal quantity of sand is at the end of a year approximately three fourths of the strength of neat cement.

4. Mixed with two parts of sand, it is half the strength of neat cement.

5. With three parts of sand, the strength is a third of neat cement.

6. With four parts of sand, the strength is a fourth of neat cement.

7. With five parts of sand, the strength is about a sixth of neat cement.

8. The cleaner and sharper the sand, the greater the strength.

9. Very strong Portland cement is heavy, of a blue-gray color, and sets slowly. Quick-setting cement has, generally, too large a proportion of clay in its composition, is brownish in color, and turns out weak, if not useless.

10. The stiffer the cement is gauged, that is, the less the amount of water used in working it up, the better.

11. It is of the greatest importance that the blocks, or stone, with which Portland cement is used, should be thoroughly soaked with water. If under water, in a quiescent state, the cement will be stronger than out of water.

12. Blocks of brickwork, or concrete, made with Portland cement, if kept under water till required for use, would be much stronger than if kept dry.

13. Salt water is as good for mixing Portland cement as fresh water.

14. Bricks made with neat Portland cement are as strong at from six to nine months as the best quality of Staffordshire blue brick, or similar blocks of Bramley Fall stone, or Yorkshire landings.

15. Bricks made of four parts or five parts of sand to one part of Portland cement will bear a pressure equal to the best picked stocks.

16. Whenever concrete is used under water, care must be taken that the water is still. Otherwise a current, whether natural or caused by pumping, will carry away the cement, and leave only the clean ballast.

17. Roman cement, though about two thirds the cost of Portland, is only about one third its strength, and is therefore double the cost, measured by strength.

18. Roman cement is very ill adapted for being mixed with sand.

Mr. Don. J. Whittemore has proposed the following formula for the ultimate tensile resistance of cements:

$$T = A\sqrt[x]{N};$$

in which T is the ultimate tensile resistance in pounds per square inch; A, an empirical coefficient, and N the age of the cement in days. This equation cannot be applied to any extended length of time, but it may be used approximately for periods possibly ranging from 7 days to 30 days. If the 7- and 28-day results obtained in the testing laboratory of the Rapid Transit Railroad Commission of New York City, given in the preceding portion of this article, be used there will result for Portland cement with close approximation, $A = 500$ and $x = 8.7$; while for natural cement $A = 120$ and $x = 3.4$. Such approximate formulæ, however, have very little practical value.

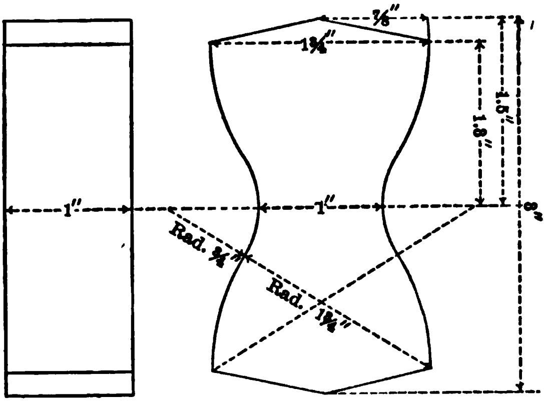


FIG. 1.

All the preceding tensile tests of cement and cement mortars, unless otherwise stated, were made with the shape of briquette shown in Fig. 1, which was recommended for use in the report of the "Committee on a Uniform System for Tests of Cement" of the American Society of Civil Engineers. That report was made in 1885, and the briquette recommended has become the standard in American practice for the testing of cements and mortars.

Adhesion between Bricks and Cement Mortar.

General Q. A. Gillmore ("On Limes, Hydraulic Cements, and Mortars") cemented Croton bricks together crosswise and then separated them by a pull. He used pure cement paste and mortars of various proportions, by volume, of cement to sand, but never more sand than 1 volume of cement to 2 volumes of sand. Nearly all the cement was

Rosendale, although some specimens were prepared with Hancock (Maryland) or James River cement. Bricks so cemented in pairs were kept 320 days and then separated. Reviewing the results, Gen. Gillmore says: "In tearing the bricks apart, at the expiration of the time specified, in a majority of cases the surface of contact of the brick and mortar remained intact, the adhesion to the brick overcoming the cohesive strength either of the bricks themselves, or of the mortar composing the joint between them. The results, therefore, although interesting for other reasons, furnish no entirely satisfactory measure of the power of adhesion."

Also, "at the age of 320 days (and perhaps considerably within that period) the cohesive strength of pure cement mortar exceeds that of Croton front bricks. The converse is true when the mortar contains fifty per cent., or more, of sand."

TABLE IX.

Mortar or Paste.	Materials Cemented.	Adhesion per Square Inch in Pounds.	Ratio of Adhesion to Resistance of Pure Cement.
Pure cement	Croton bricks	30.8	1.00
1 vol. cement, 1 vol. sand	" "	15.7	0.51
1 " " 2 " "	" "	12.3	0.40
1 " " 3 " "	" "	6.8	0.22
1 " " 4 " "	" "	5.2	0.17
1 " " 5 " "	" "	4.3	0.14
1 " " 6 " "	" "	3.3	0.11
Pure cement	Fine cut granite	27.5	1.00
1 vol. cement, 1 vol. sand	" " "	20.8	0.76
1 " " 2 " "	" " "	12.6	0.46
1 " " 3 " "	" " "	9.2	0.33
1 " " 4 " "	" " "	7.9	0.29

Table IX contains the results of another series of experiments by General Gillmore, made for the purpose of determining the adhesion to Croton front bricks and fine cut granite of mortars containing different proportions of sand. "The bricks were used wet, and were well pressed together by hand. They were wetted with fresh water every alternate day for 29 days, the age of the mortar when tested. Each result is the average of five trials."

Table X exhibits other experimental results on the adhesive resistance of cements and mortars to both stone and brick; it is taken from a similar table prepared by Mr.

TABLE X.

ADHESIVE RESISTANCE OF CEMENTS AND MORTARS.

Reference No.	Age in Days When Tested.	Kind of Cement Used.	Materials Cemented together.	Average Ahhesive Strength in Pounds per Square Inch.					Authority.
				Neat Cement.	Cement 1, Sand 1.	Cement 1, Sand 2.	Cement 1, Sand 3.	Cement 1, Sand 4.	
1	7	Portland	Sawed limestone	57	—	—	—	—	I. J. Mann, 1883
2	"	"	Cut granite	41	—	—	—	—	" "
3	"	"	Polished marble	38	—	—	—	—	" "
4	"	"	Bridgewater brick	19	—	—	—	—	" "
5	"	"	* Brick	168	102	38	20	9	Prof. Warren, 1887
6	"	"	† Brick	—	117	53	26	16	" " "
7	28	"	* Brick	213	105	45	24	14	" " "
8	"	"	† Brick	—	146	73	48	45	" " "
9	30	"	Sawed limestone	78	—	—	—	—	I. J. Mann, 1882
10	"	"	Cut granite	§ 97	—	—	—	—	" "
11	"	"	Polished marble	§ 71	—	—	—	—	" "
12	"	"	Bridgewater brick	§ 66	—	—	—	—	" "
13	"	"	Sandstone	49	—	—	—	—	" "
14	42	"	‡ Brick	68.8	—	46.9	—	—	Bauschinger, 1873
15	48	"	‡ Brick	—	—	—	24.2	—	" "
16	56	"	‡ Brick	—	54	56.9	—	—	" "
17	95	"	‡ Brick	—	—	—	14.2	—	" "
18	110	Hydraulic lime	‡ Brick	—	—	—	12.8	—	" "
19	180	Portland	Sawed slate	§ 62	—	—	—	—	I. J. Mann, 1883
20	"	"	Portland stone	§ 55	—	—	—	—	" "
21	"	"	Polished marble	§ 75	—	—	—	—	" "
22	1 yr.	"	Gault-clay brick pressed	45	44	—	—	—	J. Grant, 1871
23	"	"	Stock brick in air	78	63	—	—	—	" "
24	"	"	Stock brick in water	96	70	—	—	—	" "
25	"	"	Staf. blue brick in air	48	47	—	—	—	" "
26	"	"	Staf. blue brick in water	40	29	—	—	—	" "
27	"	"	Fareham red brick in air	126	83	—	—	—	" "
28	"	"	Fareham red brick in water	123	62	—	—	—	" "

* Clean river sand used in mixture. † Crushed sandstone used in mixture

‡ Fine river sand used in mixture.

§ Coarse particles in cement sifted out before testing.

Emil Kuichling in his report to the Executive Board of the City of Rochester, N. Y., for 1888. Tables IX and X contained at that time practically all the reliable information of this class available.

Other experimental determinations of the adhesive resistance of natural and Portland cement mortars to brick and stone may be found in the report of the Chief of Engineers, U. S. A., for 1895. At the age of 28 days the adhesive resistance of neat Portland cement to the surface of sawn limestone was about 270 pounds per square inch; about 240 pounds per square inch with a mortar of 1 cement to ½ sand; about 225 pounds per square inch with a mortar of 1 cement to 1 sand, and about 170 pounds per square inch with a mortar of 1 cement to 2 sand.

Table XI exhibits the average results of three and six months' tests of the adhesion of Portland and natural cement mortars to bricks which were cemented to each other at right angles and then pulled apart normally at the ends of the periods named. These average results are taken from the same report of the Chief of Engineers, U. S. A., for 1895.

TABLE XI.

AVERAGE ADHESIVE RESISTANCE OF BRICKS CEMENTED TOGETHER AT RIGHT ANGLES TO EACH OTHER.

Cement.	Mortar.	Adhesion, Pounds per Square Inch.
Portland	Neat	60
"	1 c., ½ s.	60
"	1 c., 1 s.	40
"	1 c., 2 s.	20
"	1 c., 3 s.	20
Natural	Neat	55
"	1 c., ½ s.	50
"	1 c., 1 s.	45
"	1 c., 2 s.	30
"	1 c., 3 s.	15

There will also be found in that report average values of the shearing adhesion of plain 1-inch round bolts to neat Portland cement and to Portland cement mortars of 1 month's age, the bolts having been embedded at various depths from 2 to 10 inches in the mortars. The shearing adhesion for the neat cement varied from a maximum of 345 pounds per square inch for a depth of insertion of 4 inches down to 230 pounds per square inch for a depth of insertion of about 8¾ inches. In the case of the Portland cement mortar of 1 cement to 2 sand the shearing adhesion varied from a maximum of 280 pounds per square inch for a depth of insertion of the bolt of 2½ inches down to 250 pounds per square inch for a depth of insertion of about 7¾ inches. When the bolt was embedded in the Portland cement mortar of 1 cement to 4 sand the shearing adhesion ranged from a maximum of about 145 pounds per square inch for a depth of insertion of 10 inches to a minimum of about 70 pounds per square inch for a depth of insertion of 2 inches. These values of shearing adhesion are important results in the theory and design of concrete-steel members.

The Effect of Freezing Cements and Cement Mortars.

There have been many attempts made to determine the effect of freezing neat cements and cement mortars after having been mixed for use at various ages and under various conditions. Some valuable data have been accumulated, but the conditions attending such investigations are so complicated and so difficult to be analyzed quantitatively that many most discordant conclusions have been reached. Different results will follow if the freezing is done immediately after the mixing of the cement or mortar, or after the initial set has taken place, or after the considerable hardening which takes place at the age of

12 to 24 hours. Probably the best data in this connection arise from an engineer's practical experience in laying masonry when the temperature of the air is below the freezing-point. Under such circumstances it is rarely the case that anything more than surface freezing takes place before the hardening of Portland cement. With the slower action of the natural cements similar conditions do not exist. It is undoubtedly prejudicial even with Portland cements to have alternate freezing and thawing take place at comparatively short intervals of time. On the other hand, the great majority of laboratory investigations indicate that Portland cement or cement mortars may be severely frozen and remain so for long periods of time without essential injury. It is probable that setting usually proceeds during a frozen condition, but at an exceedingly slow rate, and that the operation of setting is actively renewed after thawing.

While it has been stated in some quarters that natural cements may be frozen similarly and thawed without essential injury, there is considerable laboratory evidence as well as that of practice which indicates that conclusion to be erroneous, especially if it be given any considerable application. There may be cases in which natural cements can be or have been frozen without essential injury, but the author's experience in extended practical operations in masonry construction induces him to believe that any natural cement severely frozen before being thoroughly hardened is so seriously injured as to be practically destroyed. On the other hand, his extended observations not only on his own work, but on those of others, lead him to believe that, as a rule, Portland cement will not be sensibly injured under the conditions of actual masonry construction by being frozen. It is customary in most large works to permit no masonry to be laid at a temperature much below about 26° Fahr. above zero, but with precautions easily attained it is certain that concrete and other masonry laid in Portland cement mortar may properly and safely be put in place several degrees below that temperature.

It has also been stated in some quarters that natural cements and some Portlands have been actually improved by being frozen. Such conclusions should be received with exceeding caution. The author believes that there is no conclusive evidence that any cement or cement mortar can be improved by freezing.

In cold weather it is customary on some works to use salt water for mixing mortars and concretes, and that practice when suitably conducted may be resorted to with safety and propriety. Such solutions generally run from 2 to 8 or 10 per cent. by weight of salt. Occasionally, also, soda is dissolved in water at the rate of 2 pounds per gallon. Before using this solution an equal volume of water is added so that the final solution contains about 1 pound of soda to a gallon of water. This solution expedites the setting of the cement with a view to accomplishing a safe degree of hardening before the mortar is frozen. It is doubtful whether this practice should be encouraged.

The Linear Thermal Expansion and Contraction of Concrete and Stone.

Satisfactory investigations regarding the expansion and contraction of concrete and stone are exceedingly few in number, and the data by which variations in the dimensions of large masses of masonry due to temperature changes can be computed are correspondingly meagre. Professor William D. Pence, of Purdue University, has made such investigations and presented the results in a valuable paper read before the Western Society of Engineers, November 20, 1901. In his experimental work he compared the thermal linear changes of concrete bars and bars of steel and copper, basing the coefficients of expansion of the concrete and mortar on the relative changes of the two materials for the same range of temperature. These experiments were conducted with great care, but the resulting values might perhaps have been at least better defined had two materials been employed with a greater difference in their rates of thermal expansion and contraction. Professor Pence employed two kinds of concrete and one bar of Kankakee limestone, seven experiments having been performed on a concrete of 1 Portland cement, 2 sand, and 4 broken stone; one on a concrete of 1 Portland cement, 2 sand, and 4 gravel; and three on a concrete composed of 1 cement and 5 of sand and gravel, making the mixture essentially equivalent to the preceding concrete of 1 cement, 2 sand, and 4 gravel. The maximum, mean, and minimum coefficients of linear expansion per degree Fahr. found in these tests were as follows:

Kind of Concrete.	Maximum.	Mean.	Minimum.
Broken stone, 1 : 2 : 4..........	.0000057	.0000055	.0000052
Gravel, 1 : 2 : 4................	——	.0000054	——
Gravel, 1 : 5...................	.0000055	.0000053	.0000052
Kankakee limestone............	——	.0000056	——

Between January and June, 1902, Messrs. J. G. Rae and R. E. Dougherty, graduating students in Civil Engineering at Columbia University, with the aid of Professor Hallock of the Department of Physics of the same university, determined with great care by the most accurate direct measurements the coefficients of linear thermal expansion of one bar of concrete of 1 Portland cement, 3 sand and 5 gravel, and one bar of mortar of 1 Portland cement and 2 sand, each bar being 4 inches by 4 inches in cross-section and about 3 feet long, both bars being tested at the age of about 5½ years. The coefficients of linear thermal expansion for each degree Fahr. found in these investigations were as follows:

For 1:3:5 concrete	.00000655
" 1:2 mortar	.00000561

It is believed that these last two determinations were made with the utmost accuracy attainable at the present time in an unusually well equipped physical laboratory and under most favorable conditions.

When it is remembered that the coefficient of linear thermal expansion of such iron and steel as are used in engineering structures is about .0000066,* it is apparent that structures of combined concrete or other masonry and steel may be expected to act under thermal changes essentially as a unit, a conclusion which is justified at the present time by extended experience.

Art. 47.—Timber in Tension.

The ultimate resistance of timber in general is much affected by the moisture which it contains, except that the amount of moisture does not appear to affect sensibly the ultimate tensile resistance. At this point, therefore, no further attention will be given to the effect of moisture or sap on the tensile resistance, but the influence of moisture

---

* A large number of determinations of the thermal expansion of iron and steel per degree Fahr. may be found in the U. S. Report of Tests of Metals and Other Materials for 1887. The maximum, mean, and minimum for steel bars are as follows:

.000006756	.000006466	.00000617

Other coefficients of thermal expansion are also given as follows:

Wrought iron	.00000673
Cast iron	.000005926
Copper	.000009129

on the compressive and bending resistances will be fully set forth in the articles devoted to timber in compression and bending.

There are but few results of investigations which lead to satisfactory coefficients of elasticity for tension. Among those are some determinations of that coefficient for Douglas fir or Oregon pine in the "U. S. Report of Tests of Metals and Other Materials" for 1896. There were seven tests, which gave the following results in pounds per sq. in.:

Maximum.	Mean.	Minimum.
2,662,000	2,065,000	1,739,000

The timber was not well seasoned. It seems probable from such tensile tests as have been made that the coefficient of elasticity for the more common structural timbers like pine, spruce, and oak may be taken at values running from 2,000,000 down to 1,200,000 pounds per square inch.

In determining the tensile resistance, and, indeed, other resistances of timber, the size of the specimen plays a more important part, probably, than in any other class of materials used by the engineer. Small specimens, such as are usually employed in tensile tests, are inevitably so selected as to eliminate such defects as decay and decayed or other knots, wind shakes, season cracks, and other deteriorating features, so that the results exhibit physical properties belonging to the best parts of full-size sticks. In engineering practice, on the other hand, large pieces of timber must be used as furnished in the timber market. However close the inspection may be such pieces invariably include within their volumes many spots of weakness due to those features which in the small specimen are carefully excluded. It is of the utmost consequence, therefore, in dealing with physical data belonging to timber to realize that results determined by the testing of small specimens are almost without exception materially misleading in consequence of reaching higher values than those which can possibly belong to the average stick used in structural work. These observations must be carefully remembered in considering the experimental data which follow.

There are comparatively few general data available regarding the tensile tests of timber as it is employed in structural work, and almost no tensile tests giving values of the coefficient of elasticity in tension which can be accepted for use. In Art. 69, on timber beams, there will be found some very satisfactory determinations of the coefficient of elasticity for bending which may be taken as representing the stiffness of timber under stress.

Between 1891 and 1895 Professor J. B. Johnson of Washington University made a mass of tests of American timber for the Forestry Division of the U. S. Department of Agriculture. Among his results will be found a large number belonging to long-leaf yellow pine (Pinus palustris) in tension. These results, although not much affected by moisture, are for specimens containing 15 per cent. of moisture and about 1 square inch in cross-sectional area. They are as follows, expressed in pounds per square inch:

Maximum.	Average.	Minimum.
22,792	17,359	13,220

These ultimate resistances are excessively high for the reasons given, and they should not be used for the commercial timber employed in engineering practice.

The preceding values belong to specimens taken from twenty-six different trees, some of which had been tapped for turpentine while others had not; the investigations of the Forestry Division of the U. S. Department of Agriculture having shown that the operation of "boxing," or tapping the trees for turpentine, has no sensible effect upon the resistance of the timber.

The ultimate tensile resistances of many woods, domestic and foreign, are given in Table I, as well as the specific gravities.

The column "*B*" will be explained hereafter, in the chapter on transverse resistance or bending.

TABLE I.

Kind of Timber.	Experimenter.	Specific Gravity.	Ultimate Resist'nce in Pounds per Square Inch.	*B*, in Pounds per Square Inch.
Oak, English	Laslett	0.858	3,837	Values for this column will be found computed in Art. 63.
Oak, English	"	0.893	7,571	
Oak, French	"	0.976	8,102	
Oak, Dantzic	"	0.838	4,217	
Oak, American white	"	0.969	7,021	
Oak, American, Baltimore	"	0.742	3,832	
Oak, African (or Teak)	"	0.971	7,052	
Teak, Moulmein	"	0.777	3,301	
Iron wood, Burmah	"	1.176	9,656	
Chow, Borneo	"	1.134	7,199	
Greenheart, Guiana	"	1.141	8,820	
Sabicu, Cuba	"	0.917	5,558	
Mahogany, Spanish	"	0.765	3,791	
Mahogany, Honduras	"	0.659	2,998	
Mahogany, Mexican	"	0.655	3,427	
Eucalyptus, Australia:				
Tewart	"	1.169	10,284	
Mahogany	"	0.996	2,940	
Iron-bark	"	1.150	8,377	
Blue gum	"	1.049	6,048	
Ash, English	"	0.750	3,780	
Ash, Canadian	"	0.588	5,495	
Beech	"	0.705	4,853	
Elm, English	"	0.642	5,460	
Rock elm, Canada	"	0.748	9,182	
Hornbeam, England	"	0.819	6,405	
Fir, Dantzic	"	0.603	3,231	
Fir, Riga	"	0.553	4,051	
Fir, spruce, Canada	"	0.484	3,934	
Larch, Russia	"	0.649	4,203	
Cedar, Cuba	"	0.469	2,870	
Red pine, Canada	"	0.553	2,705	
Yellow pine, Canada	"	0.551	2,759	
Yellow pine, Canada	"	0.552	2,259	
Pitch pine, American	"	0.659	4,666	
Kauri pine, New Zealand	"	0.544	4,040	
Georgia pine, American	Hatfield	——	16,000	
Locust, American	"	——	24,800	
White oak, American	"	——	19,500	
Spruce, American	"	——	19,500	
White pine, American	"	——	12,000	
Hemlock	"	——	8,700	

The table gives average results. Those determined from experiments of Mr. Laslett are of English origin ("Timber and Timber Trees, Native and Foreign," by Thomas Laslett, 1875); the others are from American experiments by the late R. G. Hatfield ("Transverse Strains," 1877). Mr. Laslett's specimens were 2 inches square in cross-section, and generally were 30 inches long, while those of Mr. Hatfield were about 0.35 inch round.

It will be observed that Mr. Hatfield reached far higher results than Mr. Laslett. This disagreement may be due to the larger cross-sectional area of the latter's specimens, which certainly brings his (Mr. Laslett's) results more nearly in accordance with what might be expected from such pieces as are ordinarily used by engineers. Mr. Hatfield's specimens were far too small for technical purposes.

TABLE II.

DIAMETER OF TEST SPECIMENS, 1 INCH.

Number.	Kind of Wood.	Ultimate Resistance per Square Inch in Pounds.			No. of Tests.
		Greatest.	Mean.	Least.	
1	Yellow pine........	17,922	15,478	12,066	4
2	Oregon pine........	——	13,810	——	1
3	Oregon spruce......	——	16,160	——	1
4	White pine........	11,299	8,916	5,300	4
5	Spruce............	17,044	14,283	11,600	4
6	White wood........	7,466	6,787	6,107	2
7	Gum wood.........	19,400	14,313	4,586	4
8	White maple.......	15,714	11,164	7,312	3
9	Black walnut......	14,650	11,492	9,286	3
10	Red birch.........	22,838	18,682	13,885	3
11	White ash.........	27,532	24,120	18,961	3
12	Brown ash.........	11,733	11,632	10,667	3
13	White oak.........	22,703	17,410	12,670	4
14	Red oak...........	12,133	10,124	7,600	3
15	Yellow oak........	20,520	20,390	20,260	2
16	Hickory...........	19,610	15,995	12,400	8

All specimens were of well-seasoned wood.

The values given in Table II were found by Col. Laidley, U. S. Army, in the Government machine at Watertown, Mass. (Ex. Doc. No. 12, 47th Congress, 2d Session). Two of the specimens were about 0.63 inch in diameter, and one 1.25 inches. All the rest possessed diameters of about one inch each.

Such small specimens as those of Hatfield and Laidley, which were probably selected, give much larger results than would be found for large pieces of ordinary lumber; these considerations are highly prejudicial to the technical value of the results.

TABLE III.

Kind of Timber.	Ultimate Resistance, Pounds per Square Inch.		Working Stresses, Pounds per Square Inch.	
	With Grain.	Across Grain.	With Grain.	Across Grain.
White oak	10,000	2,000	1,000	200
White pine	7,000	500	700	50
Southern long-leaf or Georgia yellow pine	12,000	600	1,200	60
Douglas, Oregon, and yellow fir	12,000	—	1,200	—
Washington fir or pine (red fir)	10,000	—	1,000	—
Northern or short-leaf yellow pine	9,000	500	900	50
Red pine	9,000	500	900	50
Norway pine	8,000	—	800	—
Canadian (Ottowa) white pine	10,000	—	1,000	—
Canadian (Ontario) red pine	10,000	—	1,000	—
Spruce and Eastern fir	8,000	500	800	50
Hemlock	6,000	—	600	—
Cypress	6,000	—	600	—
Cedar	8,000	—	800	—
Chestnut	9,000	—	900	—
California redwood	7,000	—	700	—
California spruce	—	—	—	—

Reviewing all the experimental work which has been done up to the present time (1902) in determining the ultimate tensile resistance of timber, and keeping in view experience with the resistance of full-size timber sticks in completed structures, the best representative series of values of the ultimate and working tensile intensities of timbers is that recommended by the Committee on "Strength of Bridge and Trestle Timbers" of the Association of Railway Superintendents of Bridges and Buildings at the Fifth Annual Convention in New Orleans, 1895. That series is given in Table III.

The values given in the table belong to just such timber as is used in engineering structures, and they are as high as should be taken for practical purposes.

It will be noticed that the ultimate tensile resistance of the various timbers across the grain, so far as they are given, are but small fractions of the ultimate resistances along the grain. A corresponding large decrease in resistance across the grain will also be found in connection with the compressive resistance of the same timbers. The working resistances given in this table are those employed in the great bulk of engineering timber structures.

CHAPTER VII.

COMPRESSION.

Art. 48.—Preliminary.

WITH the exception of material in the shape of long columns, but few experiments, comparatively speaking, have been made upon the compressive resistance of constructive materials.

Pieces of material subjected to compression are divided into two general classes—"short blocks" and "long columns"; the first of these, only, afford phenomena of *pure compression.*

A "short block" is such a piece of material that if it be subjected to compressive load it will fail by pure compression.

On the other hand, a long column (as has been indicated in Art. 24) fails by combined compression and bending.

Short blocks only will be considered in the articles immediately succeeding, while long columns will be separately considered further on.

The length of a short block is usually about three times its least lateral dimension or less.

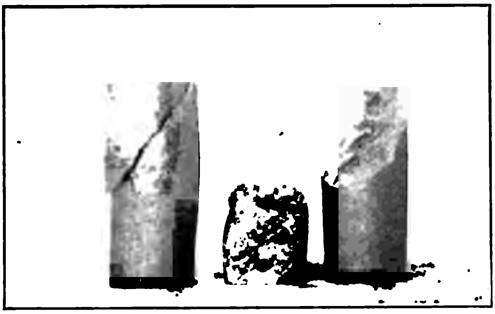
It has already been shown in Art. 4 that the greatest shear in a short block subjected to compression will be found in planes making an angle of 45° with the surfaces of the block on which the compressive force acts, i.e., with its ends. If the material is not ductile this shear will frequently cause wedge-shaped portions to separate from the block. But the friction at these end surfaces, and in the surfaces of failure will prevent those wedge portions shearing off at that angle. In fact the friction will cause the angle of separation to be considerably larger than 45°; let it be called α. Then, in order that there may be perfect freedom in failure, the length of the block must not be less than its least width or breadth multiplied by $2 \tan \alpha$. In some cases, α has been found to be about 55°, for which value.

$$2 \tan \alpha = 2 \times 1.43 = 2.86.$$

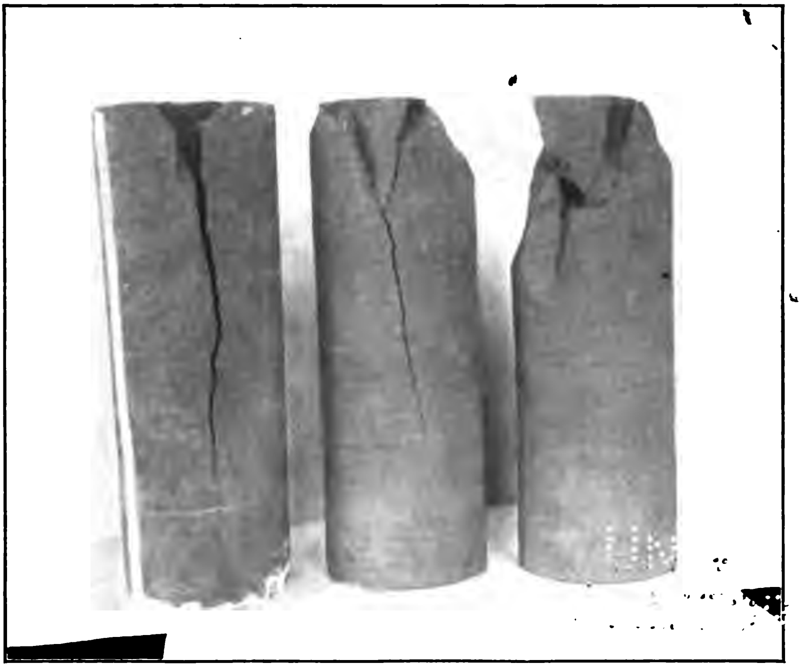
It was shown in the first section of Art. 41, that the "ultimate resistance" to tension is in reality a mean, and not the greatest intensity which the material exerts. The same course of reasoning will show that it is, also, in general, impossible to subject a short block to a uniform intensity of compression throughout its mass, and that the "ultimate resistance to compression" is a mean, usually considerably less than the greatest intensity which exists at the centre of a normal section. As the inner portion will be supported laterally by that outside of it, large blocks of brittle material may give greater intensities of ultimate resistance than small ones.

Art. 49.—Wrought Iron.

It is difficult to fix the point of failure of a short block of wrought iron or other ductile material. An excessive compressive force causes the material to increase very considerably in lateral dimensions, or to "bulge" out, so that every increase of compressive force simply produces an increased area of resistance, while the material never truly fails by crumbling or shearing off in wedges.



Failure of short cylinders of cast iron showing the shearing of the metal on the plane of maximum shear.



View exhibiting the failure of short cylinders of Connecticut brown sandstone.

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A short block of wrought iron is usually considered to fail when its length is shortened by five to ten per cent.

If p_1 is any intensity of stress while l_1 is the compressive strain, or shortening per unit of length caused by p_1, then, according to eq. (2) of Art. 2, the coefficient of elasticity for compression at the intensity p_1 will be

$$E_1 = \frac{p_1}{l_1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

This ratio is not constant for all degrees of stress and strain, though for wrought iron, within the elastic limit, the divergences from a mean value are not great. Table I contains coefficients of elasticity calculated by Prof. De Volson Wood, in the manner shown by eq. (1), from the data determined by Mr. Eaton Hodgkinson, and given in his work before cited. (See Prof. Wood's "Treatise on the Resistance of Materials.")

TABLE I.

Ap_1.	Ll_1.	E_1.	Ll_1.	E_1.
Pounds.	Inch.	Pounds.	Inch.	Pounds.
5,098	0.028	20,796,500	0.027	21,864,000
9,578	0.052	21,049,000	0.047	23,595,000
14,058	0.073	21,979,000	0.067	24,273,000
16,298	0.085	21,343,000	——	——
18,538	0.096	22,156,000	0.089	24,108,000
20,778	0.107	22,160,000	0.100	24,038,000
23,018	0.119	23,587,000	0.113	23,587,000
25,258	0.130	22,095,000	0.128	23,679,000
27,498	0.142	22,111,000	0.143	22,259,000
29,738	0.152	21,938,000	0.163	21,139,000
31,978	0.174	20,979,000	0.190	19,478,000

The results belong to two square bars, and E_1 is in pounds per square inch. A is the area of cross-section; it was 1.0506 square inches for the first bar and 1.0363 square inches for the other. Hence the bars were about one inch square. They were also ten feet long ($L = 10.00$

feet) and required lateral support to be kept in alignment so as to act like short blocks.

The table shows that the values of E_1 increase with p_1, when the latter is small; an opposite result was found for tension.

What may be called the elastic limit is found for $p_1 =$ 30,000 pounds per square inch (nearly). Hence it is seen that the greatest value of E_1 is found for p_1, equal to one half to two thirds the elastic limit.

The same general remarks in regard to the elastic limit which were made in connection with tension may be also applied to the compressive elastic limit.

The "Steel Committee" of British civil engineers, in 1870, made some experiments on twelve bars of Lowmoor wrought iron, 1.5 inches in diameter and 120 inches long. These twelve experiments were divided into four sets of three each, and the table gives the means of each of these sets or groups. The coefficients are computed at the elastic limit.

TABLE II.

Pounds per Square Inch for	
Elastic Limit.	Coefficient of Elasticity.
Pounds.	Pounds.
29,800	29,091,000
25,800	29,091,000
29,100	28,718,000
26,200	28,000,000

As a mean value, the coefficient of elasticity for wrought iron in compression may be taken at 28,000,000 pounds per square inch. For every ton (2000 pounds) of compression per square inch, therefore, a piece of wrought iron will be shortened by an amount equal to

$$\frac{2000}{28,000,000} = \frac{1}{14,000} \text{ of its length.}$$

Table III contains the results of some experiments made by Mr. Kirkaldy on specimens of Swedish iron in 1866. The last column gives the per cent. of compression of original length which the piece suffered at the point called the "ultimate compressive resistance." The results show well the great increase of resistance which a short block of ductile material offers with the increase of compression.

TABLE III.

Section of Specimen.	Length.	Pounds per Square Inch for		Per Cent. Compression.
		Elastic Limit.	Ultimate Resistance.	
In.	Ins.	Lbs.	Lbs.	
1.5 ○	1.5	24,050	148,800	45
1.5 ○	1.5	21,200	28,100	4
1.5 ○	3	23,300	84,900	33
1.0 □	1	——	184,100	53

Table IV gives the results of experiments on some very short lengths of Phœnix and Keystone columns. The first six results are for Phœnix sections from experiments by the Phœnix Iron Co. in 1873; the two following are for the same section from experiments made at Watertown, Mass., in 1879; while the last result belongs to a Keystone section experimented upon by Mr. G. Bouscaren in 1875. Unfortunately the amount of compression or shortening, in each instance, was not recorded.

TABLE IV.

Length, Inches.	Ratio of Length to Diameter of Section.	Area of Section, Square Inches.	Ultimate Resistance in Pounds per Square Inch.
8.00	1.46	6.97	60,570
8.00	1.46	6.97	60,390
4.00	0.92	5.62	65,870
4.00	0.92	5.62	65,870
4.00	1.01	2.92	56,890
4.00	1.01	2.92	55,560
8.00	1.00	11.90	57,130
8.00	1.00	11.90	57,300
9.00	1.12	14.25	51,500

Reviewing the results given in Tables II, III, and IV, it is seen that the "elastic limit" of wrought iron in compression, in short blocks, may be taken from 0.4 to 0.5 its ultimate compressive resistance, while the latter may be taken at about 55,000 pounds per square inch.

Art. 50.—Cast Iron.

The irregular elastic behavior of cast iron, as seen in tension, will also be discovered in compression. Table I contains results computed from the data obtained by Captain Rodman by testing solid cylinders 10 inches long and 1.382 inches in diameter. The second column belongs to a specimen cylinder taken from a 10-inch columbiad, and the third or last to a trial cylinder of remelted Greenwood and Salisbury iron. Neither specimen can be considered to possess a true elastic limit, but what is ordinarily so termed may be taken at about 20,000 pounds per square inch.

In the first specimen the first permanent set took place at 3000, and in the second at 5000 pounds per square inch.

For a bar ten feet long and one inch square Mr. Eaton Hodgkinson found the following values:

Greatest.	Mean.	Least.
13,216,000	12,134,100	10,837,100

all in pounds per square inch. The greatest value was found at 2240, and the least at 38,080 pounds per square inch.

TABLE I.

Intensity of Stress.	Coefficient of Elasticity in Pounds per Square Inch.	
1,000	6,896,600	——
2,000	8,888,900	33,333,300
3,000	9,836,100	18,750,000
4,000	10,666,700	13,793,100
5,000	10,752,700	13,888,900
6,000	11,320,800	12,766,000
7,000	11,382,100	13,725,500
8,000	11,510,800	13,559,300
9,000	11,920,500	13,432,800
10,000	12,121,200	13,333,300
11,000	12,290,500	13,095,200
12,000	12,182,700	13,186,800
14,000	12,444,400	13,207,500
16,000	12,260,500	12,903,200
18,000	11,920,500	12,857,100
20,000	11,695,900	12,578,600
22,000	11,253,200	12,290,500
26,000	10,236,200	11,607,200
30,000	8,596,000	10,101,000
35,000	——	7,658,600
40,000	——	5,333,300

Since the coefficient of elasticity measures the *stiffness* of a body, and since the coefficient of elasticity for wrought iron in compression has been seen to be at least twice as great as that of cast iron in the same condition, wrought iron is at least twice as stiff, compressively, as cast metal. A bar of the latter material will be compressed by 2000 pounds per square inch, about

$$\frac{2000}{12,000,000} = \frac{1}{6000} \text{ of its length.}$$

As long ago as 1840 Mr. Eaton Hodgkinson tested specimens of British cast irons 0.75 inch in diameter and 0.75 and 1.5 inches long with results ranging from 117,605 pounds down to 56,445 pounds per square inch and with an average value of 86,284 pounds per square inch.

Sir William Fairbairn also tested English cast irons after successive remeltings from one to eighteen times, and found the ultimate resistance to increase from 98,560 pounds at the first remelting to 197,190 pounds per square inch after the eighteenth remelting.

It is thus seen that the ultimate compressive resistance of cast iron may vary between wide limits, but experimental investigations on the whole show that the ultimate compressive resistance of such metal as is used for ordinary engineering castings should not be taken higher than about 80,000 pounds per square inch, but that 100,000 pounds per square inch may be taken for the better and stronger grades used for special purposes where higher degrees of excellence are required.

Major Wade tested a number of specimens of cast iron of different numbers of fusions, in order to determine the ultimate compressive resistance. His specimens were from 0.5 to 0.6 inch in diameter, and from 1.25 to 1.5 inches long (nearly). The results were as follows in pounds per square inch.

Fusion.	No. of Experiments.	Greatest	Mean.	Least.
2d.	4	114,504	99,770	84,529
3d.	2	140,415	139,540	138,666
2d and 3d. . .	2	169,427	168,589	167,752
2d.	2	140,415	136,868	133,321
3d.	1	168,251	168,251	168,251
2d.	5	163,528	154,576	144,141
3d.	4	174,120	167,030	156,863

The working resistance of cast iron in compression for short blocks and massive castings may be taken from 8000 to perhaps 12,000 pounds per square inch, according to the conditions under which the castings are to be employed.

Art. 51.—Steel.

Table II of Art. 43 contains the results found by Prof. Ricketts in testing cylindrical specimens of mild steel in compression. These specimens were six inches long between carefully faced ends, and, as the table shows, their diameter was about 0.75 inch. The coefficients of elasticity for compression were found by measurements very carefully made with a micrometer on a length of four inches. The elastic limits, however, were determined by operating with a cylinder two inches long, and were taken at those points where the material of the specimens ceased to hold up the scale beam, and may have been somewhat above that point where the ratio between stress and strain ceases to be essentially constant.

The coefficients of elasticity are found to be quite uniform, irrespective of the per cents of carbon, within the limits of the table, and they are seen to be a very little less than the coefficients for tension. The difference is so small that no essential error will arise if, for all engineering purposes, they are assumed the same.

A comparison of the elastic limits for tension and compression presents some irregularities; yet with the exception of the high percentages of carbon in the last two grades of Bessemer metal, the two sets of elastic limits as wholes are not very different from each other. In the Bessemer steel with the two high per cents of carbon, the tensile elastic limits are materially higher than those for compression. The following very important conclusion results from this comparison of the elastic limits for the mild structural steels: since these elastic limits are essentially equal it is not only permissible but wholly rational to increase the working resistances of mild steel bridge columns over those for iron in at least the same proportion that the tensile working stress of the same steel is increased over that of iron in tension. Experiments on a sufficient number of full-size steel columns are yet lacking to verify this conclusion.

It appears from such data on the compressive resistance of steel as exist that not only the coefficient of elasticity but, also, the limit of elasticity in compression may be taken the same as that for tension for the same grade of steel. This was practically true in the older investigations of Kirkaldy, and it is essentially confirmed in the few later investigations available.

The ultimate compressive resistance of steel, like the ultimate tensile resistance, varies with the content of carbon, being comparatively low with a small percentage of carbon, and correspondingly large with a high percentage of that element. It is also much affected by the operations of tempering and annealing.

Chief Engineer Wm. H. Shock, U.S.N., 1868, gives the following results for Parker Bros. "Black Diamond" steel:

Normal untempered steel: *Ult. Resist.* from 100,100 to 112,400 pounds per square inch.

Heated to light cherry-red and plunged in oil at 82° Fahr.; *Ult. Resist.* from 173,200 to 199,200 pounds per square inch.

Heated as before and plunged in water at 79° Fahr., with final temper (plum-blue) drawn on heated plate: *Ult. Resist.* from 325,400 to 340,800 pounds per square inch.

Heated as before and plunged in water at 79° Fahr., and tested at maximum hardness: *Ult. Resist.* from 275,640 to 400,000 pounds per square inch. In each of these cases there were three tests.

The following values (each is a mean of 8 tests) were found by the United States Test Board, "Ex. Doc. 23, House of Rep., 46th Congress, 2d Session," for small annealed specimens of tool steel, of about one inch in length and 0.715 inch in diameter:

Ult. comp. per sq. in. of original section.	175,992; 174,586; 183,938; 193,413; 193,197; 174,586; 193,517; 174,895 pounds per square inch.
Final comp. per sq. in. of final section.	134,717; 127,579; 149,881; 139,196; 145,751; 128,834; 125,126; 140,489 pounds per square inch.

The final lengths varied from 56 to 89 per cent. of the originals.

Table I contains the results of Major Wade's experiments on specimens of cast steel in 1851. The results are seen to be very high.

TABLE I.

Description.	Length over Diameter.	Ultimate Resistance in Pounds per Sq. In.
Not hardened....................	2.55	198,944
Hardened, low temper...........	2.47	354,544
Hardened, mean temper..........	2.52	391,985
Hardened, high temper...........	2.48	372,598

All specimens about 1 inch long and 0.4 inch in diameter.

A piece of the Hay steel used by Gen. Smith in the Glasgow, Mo., bridge, about 1¼ inches square and 3⅛ inches long, gave an ultimate compressive resistance of 139,350 pounds per square inch ("Annales des Ponts et Chaussées," Feb., 1881).

There is the same uncertainty as to the point at which compressive failure takes place in steel which attaches to the ultimate compressive resistance of all ductile metals and which was commented upon in Art. 49. It is probably safe, however, if not entirely correct, to take the ultimate compressive resistances of different grades of steel equal to their ultimate tensile resistances in the absence of explicit determinations; and a similar observation may be applied to the working resistances in pure compression of same grades of steel.

Art. 52.—Copper, Tin, Zinc, Lead, and Alloys.

Table I shows some coefficients of elasticity (i.e., ratios between stress and strain), computed from data determined by Prof. Thurston, and given by him in the "Trans. Amer. Soc. of Civ. Engrs.," Sept., 1881. The gun bronze contained copper, 89.97; tin, 10.00; flux, 0.03. The cast copper was cast very hot.

TABLE I.

Stress in Pounds per Square Inch.	Coefficients of Elasticity in Pounds per Square Inch.	
	Gun Bronze.	Cast Copper.
1,620	——	1,254,000
3,260	3,622,000	1,415,000
6,520	4,075,000	1,651,000
9,780	6,113,000	1,795,000
13,040	6,520,000	1,824,000
16,300	5,433,000	1,842,000
19,560	5,148,000	1,845,000
22,820	3,935,000	1,735,000
26,080	2,308,000	1,503,000
29,340	——	1,144,000
32,600	1,073,000	815,000
48,900	463,600	332,500

The ratios of stress over strain are far from being constant. Strictly speaking, therefore, there is no elastic limit in either case. In that of the gun bronze, however, it may be approximately taken at 20,000 pounds per square inch (Prof. Thurston takes it 22,820), and in that of the copper at 25,000 pounds. The test specimens were two inches long and turned to 0.625 inch in diameter.

At 38,000 pounds per square inch the gun bronze specimen was shortened about 41 per cent. of its original length, while its diameter had become 0.77 inch.

TABLE II.

Composition.		Pounds per Square Inch Causing a Shortening of			Greatest Load in Pounds per Square Inch.	Per Cent. of Shortening Caused by Greatest Load.	Ultimate Crushing Resistance in Lbs. per Square Inch.	Manner of Failure.
Copper.	Tin.	5 Per Cent.	10 Per Cent.	20 Per Cent.				
07.83	1.92	29,340	34,000	46,000	46,260	0.37	34,000	Flattened
95.96	3.80	39,200	42,050	52,150	52,150	0.30	42,050	"
92.07	7.76	31,500	42,000	65,000	84,100	0.45	42,000	"
90.43	9.50	32,000	38,000	60,000	61,930	0.34	38,000	"
87.15	12.77	39,000	53,000	80,000	89,640	0.39	53,000	"
80.90	18.92	65,000	78,000	103,490	103,490	0.20	78,000	"
76.60	23.23	101,040	——	——	114,080	0.09	114,080	Crushed
69.90	29.85	——	——	——	146,680	0.04	146,680	"
65.31	34.47	——	——	——	84,750	0.03	84,750	"
61.83	37.74	——	——	——	39,110	0.02	39,110	"
47.72	51.09	——	——	——	84,750	0.02	84,750	"
44.62	55.15	——	——	——	35,850	0.01	35,850	"
38.83	60.79	——	——	——	39,110	0.02	39,110	"
38.37	61.32	——	——	——	29,340	0.01	29,340	"
34.22	65.80	19,560	——	——	19,560	0.06	19,560	"
25.12	74.51	17,930	17,930	17,030	17,930	0.28	17,930	"
20.21	79.62	16,300	16,300	16,300	16,300	0.29	16,300	"
15.12	84.58	6,520	6,520	6,520	9,450	0.51	6,520	Flattened
11.48	88.50	10,100	10,100	10,100	14,020	0.50	10,100	"
8.57	91.39	6,500	——	——	9,780	0.06	9,780	"
3.72	96.31	6,520	6,520	6,520	9,780	0.34	9,780	"
0.74	99.02	6,520	6,520	6,520	9,780	0.36	9,780	"
0.32	99.46	6,520	6,520	6,520	9,780	0.38	9,780	"
Cast copper		26,000	39,000	51,000	74,970	0.45	39,000	"
"	"	33,000	45,500	58,670	78,230	0.43	45,500	"
"	"	34,000	42,000	58,000	71,710	0.32	42,000	"
"	"	30,000	36,000	50,000	104,300	0.52	36,000	"
"	"	30,000	37,000	50,000	91,270	0.48	37,000	"
"	"	35,000	48,000	65,000	97,790	0.41	48,000	"
Cast tin		6,030	6,400	6,530	7,500	0.44	6,400	"

The copper specimen failed at 71,700 pounds per square inch, having been shortened about one third of its length.

The results of a series of tests by Prof. Thurston, in connection with the United States testing commission, are given in Table II; they were abstracted from "Mechanical and Physical Properties of the Copper-tin Alloys," United

States Report, edited by Prof. R. H. Thurston, 1879. All the specimens were 0.625 inch in diameter and 2 inches long. Scarcely one of them can be said to possess an elastic limit.

The series of alloys presents some interesting results. About the middle third of the series are seen to be brittle compounds giving (as a rule) high ultimate compressive resistances, while the other two thirds are ductile, and give at the copper end high results, and low ones at the tin end.

It will be observed that Prof. Thurston took the load per square inch which gave a shortening of 10 per cent. of the original length as the ultimate resistance to crushing of the ductile alloys and metals, since such materials cannot be said to completely fail under any pressure, but spread laterally and offer increased resistance.

TABLE III.

Per Cent. of		Pounds per Square Inch for		Per Cent. of Shortening.	Manner of Failure.
Copper.	Zinc.	E_1.	Ultimate Resistance.		
96.07	3.79	305,500	29,000	10.0	Flattened
90.56	9.42	342,100	30,000	10.0	“
89.80	10.06	——	29,500	10.0	“
76.65	23.08	656,500	42,000	10.0	“
60.94	38.65	1,772,500	75,000	10.0	“
55.15	44.44	——	78,000	10.0	“
49.66	50.14	1,345,500	117,400	10.0	“
47.56	52.28	1,500,000	121,000	10.0	“
25.77	73.45	4,232,800	110,822	5.85	Crushed
20.81	77.63	2,485,000	52,152	2.75	“
14.19	85.10	897,000	48,892	10.8	“
10.30	88.88	——	49,000	10.0	Flattened
4.35	94.59	——	48,000	10.0	“
0.00	100.00	318,500	22,000	10.0	“

Table III contains the results of Prof. Thurston's tests of the copper-zinc alloys made while he was a member of the United States Board. The data are taken from "Ex.

Doc. 23, House of Representatives, 46th Congress, 2d Session." The specimens were two inches long and 0.625 inch in diameter of circular cross-section.

The values of E_1 (ratios of stress over strain) are computed for about one quarter the ultimate resistance. This ratio is so very variable for different intensities of stress that these alloys can scarcely be said to have a proper "elastic limit."

Two specimens of tobin bronze, each .75 inch in diameter and 1 inch long, tested by the Fairbanks Company of New York City in 1891, were compressed about .8 per cent. at 45,000 pounds per square inch, and a little over 10 per cent. at 90,000 pounds per square inch. Tobin bronze contains 58.2 per cent. copper, 2.3 per cent. tin, and 39.5 per cent. zinc.

Art. 53.—Glass.

The following results are taken from Sir Wm. Fairbairn's "Useful Information for Engineers," second series. The cylinders were about 0.75 inch in diameter and annealed.

TABLE I.

Kind of Glass.	Specimen.	Height of Specimen.	Crushing Resistance, Lbs. per Sq. In.
		Inch.	
Flint....................	Cylinder	1.00	23,480
"	"	1.00	34,850
"	"	1.60	20,780
"	"	2.05	32,800
Green..................	"	1.00	22,580
"	"	1.50	35,030
"	"	2.00	38,020
Crown..................	"	1.00	23,180
"	"	1.50	38,830
Flint....................	Cube	1.15	14,240
"	"	1.16	13,200
"	"	1.10	13,260
"	"	1.10	11,820
Green..................	"	1.00	20,470
"	"	1.00	19,950
Crown..................	"	0.90	21,870

It will be observed that the cubes give considerably less resistance than the cylinders.

All the glass was annealed, but Fairbairn remarks that the cubes may have been only imperfectly so, since they were cut out of the interior of larger masses, while the cylinders were cut from rods as they were drawn. The latter, also, thus retained their natural skins, which may have increased their resistances.

At the instant of failure the specimens were shattered into a great number of small pieces.

Art. 54.—Cement—Cement Mortar—Concrete.

The ultimate compressive resistances of mortars and concrete determine the carrying power of many engineering works, and it is of the greatest importance to ascertain those resistances and the conditions under which they may be made the greatest possible. Obviously, the carrying power in compression of both mortars and concretes will depend upon a considerable number of elements such as the character of the cement, the proportions of mixture of the sand and cement for mortar or of the cement, sand, and gravel or broken stone for concrete, the thoroughness of the admixture, the amount of water used, the conditions under which the mortar and concrete are maintained while the operation of setting is taking place, the temperature, and other various influences.

Table I is made up from data contained in the "Report of Tests of Metals and Other Materials," as carried on by U. S. officers at the Watertown Arsenal for 1898. The results belong to the various brands of Portland- and Silica-cement concrete exhibited in the table, considerably extended so as to include a great variety of data. It will be noticed that there are three different brands of cement and a great variety of proportions of cement, sand, and broken stone for 12-inch cubes of both concrete and mortar which give the table exceptional value, especially in view of the number of conditions under which the cubes were kept and the three grades of mortar—"Dry," "Plastic," and "Excess." The mortar of "Dry" consistency contained relatively a small amount of water and just enough to give in ordinary engineering practice the name which it carries. That mortar which is indicated as carrying an "Excess" of water would be considered wet or even sloppy in practice, while the "Plastic" mortar may be considered as midway in consistency between the other two. The age of the cubes when tested varying from about a year and a half to a year and three quarters also enhances the value of the results.

Among the important elements of the table is that of the coefficient of elasticity which runs in general from somewhat less than 2,000,000 pounds per square inch to over 2,500,000 pounds per square inch, the smaller values in general belonging to the lean concretes. These results and those in the tables which follow appear to indicate that the coefficient of elasticity for first-class concrete masonry need not be taken less than about 2,500,000 pounds per square inch, and some of the results appear to indicate that even 3,000,000 pounds per square inch would be justifiable. Indeed, in carefully fabricated concrete masonry, such as is used in combined concrete and steel construction with a concrete as rich as 1 cement, 2 sand, and 4 gravel or broken stone, 3,000,000 pounds per square inch is probably as reasonable a value as can be used.

In Table I it should be carefully noted that the weight of mortar per cubic foot is materially less than that of the concrete. Obviously this condition is due to the increased weight of the greater amount of stone in the con-

TABLE I.

TESTS OF 12-INCH CUBES OF PORTLAND-CEMENT CONCRETE AND MORTAR.

THOMAS MILLEM & SONS WAYLAND PORTLAND CEMENT.

Composition.	In Mold. Hours.	In Water. Months.	Buried in Sand. Years.	In Cool Cellar. Years.	In Air Covered with Sacki'g, Wet Several Times a Day. Months.	Expos'd Out of Doors. Years.	Under Cover in Dry Place. Months.	Age when Tested. Years.	Consistency of Mortar.	Mortar Per Cent. of Broken Stone.	Weight per Cubic Foot.	Pounds per Square Inch.	
												Ult. Resistance.	Coefficient of Elasticity.
1 c., 1 s., 4.5 b. st. ..	65	4.16	1.11	——	——	——	3.17	1.73	Dry	33	148.3	4,481	2,273,000
" " " ...	65	——	——	1.45	——	——	.63	1.51	"	"	145.3	4,126	——
" " " ...	65	——	——	——	——	1.44	1.47	1.58	"	"	147.8	4,345	——
" " " ...	65	——	——	——	3.13	1.19	1.27	1.57	"	"	146.5	4,118	——
1 c., 1 s., 6.5 b. st. ..	47	4.1	1.11	——	——	——	3.5	1.75	Plastic	33	148.9	4,510	1,667,000
" " " ...	47	——	——	1.44	——	——	.83	1.52	"	"	146.8	4,090	——
" " " ...	47	——	——	——	——	1.43	1.77	1.6	"	"	147.9	3,849	——
" " " ...	47	——	——	——	3.1	1.19	1.07	1.54	"	"	147.8	3,839	——
1 c., 1 s., 5.23 b. st. ..	23	4.07	1.11	——	——	——	3.53	1.75	Excess	33	146.4	4,253	1,786,000
" " " ..	23	——	——	1.44	——	——	.93	1.53	"	"	144.5	3,417	1,562,000
" " " ..	23	——	——	——	——	1.44	1.7	1.6	"	"	146.4	3,525	——
" " " ..	23	——	——	——	3.2	1.19	.97	1.53	"	"	146.2	3,858	——
1 c., 2 s., 6.62 b. st. ..	63	4.16	1.11	——	——	——	3.23	1.74	Dry	33	146.1	3,368	2,083,000
" " " ..	63	——	——	1.45	——	——	.63	1.51	"	"	143.2	2,795	——
" " " ..	63	——	——	——	——	1.45	1.77	1.61	"	"	144.6	2,616	——
" " " ..	63	——	——	——	3.17	1.19	1.13	1.56	"	"	144.3	2,774	——
1 c., 2 s., 7.39 b. st. ..	45	4.1	1.11	——	——	——	3.5	1.75	Plastic	33	146.1	3,132	2,778,000
" " " ..	45	——	——	1.44	——	——	.63	1.51	"	"	142.4	2,568	——
" " " ..	45	——	——	——	——	1.44	.73	1.52	"	"	146.8	2,671	——
" " " ..	45	——	——	——	3.07	1.19	1.23	1.56	"	"	145	2,736	——

1 c., 2 s., 6.23 b. st..	21	4.07	1.11	—	—	—	3.47	1.74	Excess	33	145.2	3,164	2,083,000
" " " ..	21	—	—	1.44	—	—	.77	1.51	"	"	142.9	2,649	—
" " " ..	21	—	—	—	—	1.44	1.87	1.6	"	"	144.3	2,822	—
" " " ..	21	—	—	—	3.07	1.19	1.03	1.54	"	"	143.6	2,751	—
1 c., 3 s., 9.38 b. st..	69	4.13	1.11	—	—	—	3.5	1.75	Dry	33	140.2	2,116	2,500,000
" " " ..	69	—	—	1.45	—	—	.83	1.53	"	"	137.7	1,868	—
" " " ..	69	—	—	—	—	1.44	1.7	1.6	"	"	141.1	2,271	—
" " " ..	69	—	—	—	3.1	1.19	1.87	1.61	"	"	140.8	1,968	—
1 c., 3 s., 9.57 b. st..	27	4.1	1.11	—	—	—	3.17	1.71	Plastic	33	145.2	2,781	2,083,000
" " " ..	27	—	—	1.44	—	—	.63	1.5	"	"	142.2	2,179	—
" " " ..	27	—	—	—	—	1.44	.63	1.5	"	"	144.5	1,846	—
" " " ..	27	—	—	—	3.07	1.19	.7	1.51	"	"	144.4	2,023	—
1 c., 4 s., 12.8 b. st..	41	4.07	1.11	—	—	—	2.03	1.62	Plastic	33	144.3	1,761	1,250,000
" " " ..	41	—	—	1.44	—	—	.8	1.52	"	"	139.3	1,480	—
" " " ..	41	—	—	—	—	1.44	1.47	1.58	"	"	141.6	1,569	—
" " " ..	41	—	—	—	3.07	1.19	1.23	1.55	"	"	139.5	1,592	—
1 c., 3 s., 8.34 b. st..	48	3.8	1.11	—	—	—	2.4	1.63	Excess	40	142.9	2,187	1,667,000
" " " ..	48	—	—	1.51	—	—	1.88	1.59	"	"	138.6	2,073	—
" " " ..	48	—	—	—	—	1.51	1.07	1.57	"	"	142.8	1,884	—
" " " ..	48	—	—	—	2.08	1.2	.97	1.51	"	"	142.3	1,918	—
1 c., 4 s., 10.97 b. st.	45	3.8	1.11	—	—	—	3.6	1.73	Excess	40	142.2	1,869	1,562,000
" " " ..	45	—	—	1.51	—	—	1.87	1.59	"	"	140.1	1,830	—
" " " ..	45	—	—	—	—	1.51	.73	1.49	"	"	144.2	1,372	—
" " " ..	45	—	—	—	2.08	1.2	.97	1.51	"	"	142.4	1,608	—
1 c., 5 s., 11.25 b. st.	47	3.93	1.11	—	—	—	3.4	1.73	Dry	40	141.4	1,832	1,389,000
" " " ..	47	—	—	1.52	—	—	.8	1.51	"	"	137.7	1,486	—
" " " ..	47	—	—	—	—	1.52	1.7	1.59	"	"	138	1,334	—
" " " ..	47	—	—	—	2.93	1.2	1.27	1.55	"	"	141.6	1,586	—
1 c., 5 s., 12.9 b. st..	27	3.8	1.11	—	—	—	3.6	1.73	Excess	40	141.8	1,705	1,316,000
" " " ..	27	—	—	1.51	—	—	1.87	1.59	"	"	138.3	1,506	—
" " " ..	27	—	—	—	—	1.51	.63	1.48	"	"	141.9	1,008	—
" " " ..	27	—	—	—	2.8	1.2	1.87	1.59	"	"	139.5	1,381	—
1 c., 5 s., 15.2 b. st..	23	4.03	1.11	—	—	—	3.4	1.74	Excess	33	141.2	1,470	1,250,000
" " " ..	23	—	—	1.44	—	—	2	1.61	"	"	138.9	1,572	—
" " " ..	23	—	—	—	—	1.44	1.7	1.59	"	"	140.2	1,351	—
" " " ..	23	—	—	—	3.03	1.2	1.23	1.47	"	"	139	1,371	—

TABLE I.—*Continued.*

TESTS OF 12-INCH CUBES OF PORTLAND-CEMENT CONCRETE AND MORTAR.

GLENS FALLS PORTLAND CEMENT, "IRONCLAD" BRAND.

Composition.	In Mold. Hours.	In Water. Months.	Buried in Sand. Years.	In Cool Cellar. Years.	In Air Covered with Sacki'g, Wet Several Times a Day. Months.	Expos'd Out of Doors. Years.	Under Cover in Dry Place. Months.	Age when Tested. Years.	Consistency of Mortar.	Mortar Per Cent. of Broken Stone.	Weight per Cubic Foot.	Pounds per Square Inch.	
												Ult. Resistance.	Coefficient of Elasticity.
1 c., 2 s., 6.25 b. st..	42	3.63	1.11	——	——	——	3.4	1.75	Dry	40	145.1	4,000	3,125,000
" " " ..	42	——	——	1.41	——	——	1.03	1.48	"	"	143.3	2,889	——
" " " ..	42	——	——	——	——	1.41	1.77	1.56	"	"	143.8	2,986	——
" " " ..	42	——	——	——	2.63	1.2	.97	1.51	"	"	144.5	2,992	——
1 c., 2 s., 7.25 b. st..	41	3.7	1.11	——	——	——	3.93	1.75	Excess	33	144.6	2,979	3,125,000
" " " ..	41	——	——	1.42	——	——	1.03	1.51	"	"	141.4	2,196	——
" " " ..	41	——	——	——	——	1.42	1.7	1.56	"	"	143.8	2,086	——
" " " ..	41	——	——	——	2.7	1.2	1.07	1.51	"	"	143.6	2,484	——
1 c., 2 s., 8.04 b. st..	48	3.77	1.11	——	——	——	3.77	1.74	Dry	33	145.6	3,283	2,500,000
" " " ..	48	——	——	1.42	——	——	1.87	1.59	"	"	142.1	2,637	——
" " " ..	48	——	——	——	——	1.42	1.77	1.57	"	"	143.2	2,238	——
" " " ..	48	——	——	——	2.77	1.2	1.07	1.52	"	"	144	2,651	——
1 c., 3 s., 8.32 b. st..	39	3.63	1.11	——	——	——	3.53	1.71	Dry	40	141.3	2,170	2,083,000
" " " ..	39	——	——	1.41	——	——	.83	1.49	"	"	139.4	1,716	——
" " " ..	39	——	——	——	——	1.41	1.47	1.54	"	"	142.8	1,845	——
" " " ..	39	——	——	——	2.63	1.2	1.3	1.52	"	"	142.8	1,985	——
1 c., 3 s., 9.78 b. st..	45	3.77	1.11	——	——	——	3.6	1.73	Dry	33	143.3	1,863	1,667,000
" " " ..	45	——	——	1.42	——	——	1.87	1.59	"	"	137.9	1,348	——
" " " ..	45	——	——	——	——	1.42	.7	1.49	"	"	143	1,256	——
" " " ..	45	——	——	——	2.77	1.2	1.13	1.52	"	"	141.9	1,535	——

1 c., 3 s., 10.23 b. st.	63	3.67	1.11	—	—	—	2.37	1.62	Excess	33	142.6	1,919	1,667,000
" " "	63	—	—	1.42	—	—	1	1.51	"	"	138.8	1,468	—
" " "	63	—	—	—	—	1.42	1.73	1.57	"	"	141.2	1,660	—
" " "	63	—	—	—	2.67	1.2	1.7	1.57	"	"	141	1,825	—
1 c., 4 s., 10.46 b. st.	48	3.6	1.11	—	—	—	3.6	1.72	Dry	40	142.6	2,034	1,923,000
" " "	48	—	—	1.41	—	—	1.07	1.5	"	"	138	1,572	—
" " "	48	—	—	—	—	1.41	1.43	1.53	"	"	141	1,389	—
" " "	48	—	—	—	2.6	1.2	1.27	1.52	"	"	141	1,617	—
1 c., 4 s., 11.25 b. st.	47	3.53	1.11	—	—	—	3.33	1.69	Excess	40	140.6	1,752	1,667,000
" " "	47	—	—	1.41	—	—	1.07	1.5	"	"	139.1	1,430	—
" " "	47	—	—	—	—	1.41	1.77	1.53	"	"	141.4	1,418	—
" " "	47	—	—	—	2.53	1.2	1.23	1.51	"	"	139.7	1,433	—
1 c., 4 s., 12.85 b. st.	45	3.7	1.11	—	—	—	3.7	1.73	Plastic	33	141.5	1,809	1,562,000
" " "	45	—	—	1.42	—	—	.83	1.49	"	"	139.6	1,246	—
" " "	45	—	—	—	—	1.42	1.7	1.56	"	"	139.5	1,310	—
" " "	45	—	—	—	2.7	1.2	.97	1.5	"	"	141.2	1,370	—
1 c., 4 s., 15 b. st....	45	3.63	1.11	—	—	—	3.8	1.73	Excess	33	144.9	1,841	1,667,000
" " " ...	45	—	—	1.41	—	—	.87	1.49	"	"	139.7	1,416	1,087,000
" " " ...	45	—	—	—	—	1.41	1.7	1.56	"	"	141.6	1,565	—
" " " ...	45	—	—	—	2.63	1.2	1.67	1.52	"	"	141.5	1,554	—

SILICA-SAND "CHAMPION" BRAND OF CEMENT.

1 c., 1 s., 3.9 b. st...	63	3.13	1.11	—	—	—	3.5	1.68	Dry	40	143.2	3,795	2,500,000
" " " ...	63	—	—	1.37	—	—	.87	1.53	"	"	140.6	2,762	2,778,000
" " " ...	63	—	—	—	—	1.37	1.47	1.5	"	"	144	2,804	—
" " " ...	63	—	—	—	2.13	1.2	1.27	1.49	"	"	144.6	3,138	—
1 c., 1 s., 4.2 b. st...	63	3.07	1.11	—	—	—	3.5	1.67	Excess	40	142.4	2,675	2,273,000
" " " ...	63	—	—	1.37	—	—	.77	1.44	"	"	140.7	2,145	—
" " " ...	63	—	—	—	—	1.37	.8	1.44	"	"	145	1,891	—
" " " ...	63	—	—	—	2.07	1.2	1.13	1.47	"	"	140.7	2,071	—
1 c., 1 s., 5 b. st.....	87	3.23	1.11	—	—	—	3.6	1.69	Dry	33	143.4	3,140	2,500,000
" " "	87	—	—	1.38	—	—	1.77	1.53	"	"	140.3	2,457	—
" " "	87	—	—	—	—	1.38	1.43	1.51	"	"	142.9	2,391	—
" " "	87	—	—	—	2.23	1.2	1.27	1.49	"	"	143.6	2,648	—

TABLE I.—*Continued.*

TESTS OF 12-INCH CUBES OF PORTLAND-CEMENT CONCRETE AND MORTAR.

SILICA-SAND "CHAMPION" BRAND OF CEMENT.—*Continued.*

Composition.	In Mold. Hours.	In Water. Months.	Buried in Sand. Years.	In Cool Cellar. Years.	In Air Covered with Sacki'g. Wet Several Times a Day. Months.	Expos'd Out of Doors. Years.	Under Cover in Dry Place. Months.	Age when Tested. Years.	Consistency of Mortar.	Mortar Per Cent. of Broken Stone.	Weight per Cubic Foot.	Pounds per Square Inch.	
												Ult. Resistance.	Coefficient of Elasticity.
1 c., 2 s., 5.18 b. st. .	69	3.1	1.11	——	——	——	2.33	1.57	Dry	40	141	2,598	3,125,000
“ “ “ . .	69	——	——	1.37	——	——	.77	1.44	“	“	137.5	1,895	——
“ “ “ . .	69	——	——	——	——	1.37	1.7	1.52	“	“	142	2,140	——
“ “ “ . .	69	——	——	——	2.1	1.2	1.23	1.48	“	“	140.9	2,184	——
1 c., 2 s., 6.08 b. st. .	69	3.07	1.11	——	——	——	3.53	1.7	Plastic	40	141	2,400	2,500,000
“ “ “ . .	69	——	——	1.37	——	——	.97	1.45	“	“	139.1	1,799	——
“ “ “ . .	69	——	——	——	——	1.37	.7	1.43	“	“	144.4	1,681	——
“ “ “ . .	69	——	——	——	2.07	1.2	2.03	1.54	“	“	140.8	2,267	——
1 c., 2 s., 6.92 b. st. .	69	3.13	1.11	——	——	——	3.6	1.68	Excess	33	141.2	2,150	1,923,000
“ “ “ . .	69	——	——	1.37	——	——	2	1.55	“	“	138.7	1,794	——
“ “ “ . .	69	——	——	——	——	1.37	1.73	1.52	“	“	141.5	1,744	——
“ “ “ . .	69	——	——	——	2.13	1.2	1.7	1.52	“	“	141.7	1,758	——
1 c., 3 s., 8.03 b. st. .	66	3.13	1.11	——	——	——	2.33	1.57	Dry	40	138.9	1,505	1,923,000
“ “ “ . .	66	——	——	1.37	——	——	.83	1.44	“	“	137.1	1,303	——
“ “ “ . .	66	——	——	——	——	1.37	1.7	1.52	“	“	140	1,425	——
“ “ “ . .	66	——	——	——	2.1	1.2	.97	1.45	“	“	141.5	1,365	——
1 c., 3 s., 9 b. st.	65	3.13	1.11	——	——	——	3.5	1.68	Excess	33	139.2	1,324	——
“ “ “	65	——	——	1.37	——	——	.83	1.44	“	“	138	1,081	——
“ “ “	65	——	——	——	——	1.37	1.7	1.52	“	“	139.6	1,105	——
“ “ “	65	——	——	——	2.13	1.2	1.23	1.49	“	“	140.9	1,162	——

1 c., 3 s., 9.78 b. st.	89	3.17	1.11	—	—	—	3.63	1.69	Plastic	33	140.9	1,571	1,389,000
" " "	89	—	—	1.38	—	—	1.77	1.53	"	"	138.2	1,270	—
" " "	89	—	—	—	—	1.38	1.7	1.52	"	"	140.3	1,299	—
" " "	89	—	—	—	2.17	1.2	1.23	1.49	"	"	142.8	1,206	—

MORTAR CUBES OF GLENS FALLS "IRONCLAD" BRAND PORTLAND CEMENT.

1 c., 2 s.	69	2.8	1.11	—	—	—	3.47	1.64	Dry	—	129.7	4,000	2,500,000
" "	69	—	—	1.34	—	—	2	1.52	"	—	127	3,615	—
" "	69	—	—	—	—	1.34	1.7	1.49	"	—	129	3,900	—
" "	69	—	—	—	1.8	1.2	1.03	1.43	"	—	128.3	3,034	—
1 c., 3 s.	65	2.8	1.11	—	—	—	3.53	1.65	Dry	—	125.6	2,973	2,083,000
" "	65	—	—	1.34	—	—	2.03	1.52	"	—	122.4	2,572	—
" "	65	—	—	—	—	1.34	.7	1.41	"	—	126.1	2,359	—
" "	65	—	—	—	1.8	1.2	1.13	1.44	"	—	125.6	2,731	—
1 c., 4 s.	63	2.8	1.11	—	—	—	3.53	1.65	Dry	—	121.9	1,800	1,562,000
" "	63	—	—	1.34	—	—	1.77	1.5	"	—	117.5	1,561	—
" "	63	—	—	—	—	1.34	.73	1.41	"	—	121.2	1,428	—
" "	63	—	—	—	1.8	1.2	.7	1.41	"	—	122.9	1,706	—

MORTAR CUBES OF SILICA-SAND CEMENT, "CHAMPION" BRAND.

1 c., 1 s.	45	2.8	1.11	—	—	—	3	1.6	Dry	—	127.6	3,869	1,786,000
" "	45	—	—	1.34	—	—	2.03	1.52	"	—	125.1	3,270	—
" "	45	—	—	—	—	1.34	1.73	1.49	"	—	126.4	3,638	—
" "	45	—	—	—	1.8	1.2	1.3	1.44	"	—	125.7	3,426	—
1 c., 2 s.	41	2.8	1.11	—	—	—	3	1.6	Dry	—	124.81	2,690	1,667,000
" "	41	—	—	1.34	—	—	2	1.51	"	—	121	2,188	—
" "	41	—	—	—	—	1.34	1.7	1.41	"	—	123.9	2,217	—
" "	41	—	—	—	1.8	1.2	1.3	1.46	"	—	125.7	2,342	—
1 c., 3 s.	39	2.8	1.11	—	—	—	3.03	1.6	Dry	—	119.9	1,581	1,000,000
" "	39	—	—	1.34	—	—	2.03	1.52	"	—	116	1,314	—
" "	39	—	—	—	—	1.34	1.47	1.47	"	—	119.5	1,325	—
" "	39	—	—	—	1.8	1.2	1.3	1.46	"	—	120.4	1,417	—

crete. It is equally evident that the weight of concrete will depend materially upon the weight of the gravel or broken stone used. In Table I the broken stone was a hard sandstone from Portageville, N. Y. The results of this series of tests justify the following general conclusions in reference to this particular series:

In the majority of cases the 40 *per cent. mortar cubes gave by small amount the greater ultimate compressive resistance, but the differences were not great enough to be of practical importance.*

In the majority of instances the "Dry" concrete gave a little higher ultimate resistances than the "Plastic" or "Excess" cubes, but the differences were small, and it is not believed that these results indicate in general any real advantage in actual engineering practice for the "Dry" concrete.

The ultimate resistances in reference to the treatment of the cubes after setting were greatest for those kept in water and sand, while those kept in air and covered with wet sacking came next, followed by those kept in a cool cellar, while those exposed to the weather gave the least ultimate resistance. The weight of the concrete varied as a rule inversely as the amount of sand in it.

There was no sensible difference in the weights of the two sets of cubes, one having 30 *per cent. of mortar and the other* 40 *per cent., nor did the consistency of the mortar appear to have any sensible effect upon the weight of the concrete; finally, the treatment of the concrete after setting appeared to have no sensible effect upon the weight, although those kept in a cool cellar had a little less weight than the others.*

Nearly the same set of observations may be applied to the mortar cubes except that the heaviest appear clearly to be those with the "Dry" consistency, while those containing the excess of water were the lightest.

TABLE II.

MEAN ULTIMATE COMPRESSIVE RESISTANCES OF 12-INCH PORTLAND-CEMENT CONCRETE CUBES.

Portland Cements. Brand; Composition.		Mean Ultimate Resistance, Pounds per Square Inch at Age.				Coefficient of Elasticity in Pounds per Square Inch at Age.		
		7 Days.	1 Mo.	3 Mos.	6 Mos.	1 Mo.	3 Mos.	6 Mos.
Saylors. .	1 c., 2 s., 4 b. st.	1,724	2,238	2,702	3,506	2,500,000	3,571,000	5,000,000
	1 c., 3 s., 6 b. st.	1,625	2,568	2,882	3,567	2,778,000	4,167,000	2,500,000
	1 c., 6 s., 12 b. st.	675	800	1,128	1,542	833.000	2,273,000	2,083,000
Atlas . . .	1 c., 2 s., 4 b. st.	1,387	2,428	2,966	3,953	3,125,000	4,167,000	3,125,000
	1 c., 3 s., 6 b. st.	1,050	1,816	2,538	3,170	3,125,000	2,778,000	3,571,000
	1 c., 6 s., 12 b. st.	594	1,090	1,201	1,583	1,316,000	1,136,000	1,786,000
Alpha. . .	1 c., 0 s., 2 b. st.	3,294	5,053	5,047	——	3,125,000	5,000,000	——
	1 c., 2 s:, 4 b. st.	902	2,420	3,123	4,411	2,083,000	4,167,000	3,125,000
	1 c., 3 s., 6 b. st.	892	5,150	2,355	2,750	2,083,000	3,571,000	4,167,000
	1 c., 6 s., 12 b. st.	564	1,218	1,257	1,532	1,667,000	1,786,000	1,923,000
Germania	1 c., 0 s., 2 b. st.	2,734	3,246	3.858	5,129	3,571.000	2,778,000	3,571,000
	1 c., 2 s., 4 b. st.	2,219	2,642	3,082	3,643	——	3,571,000	4,167,000
	1 c., 3 s., 6 b. st.	1,550	2,174	2,486	2,930	2,273,000	2,778,000	3,125.000
	1 c., 6 s., 12 b. st.	759	987	963	815	961,000	2,083,000	1,786,000
Alsen . . .	1 c., 0 s., 2 b. st.	3,118	3,240	3,710	5,332	2,273,000	2,273,000	3,571,000
	1 c., 2 s., 4 b. st.	1,592	2,269	2,608	3,612	2,788,000	2,778,000	4,167,000
	1 c., 3 s., 6 b. st.	1,438	2,114	2,349	3,026	2,273.000	2,778,000	3,571,000
	1 c., 6 s., 12 b. st.	417	873	844	1,323	1,562,000	1,562,000	1,786,000

10-INCH CUBES.

Alpha. . .	1 c., 0 s., 2 b. st.	——	——	5,463	6,556	——	——	5,000,000

In this table each ultimate resistance is a mean of four to six tests.

TABLE III.

MEAN ULTIMATE COMPRESSIVE RESISTANCES OF 12-INCH PORTLAND-CEMENT CONCRETE CUBES WITH LOAD TAKEN ON 8″ BY 8″.25 PLATE ON ONE FACE.

Portland Cements. Brand; Composition.		Mean Ultimate Resistance, Pounds per Square Inch at Age.			
		1 Month.	3 Months.	6 Months.	
Alpha....	1 c., 0 s., 2 b. st...	5,089	——	——	Each ultimate resistance is a mean of three tests.
	1 " 2 " 4 " ...	3,287	4,531	5,669	
Germania	1 c., 0 s., 2 b. st...	4,327	——	6,671	
	1 " 2 " 4 " ...	3,587	3,522	4,582	
Alsen....	1 c., 0 s., 2 b. st...	4,087	——	6,382	
	1 " 2 " 4 " ...	3,233	3,426	4,983	

Tables II and III contain the results taken from the "U. S. Report of Tests of Metals and Other Materials" for 1899. They exhibit the ultimate compressive resistances of cubes of Portland-cement concrete, the cements being among the well-known brands. The ages of these cubes vary from seven days to six months. The data show clearly the increase of ultimate resistance with the ages of the cubes, and the same observation applies to the three columns showing the coefficients of elasticity at one month, three months, and six months. The compositions of the different concretes of Table II are those quite generally employed in engineering practice.

Table III exhibits the ultimate resistances of the same concretes, but with the pressure applied to the 12-inch cubes on areas 8 inches by 8½ inches, this end being attained by the use of steel plates. As would be expected, the ultimate resistances are seen to be considerably greater than are found with the total load distributed over the entire surface of a cube.

The broken stone used in the cubes, the results of whose tests are given in Tables II and III, was a conglomerate from Roxbury, Mass., and the sand was coarse, clean, and sharp. The voids of the broken stone measured 49.5 per cent. of their total volume.

Table IV, taken from the same volume of the "U. S. Report of Tests of Metal and Other Materials" as Tables II and III, exhibits the ultimate compressive resistances of the mortar and concrete 12-inch cubes described therein. These results need no explanation, as they are similar to those which have already been given, but it is well to note that the last four lines of the table give results belonging to two brands of natural cement. There are also shown one test of a steel-slag cement mortar cube and one of concrete.

Table V exhibits the chemical analyses of the Portland

TABLE IV.

MEAN ULTIMATE COMPRESSIVE RESISTANCES OF MORTAR AND CONCRETE 12-INCH CUBES.

Brand; Composition.		Mean Ultimate Resistance, Pounds per Square Inch at Age of Four Months.	Weight per Cubic Foot, Pounds.	Coefficient of Elasticity, Pounds per Square Inch.
Alpha Portland	1 c., 1 s., 0 b. st.	4,371	136.5	3,571,000
	1 " 2 " 0 "	2,506	134.2	3,125,000
	1 " 3 " 0 "	1,812	133.8	1,786,000
	1 " 4 " 0 "	829	120.9	——
	1 " 5 " 0 "	484	119.5	——
	1 " 6 " 0 "	185	116.9	——
	1 " 7 " 0 "	118	111.5	——
Atlas Portland	1 " 1 " 0 "	5,570	141.5	6,250,000
Star Portland	1 " 1 " 0 "	5,045	134.5	4,167,000
Saylors Portland	1 " 1 " 0 "	3,979	134.7	3,125,000
Germania Portland	1 " 1 " 0 "	4,353	134.7	2,500,000
Alpha Portland	1 " 1* 0 "	5,306	137.3	3,571,000
Steel slag	1 " 1 s., 0 "	1,743	126.6	1,190,000
" "	1 " 2 " 4 "	1,939 †	152.1 ‡	2,500,000
Hoffman Rosendale	1 " 1 " 0 "	741	127.7	——
Norton Rosendale	1 " 1 " 0 "	643	125.2	——
	1 " 2 " 0 "	277 †	120.7	——
	1 " 2 " 4 "	332 †	146.2 ‡	——

* Granite dust. † Age, 3 months. ‡ Trap rock, broken stone.

TABLE V.

CHEMICAL ANALYSES OF PORTLAND AND STEEL-SLAG CEMENTS.

Cement.	Silica.	Oxide of Iron.	Alumina.	Lime.	Magnesia.	Sulphur Trioxide.	Carbon Dioxide.
Alpha. . .	20	2.8	10.87	58.66	3.35	1.34	2.56
Star. . . .	21.73	2.5	9.47	56.34	3.61	1.91	3.94
Standard	22.5	2.6	11.98	51.44	3.61	1.57	5.96
Alsen. . .	20.67	2.1	14.6	42.16	2.32	2.32	4.45
Steel. . . .	31.02	Trace	10.9	57.31	4.05	3.36	4.81

and steel-slag cements named in Table IV. These analyses exhibit about the usual composition of the various grades of cement to which they belong.

TABLE VI.

COMPRESSION TESTS OF 12-INCH CUBES OF PORTLAND-CEMENT CINDER CONCRETE.

Brand.	Composition.	Age when Tested, Days.	No. Cubes Tested.	Weight per Cu. Ft., Lbs.	Ultimate Resistance in Lbs. per Sq. In.			Coefficient of Elasticity, Pounds.
					Max.	Mean.	Least.	
Germania .	1 c., 1 s., 3 cinder	99 and 102	3	110.4	2,023	2,001	1,975	———
" .	1 c., 2 s., 3 "	102	3	112.8	1,701	1,634	1,589	———
" .	1 c., 2 s., 4 "	98	3	107.9	1,344	1,325	1,295	———
" .	1 c., 2 s., 5 "	98 and 101	3	106.3	1,114	1,084	1,052	———
" .	1 c., 3 s., 6 "	91	3	103.5	854	788	749	———
Alpha.	1 c., 1 s., 3 "	90	3	114.1	2,988	2,834	2,780	2,500,000
"	1 c., 2 s., 5 "	90	3	110	1,715	1,600	1,402	1,279,000
Atlas.	1 c., 1 s., 3 "	90	3	116.3	2,580	2,414	2,295	3,125,000
"	1 c., 2 s., 5 "	90	3	109.9	1,263	1,223	1,200	857,000

The results exhibited in Table VI are interesting as belonging to Portland-cement cinder concrete and they are of practical importance because such concrete is used in many buildings especially for floors, in consequence of its weighing much less than ordinary broken-stone concrete. The ages of these cinder concrete cubes is seen to run from 90 to 102 days, which is sufficient to give nearly the full ultimate resistance of such material. It is seen, however, that cinder concrete is materially less strong or capable of ultimate compressive resistance than either broken-stone or gravel concrete having the same proportions of mixture in its composition. The column giving the weight in pounds per cubic foot shows that cinder concrete weighs but about three fourths as much as that made with gravel and broken stone. The data contained in this table were taken from the "U. S. Report of Tests of Metal and Other Materials" for 1898.

Table VII, compiled from the "U. S. Report of Tests of Metal and Other Materials" for 1900, is inserted to show the comparative resistances of broken-stone and broken-slag concretes of the proportions commonly employed. It is of interest to note the results for slag concrete in which slag dust was employed instead of sand.

Table VII.

TESTS OF 12-INCH CUBES OF PORTLAND-CEMENT CONCRETE.

Brand.	Composition.	Age, Mos.	No. of Tests.	Weight per Cu. Ft., Pounds.			Ultimate Resistance, Lbs. per Sq. In.		
				Max.	Mean.	Min.	Max.	Mean.	Min.
Vulcanite	1 c., 2 s. d., 4 br. sl.	1.73	4	134	133	131.6	3,341	3,159	3,016
"	1 c., 2 s., 4 br. st.	1.73	4	145.6†	145.2	144.7	3,783	3,626	3,506
"	1 c., 2 s. d., 4 br. sl.	2.83	4	136	135.3	134.4	4,129	3,984	3,797
"	1 c. 2 s., 4 br. st.	2.83	4	147†	146.4	145.4	4,534	4,286	4,136
Grant *	1 c., 3 s., 5 br. st.	3.07	8	154	152.4	151.6	4,451	4,043	3,081
" *	" " "	3.07	4	153.1	152.6	151.5	3,516	3,187	2,930

s. d., slag dust; br. sl., broken slag. * Machine mixed. † Broken limestone.

This procedure secured considerable balancing of the broken slag and slag dust, thus materially reducing the voids and correspondingly increasing the compressive resistance, although it is seen that the sand and broken-stone concrete is sensibly stronger than that made with slag.

A large number of tests, the results of which need not be given here, have shown that gravel may advantageously be used, in the interests of economy, in the place of broken stone for concrete. On the whole, the broken-stone concrete is probably stronger than that made with gravel, but the difference is not material for all ordinary cases. The gravel should not be water-worn, but have sharp, gritty surfaces to which the setting cement may strongly bond itself. All sizes from the largest permissible down to coarse sand should be taken, and when so balanced the voids may be reduced as low as 20 per cent. of the total volume of the gravel or even lower. This balancing of the broken stone or gravel enhances both economy and resisting qualities.

TABLE VIII.

COMPRESSION TESTS OF PORTLAND-CEMENT CONCRETES GIVEN BY HENBY.

	Age, Days.	Parts of Cement, Sand, and Broken Stone.			Cement.*	Kept in	Weight, Lbs. per Cu. Ft.	Coefficient of Elasticity.	Ultimate Resistance, Lbs. per Sq. In.	Consistency.
1	30	1	2	4	A	Water	153	4,625,000	2,610	Plastic
2	30	1	2	4	A	Air	160	7,171,000	3,020	"
3	90	1	2	4	A	Air dry	140	4,421,000	1,243	Dry
4	90	1	2	4	A	" "	144	5,792,000	982	"
5	7	1	2	5	A	" "	146	3,927,000	726	Very dry
6	9	1	2	5	A	Air	152	4,930,000	423	Plastic
7	32	1	2	5	A	"	151	5,055,000	2,097	"
8	32	1	2	5	A	Water	155	7,292,000	2,830	"
9	34	1	3	6	A	Air	143	5,104,000	1,310	"
10	39	1	3	6	A	"	146	7,520,000	1,733	"
11	39	1	3	6	A	Water	152	6,646,000	2,242	"
12	60	1	3	6	A	Air	—	2,886,000	413	Very dry
13	30	1	4	8	A	Water	145	4,397,000	1,346	Plastic
14	38	1	4	8	A	"	143	4,560,000	1,282	
15	38	1	4	8	A	Air	139	2,446,000	617	Excess water
16	38	1	4	8	A	"	138	2,247,000	797	" "
17	90	1	—	—	M	Water	136	6,578,000	5,280	Plastic
18	90	1	—	—	M	Air	129	3,940,000	4,580	"

* A, Atlas; M, Medusa.

TABLE IX.

COMPRESSIVE TESTS OF ATLAS (PORTLAND) CEMENT CINDER CONCRETES GIVEN BY HENBY.

	Age, Days.	Parts of Cement, Sand, and Cinders.			Kept in	Weight, Lbs. per Cu. Ft.	Coefficient of Elasticity.	Ultimate Resistance, Lbs. per Sq. In.
1	7	1	2	4	Air	116	876,000	—
2	7	1	2	4	"	111	945,000	—
3	7	1	2	4	Water	120	514,000	—
4	30	1	2	4	"	119	1,358,000	993
5	30	1	2	4	Air	112	1,626,000	1,049
6	30	1	2	4	"	106	1,399,000	976
7	30	1	2	5	"	109	1,772,000	941
8	30	1	2	5	Water	114	1,021,000	705
9	30	1	2	5	"	114	1,168,000	682
10	60	1	2	5	Air dry	114	1,055,000	573
11	60	1	2	5	"	116	1,783,000	847
12	60	1	2	5	"	119	1,152,000	670
13	30	1	3	6	Air	—	1,473,000	484
14	30	1	3	6	"	107	1,447,000	511
15	30	1	3	6	Water	118	751,000	500
16	60	1	3	6	Air dry	107	917,000	734
17	60	1	3	6	"	101	916,000	544
18	30	1	3.5	7	Air	106	533,000	405

Tables VIII and IX, taken from a paper by Mr. W. H. Henby in the "Journal of the Association of Engineering Societies" for 1900, are selected chiefly to show the coefficients of elasticity of stone and cinder concrete. The "Air-dry" specimens were simply kept in air, but the "Air" specimens were kept in air under a damp cloth. The coefficients for the thirty- and sixty-day cinder concrete correspond well with those already given for the same grade of material, but some of those in Table VIII are unusually high and suggest the possibility of some abnormal condition. The general run of the coefficients of elasticity for the stone concrete in that table, on the other hand, is fairly in accord with the values given in preceding tables.

A careful examination of all the Tables, I to IX, shows that reasonably well-made broken-stone concrete may carry a load of 300 to 500 pounds per square inch without exceeding $\frac{1}{7}$ to $\frac{1}{8}$, or possibly $\frac{1}{6}$, of its ultimate resistance, the composition of the mixture being 1 cement, 2 sand, and 4 broken stone, or perhaps 1 cement, 3 sand, and 5 broken stone. It is possible that this may be an under statement of the capacity of the concrete if the mixture is as well balanced as it should be. It is a mistake, as has been shown repeatedly by actual test, to screen out the finer portions of the broken stone or to attempt to secure an approximately even sand grain. It is conducive to an increased resistance as it is to increased economy to balance the sand, gravel, or broken stone by using all the varying sizes between the least and the greatest. Indeed, in many cases it may be advisable to use the entire product of the crusher.

The relation between the ultimate compressive resistance of concrete made with balanced material and the length of column is illustrated by the results given in Table X, which has been collated and arranged from the "U. S.

TABLE X.

COMPRESSIVE RESISTANCES OF 12″×12″ CONCRETE COLUMNS.

Height, Feet.	Age, Days.	Composition.	W'ght in Lbs. per Cu. Ft.	Ult. Resist. in Lbs. per Sq. In.	
2	47	1 cement, 3 sand, 4–1¼″ broken stone, 2–½″ broken stone	145	1,072	Hand mixed
2	47		145	917	
4	47	do.	144	1,067	
4	47	do.	144	1,132	
6	46	do.	——	844	
6	46	do.	143	1,048	
8	42	do.	145	935	
8	42	do.	145	900	
10	40	do.	142	909	
10	41	do.	143	807	
12	39	do.	144	947	
12	39	do.	144	980	
14	34	do.	145	936	
14	35	do.	145	907	
2	47	1 cement, 3 gravel, 4–1¼″ broken stone, 2–⅜″ broken stone	145	1,185	Machine mixed
2	47		147	1,183	
4	48	do.	143	980	
4	48	do.	144	936	
6	48	do.	146	1,131	
6	48	do.	146	1,200	
8	42	do.	146	1,108	
8	42	do.	146	1,086	
10	41	do.	146	1,015	
10	42	do.	146	1,000	
12	37	do.	149	1,400	
12	39	do.	148	1,500	
14	35	do.	148	858	
14	35	do.	148	807	
6	42	1 cement, 6 gravel, 8–1¼″ broken stone, 4–⅜″ broken stone	143	500	
6	42		144	467	
6	42	1 cement, 7 gravel, 8⅔–1¼″ br'k'n stone, 4⅓–⅜″ br'k'n stone	141	427	
6	42		142	436	
6	45	1 cement, 5 gravel, 6⅔–1¼″ br'k'n stone, 3½–⅜″ br'k'n stone	146	708	
6	45		146	747	
6	46	1 cement, 4 gravel, 5⅓–1¼″ br'k'n stone, 2⅔–⅜″ br'k'n stone	146	900	
6	46		145	797	
12	36	1 cement, 3 gravel, 6–⅜″ broken stone	150	1,250	Reinforced with 4–⅜″ cold-twisted steel rods embedded in the concrete
12	39		149	1,700	
9	580	1 Silica Portland cement, 2 coarse clean sand, 3 quartz gravel (½″–2″)	148	2,548	

Report of Tests of Metal and Other Materials" for 1897. The heights of column range from 2 to 14 feet. While there are some exceptions, the rule is general that, other things being equal, the ultimate resistance decreases as the length or height of column increases. On the whole, the machine-mixed material appears to be a little stronger than the hand-mixed, but the difference is not substantial except for the 8, 10, and 12 feet lengths.

Art. 55.—Bricks and Brick Piers.

The ultimate compressive resistance of bricks depends largely upon the manner in which they are tested and the care with which the surfaces pressed are filled out with a proper cushion and made truly parallel to the bearing surfaces of the testing machine. The best of bricks as produced for the market do not have opposite faces truly parallel, and hence when they are placed in a testing machine for testing to failure the pressure will be concentrated at different points and the bricks will be broken partly by bending before the full ultimate compressive resistance is developed unless the pressed surfaces are made true by some kind of a cushion. This cushioning is frequently and perhaps usually done with plaster of paris, as in the case of the tests of bricks at the U. S. Arsenal, Watertown, Mass., the results of which are given in Table II. Again, a brick tested on edge will give a less ultimate resistance per square inch than when tested flat and the resistance on end per square inch of section will be less than that on edge. When the brick is tested flatwise, even when truly surfaced with a cushion such as plaster of paris, it is a very short block and the friction of the pressed surfaces on the bearing faces of the testing machine is sufficient to give the compressed material substantial lat-

TABLE I.

COEFFICIENTS OF ELASTICITY FOR BRICKS IN COMPRESSION, COEFFICIENTS OF EXPANSION, AND ABSORPTION OF WATER.

Brick.	Description.	Coefficient of Elasticity, Pounds per Square Inch.	Total Absorption after Exposure to Boiling Water, by Volume. Per Cent.	Coefficient of Expansion in Water, per Degree Fahr.
Kansas City Hydraulic Press Brick Co.	Red brick No. 1	1,650,000	26.8	.00000326
do.	Red brick No. 2	2,130,000	25.1	.00000317
The Powhatan Clay Manufacturing Co.	Cream white, No. 1	1,730,000	26.7	.00000205
do.	Dark red, No. 2	3,180,000	18	.00000289
The Hydraulic Press Brick Co	Medium red	714,000	32	.00000317
do.	Light chocolate	2,550,000	23.2	.00000475
do.	Dark buff, speckled	890,000	23.7	.00000391
Philadelphia and Boston Face Brick Co.	Salmon color, soft burnt, for inside work	391,000	36.2	.00000335
do.	do.	517,000	34	.00000261
do.	Chocolate brown, for outside work	5,000,000	13.6	.00000298
do.	Cream color, for outside work	577,000	35.3	.00000252
do.	Buff color, for outside work	2,570,000	24.8	.00000298
Eastern Hydraulic Press Brick Co.	Shade 200	2,120,000	17.8	.00000345
do.	Shade 300	2,050,000	18.7	.00000344
do.	Shade 410	2,170,000	17.4	.00000568
Brooke Terra Cotta Co.	Columbian buff, No. 4	4,060,000	21.3	.00000242
do.	Light buff	4,820,000	16.9	.00000205
Gladding, McBean & Co.	Dark buff, hard burnt	4,450,000	21.2	.00000419
do.	Salmon, soft burnt	1,770,000 2,290,000	26.3	.00000428
Northern Hydraulic Press Brick Co.	Red brick	1,040,000	29.2	.00000316
Chicago Hydraulic Press Brick Co.	do.	1,410,000	22	.00000344
Clark Press Brick Co.	Chocolate	1,520,000	24.1	.00000744 .00000754

eral support, not permitting it to separate and crush away readily. It will be found, therefore, that when blocks are tested flatwise the ultimate resistances per square inch, as a whole, will be much higher than when tested on edge. This condition of things holds to some extent when the bricks are tested on edge, so that an endwise test will give the ultimate compressive resistance per square inch somewhat less than that found when the brick is tested on edge.

An endwise test of the brick more truly represents the ultimate compressive resistance of the material than a test either flatwise or on edge.

The coefficients of elasticity given in Table I were determined at the U. S. Arsenal at Watertown, Mass., and are taken from the "U. S. Report of Tests of Metals and Other Materials" for 1896. The intensities at which the determinations were made range from about 1000 to 4000 or 5000 pounds per square inch. The bricks were compressed on end, the opposite faces having been ground so as to be truly parallel. It will be observed that the coefficients of elasticity vary between wide limits for soft- and hard-burnt bricks, but on the whole they are about the same as the coefficients for Portland-cement mortars and concretes. It must be remembered, in considering the elastic properties of such irregular material as brick, that abnormal results will frequently be found, and those of Table I are no exception to that observation. The small values of E probably indicate some defect of material or of proper bearing in the testing-machine, or some similar irregularity.

The coefficients of thermal expansion were determined at the same time with the other physical quantities. It will be observed that while the rate of thermal expansion varies widely, its highest values are even larger than those for iron and steel. As a whole, the values of the coefficients are low, as would naturally be expected from the porous character of the material.

In the Proceedings of the Am. Soc. C. E. for March, 1903, Mr. S. M. Turrill, Assoc. Am. Soc. C. E., gives the results of a large number of tests of common building brick, 2 in. by 4 in. by 8 in. in size, manufactured at Horseheads, N. Y. The following table is fairly representative of the results of Mr. Turrill's tests, made with great care at the civil-engineering laboratories of Cornell University:

TEST OF COMMON BUILDING BRICK.

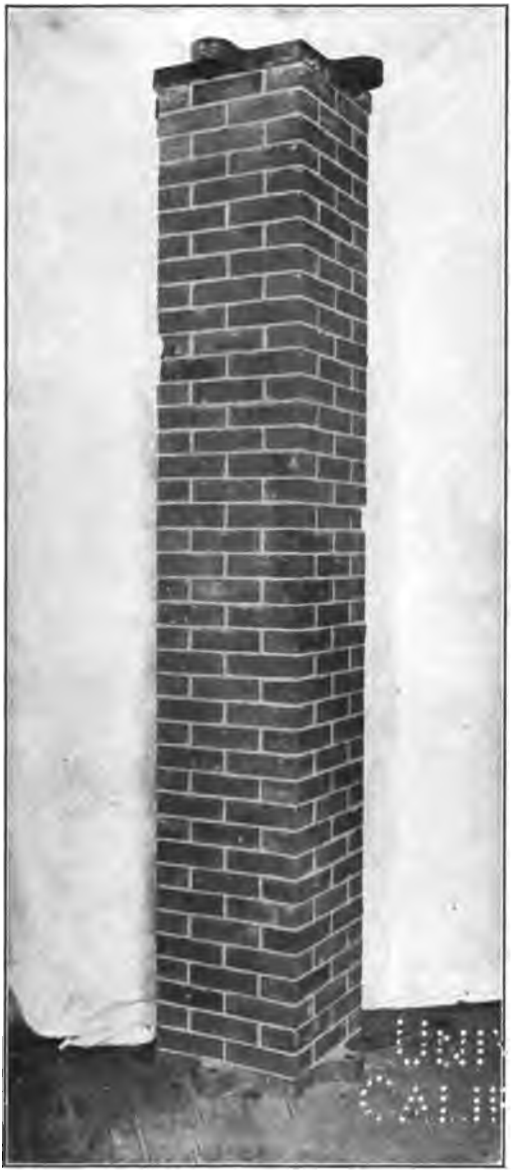
Brick Tested.	No. of Tests.	Ultimate Compressive Resistance, Pounds per Square Inch.		
		Greatest.	Mean.	Least.
On end	12	3,763	2,628	1,234
On edge	12	3,913	2,832	1,897
Flat	12	5,463	3,995	2,665

These bricks were tested in their natural condition as delivered from the kiln ready for use.

Other tests were made of the same brick saturated with water and after being reheated in a suitable oven. This latter test was designed to disclose the quality of brick after having passed through a conflagration. The saturated bricks tested on end and on edge showed material loss of resistance below that of their natural condition, but those tested flat showed large gains. The reheated bricks exhibited large gains in all three modes of testing. These bricks were obviously not of hard-burned, high-resisting character.

The coefficient of elasticity of twelve of these bricks ran from 540,000 to 1,815,000 pounds per square inch, with a mean value of 1,305,000 pounds.

The values of the ultimate resistances for a large number of classes and grades of brick are given in Table II as taken from the "U. S. Report of Tests of Metals and Other Materials" for 1894. That Table includes within its limits practically all the varieties of bricks in use at the present time. It will be seen that the ultimate resistances differ greatly for the varying grades of materials. The lower values correspond closely to the ultimate resistances of good common red brick. As will be observed, tests were made not only flatwise and on edge, but in one or two cases on end. The fifth to the eighth lines from the top



A solid 16-inch square-face brick pier laid in lime mortar. It was tested at the U. S. Arsenal, Watertown, Mass., and gave an ultimate compressive resistance of 1337 lbs. per sq. in. The pier is shown as it existed after failure.

(To face page 420.)

of the Table belong to small piers of bricks uncemented, i.e., simply placed flatwise truly on each other without mortar from two to five bricks high and tested in that condition. As would be expected, the greater the number of bricks placed in this manner the less was the ultimate resistance per square inch.

Inasmuch as tests of brick piers have shown that their ultimate compressive resistances run only from about 1000 to 4500 pounds per square inch, depending upon the character of the mortar, it is seen that in such masonry a small portion only of the compressive resistance of the bricks is developed in piers and other similar brick-masonry masses.

These latter results doubtless depend largely upon the cementing material. There is no question that the ultimate resisting capacity of brick masonry is affected greatly by the resisting capacity of the mortar, and the same general observation can be applied to other classes of masonry. There is more than this, however, affecting the carrying capacity of brick and other grades of masonry as compared with the ultimate compressive resistance of the bricks used in the one case of masonry or of the individual stones employed in the other. The texture or character of the mass of burned clay composing the brick is exceedingly variable, both in consequence of the varying mixture of the material in the bricks before being burned and in consequence of the varying degree of burning in each individual brick. Again, whatever may be the care in placing the bricks in a testing-machine, including the cushioning of the ends, it is practicably impossible to secure anything like a uniform bearing upon either the ends, sides, or beds. Their irregular dimensions and exterior surfaces and the varying quality of the materials, even in the best of brick, introduce into their resisting capacity elements of variation which are frequently so great as to lead to abnormal results. While the mortar used in forming a mass of brick masonry undoubtedly fills up many irregularities of surface, voids of considerable magnitude frequently remain unfilled. The consequence of these uncontrollable elements in a mass of brick masonry is always a material reduction of ultimate carrying capacity and frequently a large reduction. However excellent in quality, therefore, the mortar or binding material in a brick-masonry pier may be, it is inevitable that there will be not only a wide range in ultimate compressive resistance, but in all cases a material reduction below that exhibited by the individual bricks when tested by themselves.

Tests of the ultimate compressive resistance of a large number of face-brick and common-brick piers can be found in Part II of the "U. S. Report of Tests of Metals and Other Materials" for 1886. In all of these tests the mortar was composed of 1 part Rosendale cement and 2 parts sand. The heights of the piers varied from 2 ft. to 12½ ft., while the cross-sections were 8 ins. by 8 ins., 12 ins. by 12 ins., and 16 ins. by 16 ins. Some of the 12-inch and 16-inch cross-sections had a 4-inch by 4-inch or 8-inch by 8-inch core taken out of the centre. Invariably the columns of the smallest height give the highest ultimate resistances, and as a whole the highest columns give the least ultimate compressive resistances.

Fourteen of these piers were laid up with face brick, the weight of the brickwork varying from 125.2 pounds to 134.4 pounds per cubic foot. The ultimate compressive resistances of these face-brick piers varied from 2428 pounds per square inch down to 1677 pounds per square inch, the former value belonging to a column 2 ft. high and the latter, i.e. the smallest, to a column 10 ft. high and 8 ins. by 8 ins. cross-section, the highest column of the lot. These resistances varied from 12.5 per cent. to 18.1 per

TABLE II.

ULTIMATE COMPRESSIVE RESISTANCES OF BRICKS.

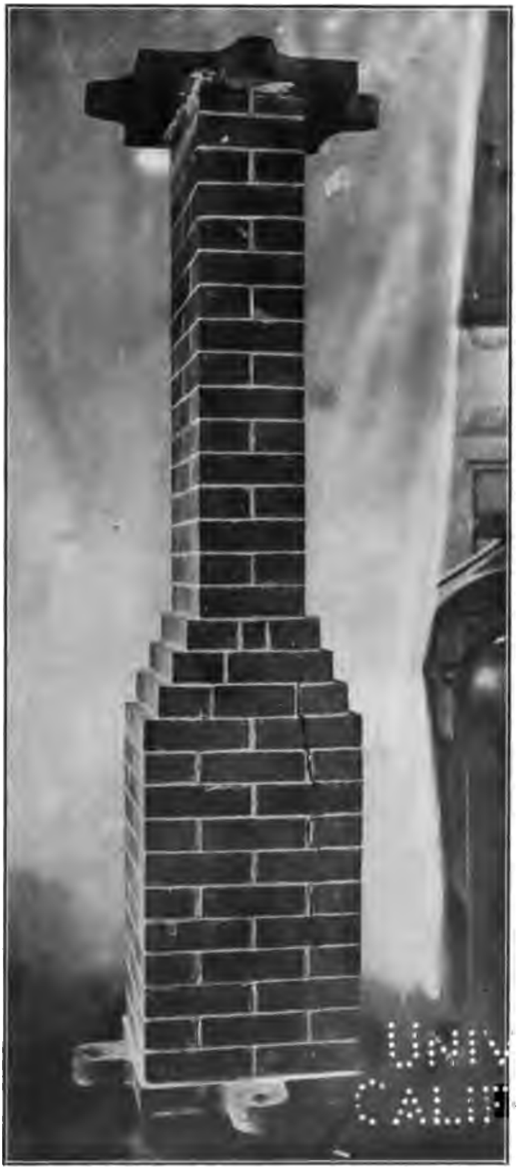
	Description.	Compressed Surface, Square Inches.	Ultimate Resistance, Pounds per Square Inch.			Remarks.
			Gr'test.	Mean.	Least.	
	Hard-burnt face brick (red).....	7.8×3.6	13,492	11,173	9,290	Tested flat
	" " " " "	" ×2.2	12,297	8,980	7,623	" on edge
	" " " " "	2.2×3.6	8,032	6,927	5,837	" " end
	" " " " "	7.8×3.8	15,837	12,469	9,100	" flat
	" " " " "	" " "	6,812	6,440	6,069	" " 2 high
	" " " " "	" " "	5,828	5,097	4,365	" " 3 "
	" " " " "	" " "	4,788	4,478	4,168	" " 4 "
	" " " " "	" " "	4,530	4,132	3,733	" " 5 "
Hydraulic Press Brick Co., St. Louis.	Medium red............	8.5×4.2	——	5,266	——	" "
	Dark red............	" " "	——	10,149	——	" "
	Paving stock............	8.2×4	——	17,558	——	
	" "	" ×2.2	——	5,992	——	" on edge
	Dark red............	8.3×4.1	——	10,643	——	" flat
	" "	8.2×4	——	17,472	——	" "
	Buff, speckled............	" ×4.1	——	9,424	——	" "
	Light chocolate..........	" " "	——	8,620	——	" "
	" " dark sp'kles	" " "	——	13,330	——	" "
	Dark buff with speckles...	" " "	——	8,907	——	" "
	Buff with dark " ..	8.4×4.2	——	8,010	——	" "
	Light buff............	8.3×4.1	——	8,144	——	" "
	Brown............	8.5×4.1	——	8,861	——	" "
A.	Brown............	8.4×4.1	——	4,794	——	" "
	Red............	" " "	——	5,574	——	" "
	"	8.3×2.3	——	5,192	——	" on edge
B.	Shades No 5 and No. 7.....	" ×4.1	13,511	13,209	12,907	" flat
C.	Dark red, three kinds......	8.2×4.1	12,372	11,030	9,686	" "
D.	Light buff............	8.1×4	15,752	15,290	14,818	" "
	" " slightly darker...	" " "	——	13,771	——	" "
	" " " " ...	" ×2.2	——	9,319	——	" "
	Buff............	" ×4	16,252	15,370	14,485	" "
	" darker............	" ×4.1	——	13,492	——	" "
	" "	" ×2.2	——	9,273	——	" on edge
	Gray............	8.2×4	13,995	13,059	12,123	" flat
	Light chocolate............	8.1×4	——	15,081	——	" "
	" "	8.2×2.2	——	9,945	——	" on edge
E.	Salmon color, soft-burnt....	8.4×4.2	3,967	3,897	3,826	" flat
	Light red............	8 ×4	9,409	8,127	6,845	" "
	" "	8 ×2.3	——	5,877	——	" on edge
	Dark red............	8 ×4	12,195	10,016	7,836	" flat
	Chocolate brown............	" " "	——	17,023	——	" "
	" "	7.7×2.2	——	7,764	——	" on edge
	Cream color............	8.1×4	3,481	3,229	2,978	" flat
	Buff............	7.9×3.9	——	10,406	——	" "
	"	" ×2.2	——	4,756	——	" on edge
	Gray............	8 ×4	——	3,840	——	" flat
F.	Buff............	8.5×4	23,135	20,617	18,098	" "
	Medium-dark buff............	" ×4.1	12,077	10,950	9,822	" "
	Light buff............	8.3×4	18,808	18,574	18,339	" "
G.	Buff............	8.2×4	8,426	8,133	7,594	" "
	"	8.1×2.4	5,972	5,000	5,829	" on edge
	White enamelled............	9 ×4.5	5,076	4,734	4,531	" flat

A, Chicago Hydraulic Press Brick Co. (medium hard-burnt).
B, Omaha Hydraulic Press Brick Co. (medium hard-burnt, red).
C, Findlay Hydraulic Press Brick Co. Findlay, Ohio (medium hard-burnt).
D, Eastern Hydraulic Press Brick Co., Philadelphia, Pa.
E, Philadelphia and Boston Face Brick Co., Boston, Mass.
F, Brooke Terra Cotta Co., Lazearville, W. Va. (Columbian buff bricks).
G, Gladding, McBean & Co., San Francisco, Cal.

cent. of the ultimate compressive resistances of the bricks used in the piers.

Thirty-eight common brick piers with the same mortar of 1 Rosendale cement to 2 sand were laid up also of the general dimensions given above, the weights of the masonry varying from 118.8 to 127.7 pounds per cubic foot. The ultimate resistances in this set of common brick piers varied from 2798 pounds per square inch to 964 pounds per square inch, the former value belonging to a pier 2 ft. high and the latter to a pier 10 ft. high, but with a 16 ins. by 16 ins. cross-section. In this series the ultimate resistance of the piers varied from 7.8 per cent. up to 17.6 per cent. of the ultimate resistances of the bricks used in building them, the higher percentage belonging to the short or low pier.

The manner of failure of brick piers is, strictly speaking, not a compression failure but largely a longitudinal splitting and lateral bulging of the separating parts, indicating that any devices effectively resisting that bulging would materially increase the carrying capacity of the masonry. A series of fourteen brick piers was tested to failure in 1899 at the testing laboratory of the College of Civil Engineering of Cornell University, an account of which is given by Mr. E. J. McCaustland in the Transactions of the Association of Civil Engineers of Cornell University for 1900. These piers were 13 ins. by 13 ins. in cross-section and 80 ins. long. They had mortar joints .3 ins. in thickness and they were about seven months old when tested. Wire netting and straps or plates were inserted at a number of the horizontal joints in some of the piers, as shown in Table III. The ultimate compressive resistance of a single brick specimen used in the construction of the piers was 3525 pounds per square inch; a low ultimate resistance which accounts for the correspondingly low ultimate compressive resistance



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An 8 × 16-in.-face brick pier with 16-in. square base laid in lime mortar. It was tested at the U. S. Arsenal, Watertown, Mass., and gave an ultimate compressive resistance of 1233 lbs. per sq. in. on the upper section and 601 lbs. per sq. in. on the lower section. The cracks due to failure are clearly seen.

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TABLE III.

ULTIMATE COMPRESSIVE RESISTANCES OF 13″ × 13″ × 80″ BRICK PIERS BUILT WITH PORTLAND CEMENT MORTAR (1 C., 2 S.) JOINTS .3″ THICK.

No.	Kind of Joint.	Ultimate Stress.		Efficiency Per Cent. of Single Brick.	Coefficients of Elasticity.
		Total.	Lbs. per Sq. In.		
1	Portland cement mortar (1:2)	194,000	1150	30	
2	Portland cement mortar (1:2)	200,000	1184		
Series of 1897	Portland cem. mortar (1:2)	Average of 3 tests	921		
3	Iron straps every 4th course	136,400	810	24	
4	Iron straps every 4th course	155,400	920		
5	Iron straps every 6th course	130,000	780	22	783,000
6	Iron straps every 8th course	142,500	843	24	
7	Wire netting every 2d course	192,000	1136	33	
8	Wire netting every 2d course	208,000	1248		994,000
9	Wire netting every course	282,000	1694	46	1,040,000
10	Wire netting every course	240,000	1440		960,000
11	Iron plate every 4th course	174,000	1030	28	
12	Iron plate every 4th course	193,500	1145		
13	Iron plate every 4th course	162,000	974		680,000
14	Iron plate every 4th course	143,000	858		830,000

of the piers shown in the table. The effect of the wire netting and the iron straps on the efficiency of the masonry, in terms of the ultimate resistance of the single brick given above, is evident. The efficiencies of the piers with the iron straps and plates are less than those of the piers with mortar joints without reinforcement, but the efficiencies with the wire netting are materially larger. Indeed the efficiency belonging to the piers having wire netting in every joint is more than 50 per cent. greater than that found for the piers with joints not reinforced. This apparently indicates that the reinforcement must be of such a character as not to laminate the joints and thus weaken them.

Coefficients of elasticity were computed on a compression or shortening of a gauged length of 56 ins. of the pier at total loads of 60,000, 100,000, and 150,000 pounds, or at pressures per square inch of 354, 590, and 885 pounds, far too high for such a purpose. These coefficients are seen to be low, as would be expected for such high intensities of pressures.

Table IV exhibits the results of testing piers of brick masonry in the Gov't testing machine at Watertown, Mass. It is taken from "Ex. Doc. No. 35, 49th Congress, 1st Session." The dimensions of piers are shown in the table;

Table IV.

CRUSHING STRENGTH OF BRICK PIERS.

No.	Height of Pier,		Section of Pier, Ins.	Composition of Mortar.	Weight per Cu. Ft., Lbs.	Ultimate Resistance, Lbs. per Sq. In.	
	Ft.	Ins.					
1	1	4	8×8	1 lime, 3 sand.	137.4	2,520	Face bricks.
2	6	8	8×8	1 " 3 "	133.5	1,877	
3	1	4	8×8	1 Portland cement, 3 sand.	136.3	3,776	
4	6	8	8×8	1 " " 3 "	133.5	2,249	
5	2	0	12×12	1 lime, 3 sand.		1,940	
6	2	0	12×12	1 " 3 "		1,900	
7	10	0	12×12	1 " 3 "	131.7	1,511	
8	10	0	12×12	1 " 3 "	125.0	1,807	
9	2	0	12×12	1 Portland cement, 2 sand.		3,670	
10	10	0	12×12	1 " " 2 "	132.2	2,253	
11	1	4	8×8	1 lime, 3 sand.	135.6	2,440	Common bricks.
12	6	8	8×8	1 " 3 "	133.6	1,540	
13	2	0	12×12	1 " 3 "		2,150	
14	2	0	12×12	1 " 3 "		2,050	
15	9	9	12×12	1 " 3 "	131.5	1,118	
16	10	0	12×12	1 " 3 "	136.0	1,587	
17	10	0	12×12	1 Portland cement, 2 sand.	131.0	2,003	
18	2	8	16×16	1 " " 2 "		2,720	
19	10	0	16×16	1 " " 2 "		1,887	
20	2	0	12×12	1 lime, 3 sand.		1,370	Bay State bricks.
21	6	0	12×12	1 " 3 "		1,133	
22	6	0	12×12	1 " 3 "	119.7	1,210	
23*	6	0	12×12	1 lime, 3 sand.	118.2	1,331	Face bricks.
24†	6	0	12×12	1 " 3 "	118.1	1,211	
25	7	10	12×12	1 " 3 "	120.3	1,174	
26	10	0	12×12	1 " 3 "	118.0	924	
27	10	0	8×12	1 " 3 "	107.0	940	
28	10	0	12×16	1 " 3 "	118.7	773	
29	6	0	12×12	1 " 3 " . 1 Rosendale cement.	120.6	1,646	
30	6	0	12×12	1 Rosendale cement, 2 sand.	123.0	1,972	
31	6	0	12×12	1 lime, 3 sand, 2 Portland cement.	120.3	1,411	
32	6	0	12×12	1 Portland cement, 2 sand.	119.7	1,702	
33	6	0	12×12	Clear Portland cement.	126.6	2,375	

* Joints broken every 6 courses.

† Bricks laid on edge.

also the kinds of mortar used and the grades of brick. The "common" and "face" brick, both hard burnt, were from North Cambridge, Mass. The other bricks were from the Bay State Brick Co., of Boston and Cambridge, Mass., and were medium burnt.

The crushing strength of three bricks of each kind between steel compression platforms was first determined as follows:

Bricks.	Crushing Resistance, Pounds per Square Inch.				Size, Inches.
				Averages.	
Face........	11,056	13,984	16,734	13,925	7.75×3.7×2.0
Common....	19,785	22,351	12,995	18,337	8.0 ×3.6×2.1
Bay State...	11,120	12,709	10,390	11,406	7.8 ×3.6×2.1

Care was taken to make the bed faces of all three bricks bear evenly against the compression platforms, and in order to accomplish this result thin sheets of brass were used for packing.

The brick piers were built of bricks "laid on beds and joints broken every course, with the exception of two 12 by 12 piers, one of which had joints broken every sixth course, and one had bricks laid on edge.

"They were built in the month of May, 1882," and "their ages when tested ranged from 14 to 24 months." They were all tested between cast-iron plates.

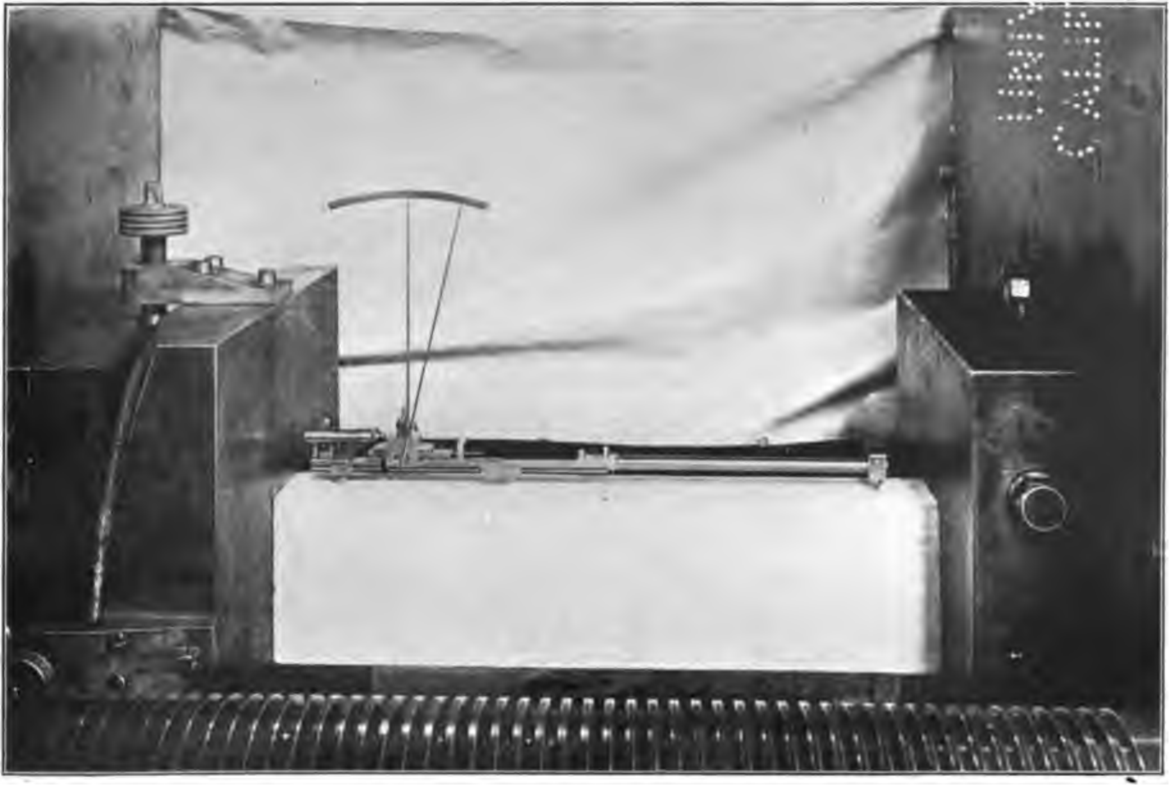
"Loads were gradually applied in regular increments, . . . returning at regular intervals to the initial load. . . . Cracks made their appearance at the surfaces of the piers and were gradually enlarged before the maximum loads were reached. Final failure occurred by the partial crushing of some of the bricks, and by the enlargement of these cracks, which took a longitudinal direction and occurred in the bricks of one course opposite the end joints of the bricks in the adjacent courses. This manner of failure was common to all the piers."

It is important to notice that the resistance of the piers varies with the strength of the mortar used in the joints.

The remarkably high results for the single bricks given above are probably due to the excellent quality of the material tested and to the great care exercised to have even bearings on the compressed beds.

Ultimate compressive tests of 4-inch cubes of terra cotta from Gladding, McBean & Co., of San Francisco, may be found in the "U. S. Report of Tests of Metals and Other Materials" for 1897. These cubes were of brown, buff and white, terra cotta and were nine in number. The maximum ultimate compressive resistance was 3165 pounds per square inch, the mean 2940 pounds per square inch, and the minimum 2669 pounds per square inch. As a rule the ultimate compressive resistance of terra cotta will be found less than that of ordinarily good soft-burnt brick.

The results of all the experimental investigations available in connection with brick masonry and experiences in the best class of engineering work indicate that masonry laid up of good hard-burnt common brick may safely carry a working load of 15 to 20 tons per square foot or 210 to 280 pounds per square inch. In the construction of this class of masonry where the duties are to be severe it is of the utmost importance that the best class of Portland cement mortar be employed, as the carrying capacity of brick masonry depends largely if not chiefly upon the character of the mortar.



The micrometer used with the 400-ton Emery testing machine at the U. S. Arsenal, Watertown, Mass. The view shows the micrometer in position for determining the compressibility of a piece of stone.

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Art. 56.—Natural Building Stones.

The ultimate compressive resistance of natural building stones is affected greatly by the condition of the rock from which the cube or other test-piece is taken. That portion of a ledge exposed to the weather may be much weakened and, in fact, even disintegrated, but the material at a short distance from the exterior surface may have the greatest resistance of which the particular kind of stone is capable of yielding. Again, the compressive resistance of stones on their natural beds is much greater than when tested on edge. In the tests which follow the test-pieces were fairly representative of such quality of stones as would pass inspection in first-class engineering work, and it is to be assumed that they were compressed on their beds unless otherwise stated.

Table I taken from the " U. S. Report of Tests of Metals and Other Materials " for 1894, exhibits the coefficients of elasticity, ultimate compressive resistances, weights per cubic foot and coefficients of thermal expansion per degree Fahr., as well as the ratio, r, between lateral and direct strains for the granites, marbles, limestones, sandstones, and other stones shown in the left-hand column. The coefficients of elasticity and of thermal expansion were determined by employing blocks of stone about 24 ins. long and 6 ins. by 4 ins. in cross-section, the gauged length being 20 inches, but the ultimate compressive resistances were found by testing 4-inch. cubes. The number of tests for each coefficient of elasticity and ultimate resistance varied from one to nine but were generally two or three. The general run of values of ultimate resistance will be found to conform as well as could be expected with results for the same kind of stones in the tables which follow.

It will be observed that the marbles are the heaviest stones, although the granites are not much lighter. There is a large difference, however, between the sandstones and the marbles or granites.

TABLE I.

NATURAL STONES IN COMPRESSION ON BEDS.

Stone.	Coefficient of Elasticity, Lbs. per Sq. In.	Ultimate Compressive Resistance, Lbs. per Sq. In.			Weight per Cu. Ft., Lbs.	Coefficient of Expansion per Degree Fahr.	r.
		Max.	Mean.	Min.			
Branford granite, Conn....	8,712,100	15,854	15,707	15,560	162	.00000398	$\frac{1}{4}$
Milford granite, Mass.......	7,676,750	25,738	23,773	19,258	162.5	.00000418	$\frac{1}{5.8}$
Troy granite, N. H.........	6,118,850	28,768	26,174	23,580	164.7	.00000337	$\frac{1}{5.1}$
Milford pink granite, Mass..	6,200,350	22,162	18,988	15,756	161.9	——	
Pigeon Hill granite, Mass...	8,095,250	20,716	19,670	17,772	161.5	——	
Creole marble, Ga.........	7,993,700	15,512	13,466	11,420	170	——	$\frac{1}{2.9}$
Cherokee marble, Ga.......	10 427,800	13,415	12,619	11,822	167.8	.00000441	$\frac{1}{3.7}$
Etowah marble, Ga........	8,792,600	14,217	14,053	13,888	169.8	——	$\frac{1}{3.6}$
Kennesaw marble, Ga......	8,217,950	10,771	9,563	8,354	168.1	——	$\frac{1}{3.9}$
Lee marble, Mass..........	——	——	——	——	——	.00000454	
Marble Hill marble, Ga....	9,950,850	11,532	11,505	11,478	168.6	.00000202	$\frac{1}{3.4}$
Tuckhoe marble, N. Y.....	15 173,200	19,223	16,203	11,640	178	.00000441	$\frac{1}{4.5}$
Mount Vernon limestone, Ky	3,278,400	11,566	7,647	5,247	139.1	.00000464	$\frac{1}{4}$
Oolitic limestone, Ind......	——	——	——	——	——	.00000437	
North River bluestone, N. Y.	5,475,300	——	22,947	——	——	.00000519	
Manson slate, Maine.......	——	——	——	——	——	——	
Cooper sandstone, Oregon...	——	——	14,920	——	——	——	
Cooper sandstone, Oregon ..	3,021,350	16,366	15,284	14,203	159.8	.00000177	$\frac{1}{11}$
Maynard sandstone, Mass...	2,034,650	10,538	9,880	9,223	133.5	.00000567	$\frac{1}{3}$
Kibbe sandstone, Mass.....	2,066,800	10,663	10,363	10,063	133.4	.00000577	$\frac{1}{3.3}$
Worcester sandstone, Mass..	2,668,750	9,869	9,763	9,656	136.6	.00000517	$\frac{1}{4.4}$
Potomac sandstone, Md...	——	——	——	——	——	.000005	
Olympia sandstone, Oregon.	——	13,441	12,665	12,061	——	.0000032	
Chuckanut sandstone, Wash.	——	12,790	11,389	10,276	——	——	
Dyckerhoff's cement.......	——	——	——	——	——	.00000578	
* Yammerthal flint limestone, Buffalo...........	——	28,951	23,724	18,496	——	——	
* Red granite.............	——	——	28,647	——	——	——	

* From Report of 1899.

The coefficients of elasticity generally range considerably higher than those for concrete in Art. 54, but the sandstones form an exception to this observation. The coefficients of thermal expansion vary between rather wide limits but they are mostly a little lower only than those determined for concrete. The coefficient for the Dyckerhoff cement is very close to those exhibited for cement mortar and concrete in Art. 46. The column headed *r*, giving the ratios between lateral and direct strains, contains interesting data. From what has been shown in Art. 4 it is apparent that the total volume of the test-pieces was considerably reduced by the compression to which the cubes were subjected.

The coefficients of elasticity were determined at intensities of pressure running from 1000 or 2000 pounds per square inch up to 8000 or 10,000 pounds per square inch.

A coefficient would first be determined at comparatively low pressures, as from 1000 to 3000 pounds per square inch, and then at higher pressures, as from 7000 to 9000 or 10,000 pounds per square inch. As a rule, the coefficients determined at the higher pressures were materially higher in value than the others, the stiffness of the stone increasing with the loads within the limits of the test. The values in the table are the means of those at the low and high pressures.

With the ordinary working values of pressures in masonry, probably not more than two thirds of the values of the coefficients of elasticity given in the table should be employed.

In the "U. S. Report of Tests of Metals and Other Materials" for 1900 there may be found the results of compressing 4-inch cubes of Tennessee marble and of granite from the Mount Waldo Quarries at Frankfort, Maine. The ultimate compressive resistances of the 4-inch Tennessee marble cubes expressed in pounds per square inch, were as follows:

Maximum.	Mean.	Minimum.
25,478	20,329	16,309

The preceding three results cover twenty tests.

The ultimate resistances in pounds per square inch of the "Black Granite" from the Waldo Quarries, as determined from four tests of 2-inch cubes, were as follows:

Maximum.	Mean.	Minimum.
32,635	30,949	29,183

Again, in the same report, the ultimate resistances in pounds per square inch of four 4-inch cubes of limestone from Carthage, Mo., are as follows·

Maximum.	Mean.	Minimum.
17,130	14,947	13,660

Tables II, III, IV, and V of this article contain the results of tests given in the "Report on the Compressive Strength, Specific Gravity, and Ratio of Absorption of the Building Stones in the United States," by Gen. Q. A. Gillmore, 1876. The specimens, whose tests are given in Table II, were 2-inch cubes. "Each cube was placed between two cushion blocks of soft pine wood, 2 inches by 2 inches square, and slightly more than 0.25 inch in thickness; one on the top and the other under the bottom; the grain of the wood being parallel in each to the other—though no difference was observed when this was changed, as regards amount of record. . . . The cubes . . . were brought to a true, smooth, and regular but not a polished surface." The third column shows whether the specimen was crushed "on bed" or "on edge."

TABLE II.

TWO-INCH CUBES.

Kind.	Locality.		Position.	Compressive Resistance, Lbs. per Sq. In.	Pounds per Cubic Foot.	Remarks.
Blue	Staten Island, N. Y.	Granite.	Bed	22,250	178.8	Cracked before bursting.
	Dix Island, Me.			15,000	166.5	Burst suddenly.
Dark	Quincy, Mass.			17,750	166.2	Cracked before bursting.
Light	Quincy, Mass.			14,750	168.7	Cracked before bursting.
Flagging	North River.			13,425	168.1	Broke suddenly.
Old Quarry.	Westerly, R. I.			17,750	165.6	" "
Old Quarry.	Westerly, R. I.			17,250	165.6	" "
Up River.	Richmond, Va.			21,250	—	" "
Up River.	Richmond, Va.			20,000	—	" "
Niantic River	New London, Conn.			12,500	166.3	" "
Niantic River	New London, Conn.		Edge	14,175	166.3	" "
Porter's Rock	Mystic River, Conn.		Bed	18,125	164.4	" "
Porter's Rock	Mystic River, Conn.		Edge	22,250	164.4	" "
Gray	Westerly, R. I.		Bed	14,687	166.9	" "
Gray	Westerly, R. I.		Edge	14,937	166.9	" "
Gray	Richmond, Va.		Bed	14,100	164.4	" "
Gray	Richmond, Va.		Bed	13,875	164.4	" "
Gray	New Haven, Conn.		Edge	7,750	162.5	Waxy-looking.
Gray	New Haven, Conn.		Bed	9,500	162.5	Broke suddenly.
Gneiss	Sachemshead Quarry, Conn		Edge	15,937	163.7	" "
Gneiss	Sachemshead Quarry, Conn		Bed	14,000	163.7	" "
Dark	Duluth, Minn.		Bed	17,750	173.7	Syenitic.
Dark	Huron Island, Mich.		Bed	18,125	164.4	
Bluish-gray.	Keene, N. H.		Bed	10,375	166.0	
Gray	Pompton, N. J.		Bed	24,040	—	Average of 3.
Glen's Falls.	Glen's Falls, N. Y	Limestone.	Bed	11,475	168.8	Burst without cracking.
Glen's Falls	Glen's Falls, N. Y		Edge	10,750	168.8	" " "
Lake	Lake Champlain, N. Y.		Bed	25,000	171.9	" " "
Lake	Lake Champlain, N. Y.		Edge	21,500	171.9	" " "
North River.	Kingston, N. Y.		Bed	13,900	168.2	" " "
North River	Kingston, N. Y.		Edge	11,050	168.2	" " "
White	Joliet, Ill.		Bed	12,775	158.8	" " "
White	Joliet, Ill.		Bed	16,900	162.5	" " "
Drab	Lime Island, Mich.		Bed	25,000	161.2	" " "
Drab	Lime Island, Mich.		Bed	16,425	159.4	" " "
Drab	Marquette, Mich.		Bed	7,825	146.3	" " "
Drab	Marquette, Mich.		Edge	7,600	146.3	Rather a clay stone.
Dark	Bardstown, Ky.		Bed	16,250	166.9	
Dark	Bardstown, Ky.		Edge	15,000	166.9	
Drab	Canton, Mo.		Bed	9,250	146.0	
Drab	Canton, Mo.		Bed	5,650	146.0	
Caen	France		Bed	3,650	118.8	
Caen	France		Bed	3,450	118.8	
East Chester.	Tuckahoe, N. Y.	Marble.	Bed	12,950	179.7	
East Chester.	Tuckahoe, N. Y.		Bed	12,050	179.7	
Vermont.	Dorset, Vt.		Bed	7,612	164.7	
Vermont.	Dorset, Vt.		Bed	8,670	167.8	
Drab	Mill Creek Quarry, Ill.		Bed	9,687	160.6	
Drab	Mill Creek Quarry, Ill.		Edge	9,787	156.9	
Drab	North Bay Quarry, Wis.		Bed	20,025	175.0	
Drab	North Bay Quarry, Wis.		Edge	13,700	175.0	
Common Ital.	Italy		Bed	11,250	168.2	
Common Ital.	Italy		Bed	13,062	168.2	

TABLE II.—*Continued.*

Kind.	Locality.		Position.	Compressive Resistance, Lbs. per Sq. In.	Pounds per Cubic Foot.	Remarks.
Brown......	Little Falls, N. Y......	Sandstone.	Bed	9,850	140.6	Broke suddenly. Hardened by years of exposure.
Brown......	Little Falls, N. Y......		Edge	9,150	140.6	
Gray........	Belleville, N. J........		Bed	11,700	141.0	
Gray........	Belleville, N. J........		Edge	10,250	141.0	
Brown......	Middletown, Conn.....		Bed	6,950	148.5	
Brown......	Middletown, Conn.....		Edge	5,550	148.5	
Pink........	Medina N. Y.........		Bed	17,250	150.6	
Pink........	Medina, N. Y.........		Edge	14,812	149.3	
Drab........	Berea, Ohio..........		Bed	10,250	131.9	
Drab........	Berea, Ohio..........		Bed	8,300	133.1	
Drab........	Berea. Ohio..........		Bed	7,250	137.5	
Drab........	Vermillion, Ohio......		Bed	8,250	135.3	
Drab........	Vermillion, Ohio......		Bed	6,000	135.3	
Purple......	Fond du Lac, Wis.....		Bed	6,250	138.8	
Purple......	Fond du Lac, Wis.....		Edge	5,110	138.8	
Purple......	Marquette, Mich......		Bed	7,450	135.0	
Purple......	Marquette, Mich......		Edge	5,730	135.0	
Red-brown. .	Seneca Freestone, O. ..		Bed	9,687	149.3	
Red-brown. .	Seneca Freestone, O. ..		Edge	10,500	149.3	
Olive green..	Cleveland, Ohio.......		Bed	6,800	140.0	
Olive green..	Cleveland, Ohio.......		Edge	7,910	140.0	
Brown......	Albion, N. Y.........		Bed	13,500	151.2	
Brown......	Albion, N. Y.........		Edge	11,350	151.2	
Pink........	Kasota, Minn.........		Bed	10,700	164.4	Calcareous.
Pink........	Kasota, Minn.........		Edge	11,675	164.4	
Light buff...	Fontenac, Minn.......		Bed	6,250	145.3	
Light buff...	Fontenac, Minn.......		Edge	7,775	145.3	
Freestone....	Dorch'ter, New Brunswick..............		Bed	9,150	——	
Freestone....	Dorch'ter, New Brunswick..............		Edge	6,050	——	
Yellow drab..	Massillon, Ohio.......		Bed	8,750	131.8	
Yellow drab..	Massillon, Ohio.......		Bed	6,725	131.8	
Craigleith....	Edinburgh, Scotland...		Bed	12,000	141.3	
Craigleith....	Edinburgh, Scotland...		Edge	11,250	141.3	

General Gillmore showed that the size of the cube tested affected very greatly the ultimate compressive resistance per unit of area of face of cube. Table III shows the results of gradually increasing the size of cubes of Berea sandstone, crushed "on bed" between wooden cushion blocks, increasing (with size of cube) from about 0.0625 inch to about 0.4 inch in thickness. The general result is marked in spite of two or three irregularities.

These results are natural consequences of the character of stone and the cubical form of the specimens. A few of

TABLE III.

BEREA SANDSTONE CUBES.

Edge of Cube.	Compressive Resistance, Pounds per Square Inch.	Edge of Cube.	Compressive Resistance, Pounds per Square Inch.
Inches.	Pounds.	Inches.	Pounds.
0.25	4,992	2.00	8,955
0.50	6,080	2.25	9,130
0.75	6,347	2.50	8,856
1.00	6,990	2.75	9,838
1.25	7,342	3.00	10,125
1.50	8,226	4.00	11,720
1.75	9,310	——	——

General Gillmore's experiments showed that such results would probably not appear if the length of the specimens had been two or three times the width or breadth.

The effect of different bearing surfaces on the ultimate compressive resistance of stone cubes is well shown by the results given in Table IV. All the results are in pounds per square inch, and belong to 2-inch cubes, with the exception of the "Sandstone, drab" specimens, which were 1.5-inch cubes. Each result is a mean of two to five tests.

The steel cushion gave the highest results by a little. A soft cushion seems to be driven into the small cavities and interstices of the specimen, and thus to produce a splitting action at the bearing surfaces. "The beds of the granite and marble cubes were rubbed to the border of polish; those of sandstone were rubbed smooth."

Again, polished and unpolished cubes give different resistances per square inch, as shown in Table V. The results there given are for 2-inch cubes pressed upon by wooden cushions.

TABLE IV.

Kind of Stone.	Ultimate Compressive Resistance, Pounds per Square Inch.			
	Steel.	Wood.	Lead.	Leather.
Granite, Millstone Point, Conn.	23,190	22,880	15,730	——
Granite, Keene, N. H.	24,000	19,830	14,480	15,730
Marble, East Chester, N. Y.	19,125	17,540	11,560	——
Sandstone, Berea, Ohio	11,260	10,290	7,380	6,730
Vermont marble, Vermont	13,280	10,850	9,200	8,190
Limestone, Sebastopol	1,075	1,075	1,075	1,075
Sandstone, drab	4,000	4,000	4,000	——
Sandstone, Massillon, Ohio	8,500	8,750	7,250	——
Sandstone, Massillon, Ohio (softer)	5,660	6,730	5,500	3,640

It is at once evident that the polished cubes gave considerably the highest resistances. This is probably due to the fact that the splitting action of the wooden cushions was reduced to a minimum on the polished surfaces.

TABLE V.

Kind of Stone.	Ultimate Compressive Resistance, per Square Inch.	
	Polished.	Unpolished.
	Pounds.	Pounds.
Granite, Quincy, Mass.	24,750	17,750
Granite, Staten Island, N. Y.	25,000	22,250
Granite, Garrison's, N. Y.	21,630	13,380
Granite, Tarrytown, N. Y.	23,750	18,250
Granite, Millstone Point, Conn.	22,880	18,750
Granite, Keene, N. H.	19,830	12,750
Granite, Westerly, R. I.	23,500	17,750
Marble, East Chester, N. Y.	17,540	12,950
Marble, Vermont	10,850	8,750

General Gillmore's experiments show, in a conclusive manner, that variety in circumstances of testing will produce a variety of results for the same section of stone specimen. Attending circumstances and dimensions of specimens, therefore, should always be given.

The results found in Table VI are taken from the "U. S. Report of Tests of Metals and Other Materials" for 1894. They relate to the various kinds of rock indicated and were found by testing 4-inch to 5-inch cubes on their beds.

TABLE VI.

State.	Stone.	Ultimate Compressive Resistance, Pounds per Square Inch.
Minnesota	Ortonville granite	20,415
"	Kasota pink limestone	10,833
"	Faribault marble	17,780
"	Duluth brownstone	4,353
"	Mankato sandstone	9,606
"	Mantorville sandstone	8,775
"	Frontinac sandstone	10,114
"	Luverne quartzite	21,556
"	" "	19,875
Iowa	Rubble rock	9,465
"	Firestone	4,834
"	Gypsum, Fort Dodge	2,899

The ultimate resistances of the sandstones are relatively low, while the higher values are found for granites, limestones, and quartzites, as is usual.

Art. 57.—Timber.

The ultimate compressive resistance, coefficient of elasticity, and other physical properties of timber in compression are affected greatly by the amount of moisture in the timber and by the size of stick. The investigations of Professor J. B. Johnson, acting for the Forestry Division of the U. S. Department of Agriculture, have shown that when the amount of moisture exceeds about 30% by weight of the timber the physical properties are not much affected by any increased saturation. The walls of the wood cells at that point seem to experience their maximum softening. Green timber may be considered as carrying about one third of its weight in moisture, and it seems to matter little whether that moisture is water or sap, timber once dried and resaturated appearing to suffer the same diminished resistance as in its original green condition. Professor Johnson's tests showed that the Southern pines increased their ultimate compressive resistance in some cases as much as 75% by the process of drying or seasoning from 33% of moisture down to 10%, the general rule being a greatly increased compressive resistance with a decrease of moisture. It follows from these results, therefore, that green timber will be much weaker in compression than seasoned timber. Ordinary air seasoning even under cover seldom reduces moisture below about 15% in weight of the timber itself, although under favorable circumstances of seasoning the moisture may sometimes drop to 12% of that weight. As a matter of precision, therefore, or accuracy, the ultimate compressive resistance of timber should always be stated in connection with the percentage of moisture carried by the timber. This will be found to be the case in all of Professor Johnson's experimental work, to which reference has already been made and the results of which are chiefly found in bulletins Nos. 8 and 15 of the Division of Forestry of the U. S. Department of Agriculture, the former being dated 1893.

The earlier tests of Professor Johnson were made on a basis of 15% moisture, but in his later work a basis of 12% moisture was adopted, and he states in Circular No. 15 that in reducing the moisture from 15% to 12% the corresponding increases in the ultimate compressive resistance in pounds per square inch of Southern pines are approximately as follows:

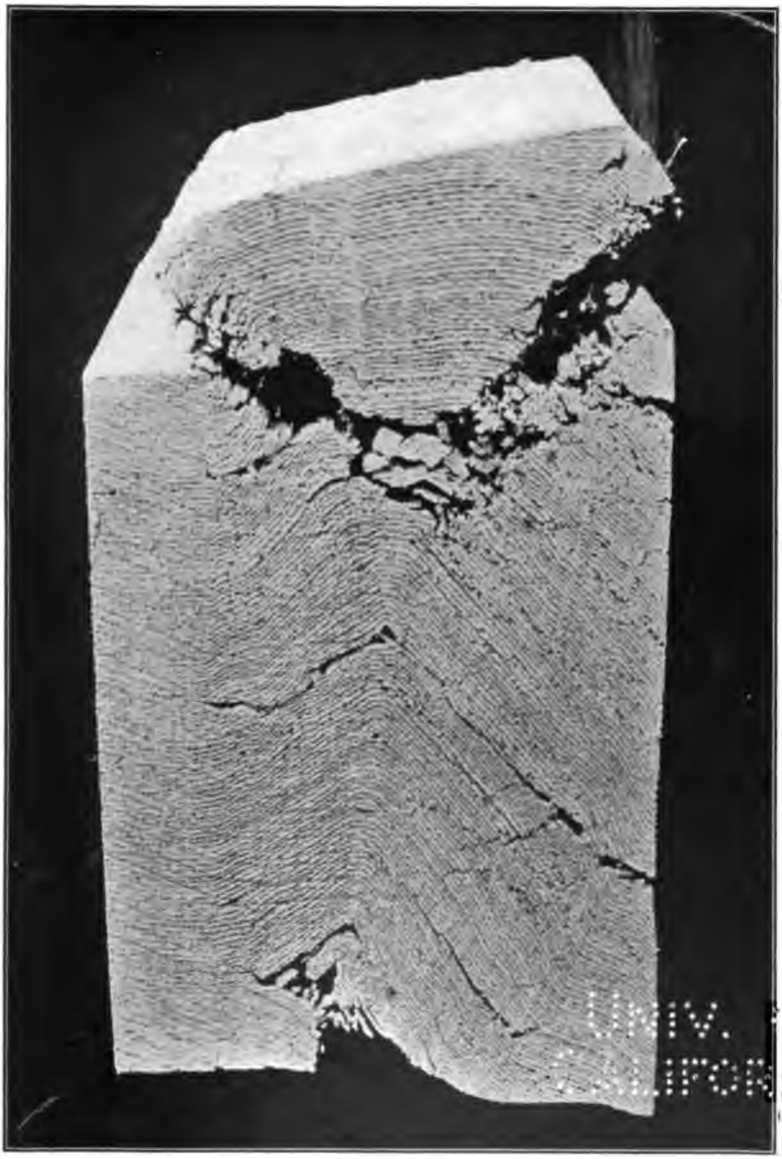
	Endwise.	Across Grain.
Long-leaf pine	1,100	180
Cuban pine	800	220
Loblolly pine	900	150
Short-leaf pine	600	60

While it is important as a matter of physics to recognize clearly the effect of moisture upon the compressive resistance of timber, it is of equal importance, and possibly of greater importance, to recognize the fact that in engineering practice, except in specially protected cases, the timber used in structures is more or less exposed and can seldom or never be depended upon to contain even as little as 15% of moisture, and with some conditions of weather and at some seasons of the year it may contain considerably more. It follows, also, that the condition of timber as to moisture in most structures will change materially from time to time. It would be unwise, therefore, and perhaps dangerous to use working compressive resistances based upon the results of tests of small pieces with moisture reduced to 15% or 12%.

Again, it has been frequently stated as a result of the timber investigations by the Forestry Division of the U. S. Department of Agriculture, that the ultimate compressive resistance of large sticks may be taken as practically identical with that belonging to small selected test pieces, the quality of the material being the same in both cases. It is possible, if the quality of material throughout all portions of every large stick were identical with the quality of small selected specimens, that the ultimate compressive resistance per square inch might be the same; but that is radically different from the facts as they are. There is probably no stick of timber whose condition is permanent at any given time. If it is seasoning, its quality is improving, but after reaching a maximum of excellence it begins to depreciate by decay or from other causes. Any large stick of timber as used by the engineer is seldom free from some point of incipient decay and it is never free from knots, large or small, wind shakes, cracks from one cause or another, or from some other defective condition, at some point. Small specimens for testing are invariably so selected as to eliminate such spots as militating against a comparatively high resistance. The inevitable result for full-size sticks is a decreased resistance materially below that of the small specimen. For all these reasons, therefore, in engineering practice it would be a radical error to accept the ultimate compressive resistance per square inch of small test specimens as practically identical with that of large sticks. Values for the latter class of timber should be determined upon pieces as large as those used in structures and under the same conditions in which they are used, which means an indefinite amount of moisture ordinarily sensibly larger than 12% or 15%.

In the "U. S. Report of Tests of Metals and Other Materials" for 1896 and 1897 there may be found results of compressive tests for coefficients of elasticity for sticks of timber as shown in Table I. Those sticks were many of them large enough to form full-size posts. They appear to have been of merchantable timber of about such quality as is used in first-class engineering works. They had the usual supply of knots and other features which, while not material defects, prevented the pieces from being of selected quality.

As also shown in the table, there were a considerable number of tests in each case. "Endwise" compression means compression parallel to the fibres of the timber, while "Tangentially" means a direction tangent to the rings of growth. That compression indicated by "Radially" was in a radial direction, i.e., passing through the centre of the tree trunk. The determinations were



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The fracture of a piece of Douglass fir or Oregon pine loaded tangentially to the rings of growth. The ultimate compressive resistance was found to be 600 lbs. per sq. in.

(*To face page* 440.)

TABLE I.

TIMBER IN COMPRESSION.

Kind of Wood.	Coefficient of Elasticity, Pounds per Square Inch.			No. of Tests.	Remarks.
	Maximum.	Mean.	Minimum.		
Douglas fir:					
Endwise.	3,461,000	2,358,000	1,915,000	4	Not well seasoned.
Tangentially	112,000	74,600	40,000	9	" " "
Radially	207,000	158,000	134,300	6	" " "
White oak:					
Endwise.	1,789,000	1,554,000	1,338,000	6	" " "
Long-leaf pine:					
Endwise.	1,890,000	1,657,000	1,488,000	4	From tops of trees
"	2,252,000	2,175,000	2,049,000	4	From butts of trees
Short-leaf pine:*					
Endwise.	1,655,000	1,469,500	1,202,000	6	Not well seasoned
Spruce:*					
Endwise.	1,623,000	1,531,000	1,437,000	10	" " "
Old yellow-pine posts:*					
Endwise.	2,300,000	2,251,000	2,207,000	12	Very dry.

* These results are means of determinations at intensities varying from 500 to 5,000 pounds per square inch.

made at intensities of pressure varying from one third to one half the ultimate resistance. It will be noticed that in the values for long-leaf pine the highest results belong to sticks from the butts of trees, while those from the tops give materially less values. It will also be observed that the values for the very dry yellow-pine posts in the last line of the table are high, showing the increased stiffness due to the absence of moisture. The coefficients of elasticity in the last five lines of the table were computed from the resilience of the compressed columns by means of a formula similar to eq. (2) of Art. 29.

Table II exhibits the results of compressive tests made by Professor Johnson on selected specimens of various American timbers in his work for the Division of Forestry. Most of the test-pieces were 4 inches square in section by 8 inches long. The tests, therefore, are for short blocks and not for columns. The number of tests for each kind of timber vary from 10 for green ash up to 1230 for long-leaf pine. The results are valuable if the size of specimen and the amount of moisture are both carefully borne in mind, but they should be used as they stand as representing

TABLE II.

ULTIMATE ENDWISE AND ACROSS-GRAIN COMPRESSIVE RESISTANCES OF TIMBER.

No.	Species.	Average Highest 10 Per Ct. of Tests.	Average Lowest 10 Per Ct. of Tests.	Average of All Tests.	Across Grain.
	Reduced to 15% Moisture.	Lbs. per Sq. In.	Lbs. per Sq. In.	Lbs. per Sq. In.	Lbs. per Sq. In.
1	Long-leaf pine	8,600	5,700	6,900	1,000
2	Cuban pine	9,500	6,500	7,900	1,000
3	Short-leaf pine	7,600	4,800	5,900	900
4	Loblolly pine	8,700	5,400	6,500	1,000
	Reduced to 12% Moisture.				
5	White pine	6,800	4,000	5,400	700
6	Red pine	8,100	4,900	6,700	1,000
7	Spruce pine	8,800	5,600	7,300	1,200
8	Bald cypress	8,500	4,200	6,000	800
9	White cedar	6,000	4,400	5,200	700
10	Douglas spruce, *a*	8,100	4,200	5,700	800
11	White oak	11,300	6,300	8,500	2,200
12	Overcup oak	8,600	6,000	7,300	1,900
13	Post oak	8,100	6,000	7,100	3,000
14	Cow oak	9,800	5,600	7,400	1,900
15	Red oak	9,200	5,500	7,200	2,300
16	Texan oak	9,800	6,900	8,100	2,000
17	Yellow oak	8,300	5,800	7,300	1,800
18	Water oak	9,000	6,300	7,800	2,000
19	Willow oak	8,700	5,500	7,200	1,600
20	Spanish oak	9,500	5,100	7,700	1,800
21	Shagbark hickory	10,900	7,500	9,500	2,700
22	Mockernut hickory	11,600	8,000	10,100	3,100
23	Water hickory	9,600	7,000	8,400	2,400
24	Bitternut hickory	11,200	7,800	9,600	2,200
25	Nutmeg hickory	11,000	7,100	8,800	2,700
26	Pecan hickory	10,400	7,300	9,100	2,800
27	Pignut hickory	12,700	8,900	10,900	3,200
28	White elm	8,800	5,000	6,500	1,200
29	Cedar elm	10,100	6,500	8,000	2,100
30	White ash	8,700	5,700	7,200	1,900
31	Green ash	9,800	6,600	8,000	1,700
32	Sweet gum	8,500	5,600	7,100	1,400

a, actual tests on "dry" material, not reduced for moisture.

ultimate values for full-size sticks only after comparison with the results of tests of such sticks.

It is essential to observe the small resisting capacity of the various timbers when compressed across the grain, the resistance in the latter condition being but a small fraction of that along the grain.

Table III contains the results of tests by Colonel Laidley, U.S.A., "Ex. Doc. No. 12, 47th Congress, 2d Session." A few other tests of short blocks from the same source will be found in the article on "Timber Columns." Unless otherwise stated, all the specimens were thoroughly seasoned.

In this table the "length" of all those pieces which were compressed in a direction perpendicular to the grain might, with greater propriety, be called the thickness, since it is measured across the grain.

In the tests (24–60) the compressing force was distributed over only a portion of the face of the block on which it was applied; thus the compressed area was supported, on the face of application, by material about it carrying no pressure. In some cases this rectangular compressed area extended across the block in one direction, but not in the other. In all such instances the ultimate resistance was a little less than in those in which the area of compression was supported on all its sides.

The "ultimate resistance" was taken to be that pressure which caused an indentation of 0.05 inch.

Nos. (44–55) show the effect of varying thickness of blocks. Within the limits of the experiments, the ultimate resistance is seen to decrease somewhat as the thickness increases.

The best series of values of the ultimate compressive resistance of timbers as actually used in large pieces and for engineering structures that can be written at the present time is that given in Table IV.

TABLE III.

No.	Kind of Wood.	Length, Inches.	Compressed Section in Inches.	Ultimate Resistance, Pounds per Square Inch.	Perpendicular to or with Grain.	Remarks.
1	Oregon pine	16.5	2.46×2.0	8,496	With	
2	Oregon pine	19.9	1.21×1.21	9,041	"	
3	Oregon pine	19.9	1.21×1.21	8253	"	
4	Oregon maple	8.0	3.63×3.63	6,661	"	
5	Oregon spruce	24.02	3.92×5.75	5,772	"	Unseasoned
6	California laurel	8.0	3.58×3.58	6,734	"	Worm-eaten
7	Ava Mexicana	8.0	3.69×3.69	6,382	"	
8	Oregon ash	8.0	3.64×3.64	5,121	"	
9	Mexican white mahogany	8.0	3.77×3.77	6,155	"	
10	Mexican cedar	8.0	3.75×3.75	4,814	"	
11	Mexican mahogany	8.0	3.75×3.75	10,043	"	
12	White maple	12.0	4.00×4.00	7,140	"	
13	White maple	12.0	4.00×4.00	7,210	"	
14	Red birch	13.0	4.26×4.26	8,030	"	
15	Red birch	13.0	4.26×4.26	7,820	"	
16	Whitewood	12.0	4.00×4.00	4.440	"	
17	Whitewood	12.0	4.00×4.00	4,330	"	
18	White pine	12.0	4.00×4.00	5,475	"	
19	White pine	12.0	4.00×4.00	5,760	"	
20	White oak	12.0	4.00×4.00	7,375	"	
21	White oak	12.0	4.00×4.00	7.010	"	
22	Ash	12.0	4.00×4.00	7.940	"	
23	Ash	12.0	4.00×4.00	7,640	"	
24	Oregon pine	1.95	3.45×3.00	1,150	Perp.	
25	Oregon maple	3.63	3.63×3.00	1,875	"	
26	Oregon spruce	3.92	5.75×4.75	710	"	Unseasoned
27	Oregon spruce	3.92	4.75×4.00	680	"	Unseasoned
28	California laurel	3.58	3.58×3.00	2,000	"	
29	Ava Mexicana	3.69	3.69×3.00	2,100	"	
30	Oregon ash	3.64	3.64×3.00	2,200	"	
31	Mexican white mahogany	3.77	3.77×3.00	2,150	"	
32	Mexican cedar	3.75	3.75×3.00	1,950	"	
33	Mexican mahogany	3.75	3.75×3.00	4,500	"	
34	White pine	3.06	6.20×4.75	875	"	
35	White pine	2.90	4.75×4.00	1,012	"	Mean of two
36	Whitewood	3.15	4.75×6.20	900	"	Mean of two
37	Whitewood	3.15	4.75×4.00	1,000	"	Mean of four
38	Black walnut	0.875	4.75×4.00	2,450	"	Mean of two
39	Black walnut	0.875	4.00×3.94	2,200	"	Mean of two
40	Black walnut	0.875	4.00×2.50	2,525	"	Mean of two
41	White oak	2.40	4.75×4.00	3,550	"	
42	Spruce	3.70	4.75×4.00	970	"	Mean of four
43	Yellow pine	3.90	4.00×4.00	1,900	"	
44	Black walnut	0.75	4.05×4.00	2,800	"	
45	Black walnut	1.00	4.05×4.00	2,560	"	
46	Black walnut	1.25	4.05×4.00	2,400	"	
47	Black walnut	1.50	4.05×4.00	2,500	"	
48	Black walnut	1.75	4.05×4.00	2,400	"	
49	Black walnut	2.06	4.05×4.00	2,360	"	
50	White pine	0.75	4.05×4.00	1,120	"	
51	White pine	1.00	4.05×4.00	1,100	"	
52	White pine	1.25	4.05×4.00	1,160	"	
53	White pine	1.50	4.05×4.00	1,070	"	
54	White pine	1.75	4.05×4.00	1,060	"	
55	White pine	2.00	4.05×4.00	1,000	"	
56	Yellow birch	4.25	4.25×3.00	2,000	"	
57	Yellow birch	4.25	5.08×3.00	1,650	"	
58	White maple	4.00	3.05×3.00	1,700	"	
59	White maple	4.00	5.08×3.00	1,900	"	
60	White oak	3.05	3.96×3.00	2,500	"	Mean of two

TABLE IV
TIMBER IN COMPRESSION.

Kind of Timber.	Ultimate Resistance, Pounds per Square Inch.			Average Working Stresses, Pounds per Square Inch.		
	With Grain.		Across Grain.	With Grain.		Across Grain.
	End Bearing.	Columns Under 15 Diams.		End Bearing.	Columns Under 15 Diams.	
White oak	7,000	4,500	2,000	1,400	900	500
White pine	5,500	3,500	800	1,100	700	200
Southern long-leaf or Georgia yellow pine	8,000	5,000	1,400	1,600	1,000	350
Douglas (Oregon) and yellow fir.	8,000	6,000	1,200	1,600	1,200	300
Washington fir or pine (red fir).	——	——	——	——	——	——
Northern or short-leaf yellow pine	6,000	4,000	1,000	1,200	800	250
Red pine	6,000	4,000	800	1,200	800	200
Norway pine	6,000	4,000	800	1,200	800	200
Canadian (Ottawa) white pine	——	5,000	——	——	1,000	——
Canadian (Ontario) red pine	——	5,000	——	——	1,000	——
Spruce and eastern fir	6,000	4,000	700	1,200	800	200
Hemlock	——	4,000	600	——	800	150
Cypress	6,000	4,000	700	1,200	800	200
Cedar	6,000	4,000	700	1,200	800	200
Chestnut	——	5,000	900	——	1,000	250
California redwood	——	4,000	800	——	800	200
California spruce	——	4,000	——	——	800	——

That table is taken from the report of the Committee of the Association of Railway Superintendents of Bridges and Buildings, to which reference was made on page 384. As stated in the article on tensile resistance, the values are applicable to timber as actually employed, and the working resistances are those of common use in connection with the design and construction of timber structures. As in the case of tension, the compressive resistances across the grain are but small fractions of those with the grain. Values are given for columns under 15 diameters in length for the reason that such columns fail essentially by compression and without the bending which characterizes long columns. The table is one of great practical value.

Bauschinger's Relation.

The experimental investigations of the Department of Forestry conducted by Professor Johnson appear to confirm at least approximately for small selected specimens the relation between the ultimate compressive resistance along the grain and the specific gravity found by Professor Bauchinger in his timber investigations. If c is the ultimate compressive resistance of timber on end, i.e., along the grain in pounds per square inch, and if g is the specific gravity of the timber, Professor Bauschinger's empirical relation between the ultimate compressive resistance and the specific gravity of the timber stands as follows:

$$c = 13800g - 900.$$

It is doubtful whether this empirical relation for such timbers as pine, larch, and fir with which Professor Bauschinger experimented can have much real value in connection with the large cross-sections of sticks used in engineering practice.

Miscellaneous Results.

Table V exhibits results of limited interest taken from the "U. S. Reports of Tests of Metals and Other Materials." They have possible applications only in unusual circumstances.

TABLE V.

MISCELLANEOUS ULTIMATE COMPRESSIVE RESISTANCES.

Kind.	Dimensions, Inches.			Ultimate Compressive Resistance, Lbs. per Sq. In.	No. of Tests.
	Length.	Breadth.	Thickness.		
Cabbage palm, Fla.	109	8.6	Diam.	756 (Endwise)	1
Redwood, Cal.	73	12.4	"	4,487 "	1
Iron bark, Aus.	24	6	5	8,738 "	1
Douglas fir, tangentially	12	10	8	613	1
" " "	9.6	9.7	9.6	600	1
" " "	9.6	9.7	9.6	700	1
" " "	7.8	8	8	800	1
" " obliquely to rings.	9.6	8	8	439	1

CHAPTER VIII.

COMPRESSION—LONG COLUMNS.

Art. 58.—Preliminary Matter.

THERE is a class of members in structures which is subjected to compressive stress, and yet those members do not fail entirely by compression. The axes of these pieces coincide, as nearly as possible, with the line of action of the resultant of the external forces, yet their lengths are so great compared with their lateral dimensions that they deflect laterally, and failure finally takes place by combined compression and bending. Such pieces are called "long columns," and the application to them, of the common theory of flexure, has been made in Art. 24.

Two different formulæ were first established for use in estimating the resistance of long columns; they are known as "Gordon's Formula" and "Hodgkinson's Formula." Neither Gordon nor Hodgkinson, however, gave the original demonstration of either formula.

The form known as Gordon's formula was originally demonstrated and established by Thomas Tredgold ("Strength of Cast Iron and other Metals," etc.), for rectangular and round columns, while that known as Hodgkinson's formula (demonstrated in Art. 24) was first given by Euler.

In 1840, however, Eaton Hodgkinson, F.R.S., published the results of some most valuable experiments made by himself on cast and wrought-iron columns (Experimental Researches on the Strength of Pillars of Cast Iron, and other Materials; Phil. Trans. of the Royal Society, Part II, 1840), and from these experiments he determined empirical coefficients applicable to Euler's formula, on which account it has since been called Hodgkinson's formula.

Mr. Lewis Gordon deduced from the same experiments some empirical coefficients for Tredgold's formula, since which time Gordon's formula has been known.

The latter has been quite generally used, but it has lately been displaced by the straight-line formula to be given later. Hodgkinson's coefficients and formula will be given farther on.

Before taking up either, however, it will be useful and convenient to determine the moments of inertia and squares of the radii of gyration of the various forms of cross-sections of the columns now in common use.

It will also be both convenient and important to determine the conditions which exist with an isotropic character of section in respect to the moment of inertia.

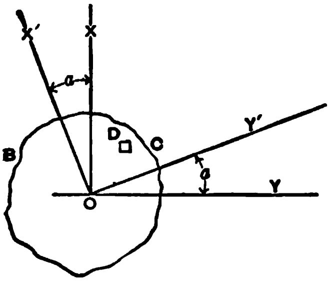


Fig. 1*a*.

In Fig. 1*a* let *BC* be any figure whose area is *A*, and whose centre of gravity is at *O*. In the plane of that figure let any arbitrary system of rectangular coordinates *X'*, *Y'* be chosen and let *XY* be any other system having the same origin; also, let x', y' and x, y be the coordinates of the element *D* of the surface *A* in the two systems. There will then result

$$x = x' \cos \alpha + y' \sin \alpha,$$
$$y = y' \cos \alpha - x' \sin \alpha.$$

The moments of inertia of the surface about the axes y and x will then be

$$\int x^2dA = \cos^2\alpha\int x'^2dA + 2\sin\alpha\cos\alpha\int x'y'dA + \sin^2\alpha\int y'^2dA,$$

$$\int y^2dA = \cos^2\alpha\int y'^2dA - 2\sin\alpha\cos\alpha\int x'y'dA + \sin^2\alpha\int x'^2dA.$$

If x and y are to be so chosen that they are ***principal*** axes, then must $\int xydA = 0$, or

$$0 = \int xydA = \sin\alpha\cos\alpha\int y'^2dA + (\cos^2\alpha - \sin^2\alpha)\int x'y'dA$$
$$- \sin\alpha\cos\alpha\int x'^2dA; \quad (1a)$$

$$\therefore \tan 2\alpha = \frac{2\int x'y'dA}{\int x'^2dA - \int y'^2dA}.$$

Hence, since $\tan 2\alpha = \tan(180 + 2\alpha)$, there will ***always*** be two principal axes 90° apart.

Now, if $\int x'y'dA = 0$, while no other condition is imposed. $\tan 2\alpha = 0$. This makes $\alpha = 0$ or 90°; i.e., $X'Y'$ are the principal axes.

If, however, $\int x'ydA = 0$, while α is neither 0 nor 90°, eq. (1a) becomes

$$\int y'^2dA - \int x'^2dA = 0;$$

or

$$\tan 2\alpha = \frac{0}{0}, \text{ i.e., indeterminate.}$$

This shows that any axis is a *principal* axis; also that

$$\int x^2dA = \int y^2dA = \int x'^2dA = \int y'^2dA.$$

Hence the surface is completely isotropic in reference to its moment of inertia, or *its moment of inertia is the same about every axis lying in it and passing through its centre of gravity.*

It has been seen that this condition exists where there are two different rectangular systems, for which

$$\int xydA = \int x'y'dA = 0;$$

but the first of these holds true if either x or y is an axis of symmetry, and the latter if either x' or y' is an axis of symmetry.

Hence, *if the surface has two axes of symmetry not at right angles to each other, its moment of inertia is the same about all axes passing through its centre of gravity and lying in it.*

Eqs. (1*a*) and the two preceding it also show that the same condition obtains *if the moments of inertia about four axes at right angles to each other, in pairs, are equal.*

In the case of such a surface, therefore, it will only be necessary to compute the moment of inertia about such an axis as will make the simplest operation.

Since a column fails partly by flexure, it is manifest that *the moment of inertia of its cross-section should be the largest possible about an axis passing through its centre of gravity and normal to the plane of flexure.*

Box Column of Plates and Angles.

Fig. 1 shows the cross-section of a box column composed of 4 plates and 4 or 8 equal-legged Ls. *FB* and *CD* intersect at the centre of gravity of the cross-section.

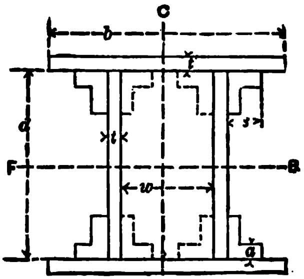


FIG. 1.

If the 4 Ls shown in dotted lines are omitted, the moment of inertia about *FB* will be

$$I=\frac{bt'^3}{6}+bt'\frac{(d+t')^2}{2}+\frac{(s+t)d^3}{6}-\left[\frac{(s-a)(d-2a)^3+a(d-2s)^3}{6}\right]. \quad (1)$$

If the dotted Ls are not omitted

$$I=\frac{bt'^3}{6}+bt'\frac{(d+t')^2}{2}+\frac{(2s+t)d^3}{6}-\left[\frac{(s-a)(d-2a)^3+a(d-2s)^3}{3}\right]. \quad (2)$$

If the 4 Ls shown in dotted lines are omitted, the moment of inertia about *CD* will be

$$I=\frac{t'b^3}{6}+\frac{a(w+2t+2s)^3}{6}+\frac{(s-a)(w+2t+2a)^3}{6}+\frac{(d-2s)(w+2t)^3}{12}-\frac{dw^3}{12}. \quad (3)$$

If the dotted angles are not omitted

$$I=\frac{t'b^3}{6}+\frac{a[(w+2t+2s)^3-(w-2s)^3]}{6}$$
$$+\frac{(s-a)[(w+2t+2a)^3-(w-2a)^3]}{6}+\frac{(d-2s)[(w+2t)^3-w^3]}{12}. \quad (4)$$

If latticing is used instead of the two plates *bt'*, *t'* becomes equal to zero, and the first term in the second member of each of the above equations disappears.

If *A* represents the area of the cross-section, and *r* the radius of gyration,

$$r^2=\frac{I}{A}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Box Column of Plates and Channels.

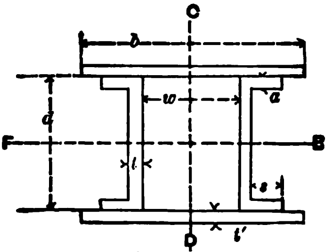


FIG. 2.

Fig. 2 shows a normal cross-section of this column. *FB* and *CD* intersect in the centre of gravity of the cross-section. As in the preceding figure, these lines are lines of symmetry. The moment of inertia about *FB* is

$$I=\frac{bt'^3}{6}+bt'\frac{(d+t')^2}{2}+\frac{(s+t)d^3-s\,(d-2a)^3}{6}. \quad . \quad (6)$$

The moment of inertia about *CD* is

$$I=\frac{t'b^3}{6}+\frac{2a(w+2t+2s)^3+(d-2a)(w+2t)^3-dw^3}{12}. \quad . \quad (7)$$

If latticing takes the place of the two plates bt', all terms in the second members of eqs. (6) and (7) involving t' will disappear. The moment of inertia about FB then becomes

$$I=\frac{(s+t)d^3-s(d-2a)^3}{6} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (8)$$

and that about CD

$$I=\frac{2a(w+2t+2s)^3+(d-2a)(w+2t)^3-dw^3}{12}. \quad . \qquad (9)$$

(*Radius of gyration*)$^2=r^2=\frac{I}{A}$, in which A is the area of whole section.

Eqs. (7) and (9) may also take the forms given in eqs. (15) and (16).

Built Column of Plates and Angles.

Fig. 3 shows a normal cross-section of this column with the two axes of symmetry, FB and CD, intersecting at its centre of gravity. The moment of inertia about FB is

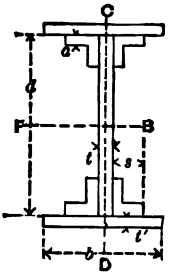


FIG 3.

$$I=\frac{bt'}{2}\left(\frac{t'^2}{3}+(d+t')^2\right)+\frac{(s+\frac{1}{2}t)d^3}{6}$$
$$-\left[\frac{(s-a)(d-2a)^3+a(d-2s)^3}{6}\right]. \qquad (10)$$

The moment of inertia about CD takes the value

$$I=\frac{t'b^3}{6}+\frac{a(2s+t)^3}{6}+\frac{(s-a)(2a+t)^3}{6}+\frac{(d-2s)t^3}{12}. \qquad (11)$$

If the two plates bt' are omitted, the terms involving t' in eqs. (10) and (11) reduce to zero.

(Radius of gyration)$^2 = r^2 = \frac{I}{A}$, in which A is area of section.

False Channel Section.

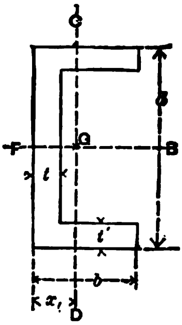


FIG. 4.

Let FB and CD intersect in the centre of gravity, G, of the section. The distance x_1 of G from the back of the channel is

$$x_1 = \frac{\frac{1}{2}[b^2 d - (b^2 - t^2)(d - 2t')]}{A}, \quad . \quad (12)$$

in which A is area of the cross-section of the channel. This is usually found by taking one tenth of the weight, in pounds, per yard of the channel. Analytically

$$A = 2bt' + t(d - 2t'). \quad . \quad . \quad (13)$$

The moment of inertia about CD then becomes

$$I' = \frac{2t'(b - x_1)^3 + dx_1^3 - (d - 2t')(x_1 - t)^3}{3}. \quad . \quad (14)$$

About FB it has the value

$$I = \frac{bd^3 - (b - t)(d - 2t')^3}{12}. \quad . \quad . \quad . \quad (14a)$$

(Radius of gyration)$^2 = r^2 = \frac{I}{A}$.

The line CD can be very quickly and accurately located by balancing the section, cut out of manilla paper, on a knife edge.

Eqs. (7) and (9) may now take the forms

$$I=\frac{t'b^3}{6}+2I'+2A\left(\frac{w}{2}+x_1\right)^2, \quad . \quad . \quad . \quad (15)$$

$$I=2\left[I'+A\left(\frac{w}{2}+x_1\right)^2\right]. \quad . \quad . \quad . \quad . \quad . \quad (16)$$

In eqs. (15) and (16) A represents the area of one channel section.

Angle Iron Section.

Fig. 5 represents this section with the two axes taken parallel to the legs, passing through the centre of gravity G. The area of cross-section is usually found from the weight per yard. Analytically

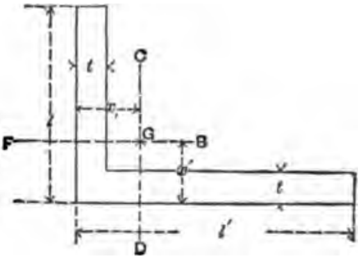


FIG. 5.

$$A=lt+(l'-t)t=(l+l'-t)t. \quad . \quad . \quad . \quad . \quad (17)$$

Again,

$$x_1=\frac{\frac{1}{2}[ll'^2-(l'^2-t^2)(l-t)]}{A}, \quad . \quad . \quad . \quad . \quad (18)$$

$$x'=\frac{\frac{1}{2}[l^2l'-(l'-t)(l^2-t^2)]}{A}. \quad . \quad . \quad . \quad . \quad (19)$$

The moment of inertia about CD is

$$I=\frac{t(l'-x_1)^3+lx_1^3-(l-t)(x_1-t)^3}{3}. \quad . \quad . \quad (20)$$

About FB

$$I=\frac{t(l-x')^3+l'x'^3-(l'-t)(x'-t)^3}{3}. \quad . \quad . \quad (21)$$

If the angle iron is equal legged, l becomes equal to l'.

$$(\textit{Radius of gyration})^2 = r^2 = \frac{I}{A}.$$

As in the case of the **C**, x_1 and x' may easily and accurately be found by balancing a model of the **L** section on a knife edge.

Latticed Column of Four Angles.

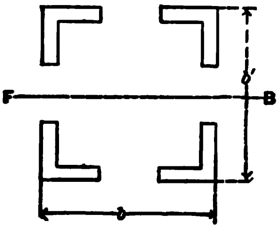


Fig. 6.

The four **L**s are held in the relative positions shown in Fig. 6 by latticing, the latter being riveted to the legs of the **L**s, but not shown in the figure. The **L**s are equal legged.

From either eq. (20) or (21) the moment of inertia of the section of any one **L**, about an axis passing through its centre of gravity and parallel to b, is

$$I_1 = \frac{t(l-x_1)^3 + lx_1^3 - (l-t)(x_1-t)^3}{3}.$$

Hence the moment of inertia of the column section of Fig. 6, about FB, is

$$I' = 4I_1 + A\left(\frac{b'}{2} - x_1\right)^2. \quad . \quad . \quad . \quad . \quad (22)$$

A is the area of the column section, or four times the area of one **L** section.

If b is different from b', the moment of inertia of the column section about an axis passing through its centre and parallel to b' will be found by simply changing b' to b in eq. (22).

$$(\textit{Radius of gyration})^2 = r^2 = \frac{I'}{A}.$$

Latticed or Closed Columns of Plates and Angles.

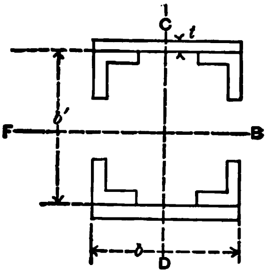


FIG. 7.

Fig. 7 represents a normal section of one of these columns. By the aid of eq. (22), the moment of inertia of the section about FB may be written

$$I = I' + \frac{bt}{2}\left(\frac{t^2}{3} + (b'+t)^2\right), \qquad (23)$$

and that about CD, remembering that in I', b' is to be changed to b:

$$I = I' + \frac{tb^3}{6}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (24)$$

If the plates are on the sides parallel to b', then b is to be changed to b' and b' to b in eqs. (23) and (24).

Fig. 8 represents the normal section of the other of these columns, in which there is no latticing, the column being perfectly closed.

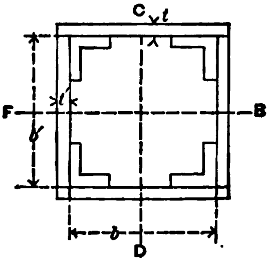


FIG. 8.

Again, using eq. (22), the moment of inertia about the axis FB is

$$I = I' + \frac{bt}{2}\left(\frac{t^2}{3} + (b'+t)^2\right) + \frac{t'(b'+2t)^3}{6}. \qquad (25)$$

The moment of inertia about CD is

$$I = I' + \frac{t'(b'+2t)}{2}\left(\frac{t'^2}{3} + (b+t')^2\right) + \frac{tb^3}{6}. \quad . \quad . \quad . \qquad (26)$$

In the I' in eq. 26, b' is to be changed to b. Ordinarily, $b = b'$ and $t = t'$.

(*Radius of gyration*)$^2 = r^2 = \frac{I}{A}$, A being area of cross-section.

Tee Section.

The axis *FB* is taken parallel to the head of the tee section, and *CD* perpendicular to it, while *G* is its centre of gravity. Analytically the area of the section is

$$A = bt + dt'. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

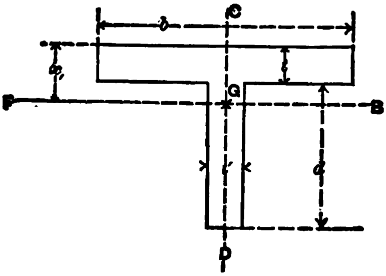


FIG 9.

The area may also be taken from the weight in the usual manner.

$$x_1 = \frac{\frac{1}{2}[(b-t')t^2 + (d+t)^2t']}{A} \quad . \quad . \quad . \quad . \quad . \quad (28)$$

The moment of inertia about *FB* is

$$\bar{I} = \frac{bx_1^3 + t'(d+t-x_1)^3 - (b-t')(x_1-t)^3}{3} \quad . \quad . \quad . \quad (29)$$

The moment of inertia about *CD* is

$$I = \frac{tb^3 + dt'^3}{12}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

$$(\textit{Radius of gyration})^2 = r^2 = \frac{I}{A}.$$

As in the other cases *FB* may be located by balancing on a knife edge.

False I Section.

If the area is not taken from the weight per yard, it may be written

$$A = bd - (b - t')(d - 2t). \quad . \quad (31)$$

The moment of inertia about CD is

$$I = \frac{2tb^3 + (d - 2t)t'^3}{12}. \quad . \quad . \quad (32)$$

About FB it has the value

$$I = \frac{bd^3 - (b - t')(d - 2t)^3}{12}. \quad . \quad (33)$$

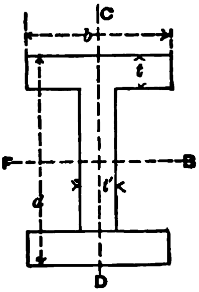


Fig. 10.

$$(Radius\ of\ gyration)^2 = r^2 = \frac{I}{A}.$$

Star Section.

Fig. 11 shows this section with the different dimensions. The area of cross-section is

$$A = bt + b't' - tt'. \quad . \quad (34)$$

The moment of inertia about FB is

$$I = \frac{t'b'^3 + (b - t')t^3}{12}. \quad . \quad (35)$$

About CD the moment of inertia has the value

$$I = \frac{tb^3 + (b' - t)t'^3}{12}. \quad . \quad (36)$$

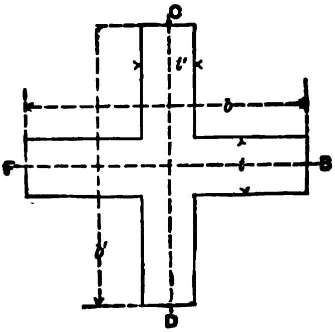


Fig. 11.

Ordinarily $t = t'$.

$$(Radius\ of\ gyration)^2 = r^2 = \frac{I}{A}.$$

Solid Rectangular Section.

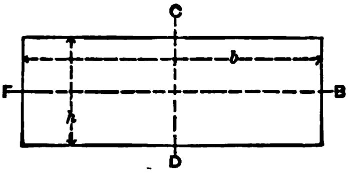


FIG. 12.

In Fig. 12 $A=bh$.

The moment of inertia about FB is

$$I=\frac{bh^3}{12}, \quad . \quad . \quad (37)$$

and about CD,

$$I=\frac{hb^3}{12}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

$$(\textit{Radius of gyration})^2=r^2=\frac{I}{A}=\frac{h^2}{12} \text{ or } \frac{b^2}{12}.$$

If the rectangular section is square, $b=h$.

Hollow Rectangular Sections.

The area of the section shown in Fig. 13 is $A=bh-b'h'$.

The moment of inertia about FB is

$$I=\frac{bh^3-b'h'^3}{12}, \quad (39)$$

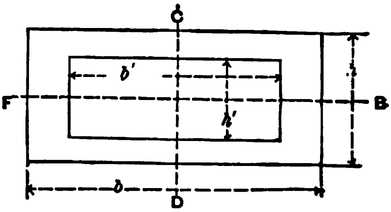


FIG. 13.

and that about CD is

$$I=\frac{hb^3-h'b'^3}{12}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

$$(\textit{Radius of gyration})^2=r^2=\frac{I}{A}.$$

All the equations of this case (except eq. (40)), just as they stand, apply directly to the rectangular cellular section of Fig. 14, considered in reference to the axis *FB*. If there were *n* cells instead of 3, the space between any adjacent two would have the width $\frac{b'}{n}$.

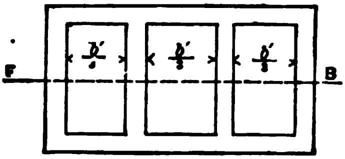


FIG. 14.

Solid and Hollow Circular Sections.

First consider a solid cylindrical column whose cross-section has the radius r_2, as shown in Fig. 15. The moment of inertia about any diameter is

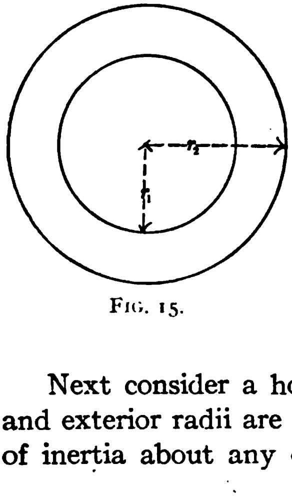


FIG. 15.

$$I = \frac{\pi r_2^4}{4}. \quad . \quad . \quad (41)$$

$$(\textit{Radius of gyration})^2 = \frac{\pi r_2^4}{4\pi r_2^2}$$

$$= \frac{r_2^2}{4} = r^2.$$

Next consider a hollow circular column whose interior and exterior radii are r_1 and r_2 respectively. The moment of inertia about any diameter is

$$I = \frac{\pi(r_2^4 - r_1^4)}{4} = \frac{A(r_2^2 + r_1^2)}{4}; \ (A = \text{area}). \quad . \quad (42)$$

$$(\textit{Radius of gyration})^2 = \frac{I}{\pi(r_2^2 - r_1^2)} = \frac{r_2^2 + r_1^2}{4} = r^2.$$

As tables of circular areas are easily accessible, it may be convenient to write

$$r^2 = \frac{\pi r_2^2}{12.566}, \quad \text{or} \quad r^2 = \frac{\pi(r_2^2 + r_1^2)}{12.566}.$$

Phœnix Section.

Fig. 16 shows the section of a four-segment Phœnix column. Let CD represent any axis taken through the centre of the column. The moments of inertia of the rectangles bl about axes through their centres of gravity and parallel to CD will be very small indeed compared with the moment of inertia of the whole section. The moment of inertia of any one of these rectangles, therefore, about CD, will be taken as equal to the product of its area by the square of the normal distance from its centre of gravity to the axis CD. The moment of inertia of the section about CD will then be

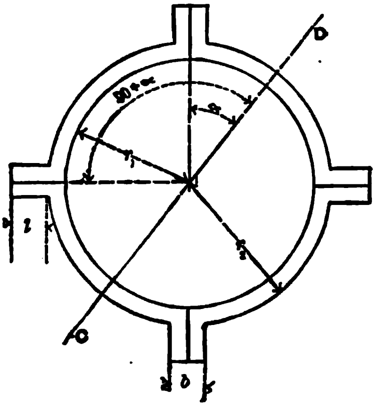


FIG. 16.

$$I = \frac{\pi(r_2^4 - r_1^4)}{4} + 2bl\left[\left(r_2 + \frac{l}{2}\right)^2 \sin^2 \alpha + \left(r_2 + \frac{l}{2}\right)^2 \cos^2 \alpha\right];$$

$$\therefore\ I = \frac{\pi(r_2^4 - r_1^4)}{4} + 2bl\left(r_2 + \frac{l}{2}\right)^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

The moment of inertia is thus seen to be the same about all axes, a result of the general principle established in the first part of this article.

The area of the cross-section is

$$A = \pi(r_2^2 - r_1^2) + 4bt. \quad . \quad . \quad . \quad . \quad (43a)$$

$$(Radius\ of\ gyration)^2 = r^2 = \frac{I}{A}.$$

The moments of inertia of six- and eight-segment columns may be found in precisely the same manner. The moments of inertia of the rectangular sections of the flanges about axes passing through their centres of gravity, being very small indeed when compared with the moment of inertia of the whole section, may be neglected without sensible error.

True I Section.

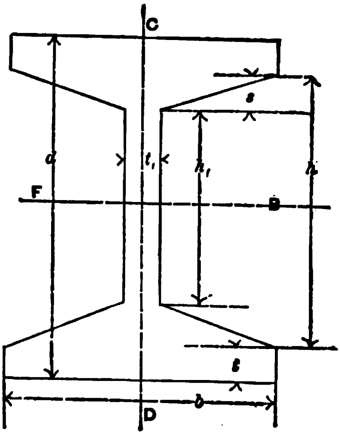


Fig. 17.

Let $r = \frac{2s}{b - t_1}$; r is then the batter, or slope, of the under side of each flange to the top or bottom of the beam; it ranges from about one third to essentially nothing.

If the area of the cross-section is not deduced from the weight, *Area of section*

$$= A = 2bt + ht_1 + s(d - t_1). \quad (44)$$

The moment of inertia about *CD* is

$$I = \frac{2tb^3 + h_1 t_1^3}{12} + \frac{r(b^4 - t_1^4)}{48}. \quad . \quad . \quad . \quad . \quad (45)$$

If t_1 is very small as compared with b, remembering that $\frac{b}{2}r$ is then essentially equal to s, there will result

$$I=\frac{(2t+\frac{1}{2}s)b^3+h_1t_1^3}{12}. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (46)$$

This formula is sufficiently accurate for all wrought-iron and steel beams.

The moment of inertia about FB is

$$I=\frac{bd^3-\frac{1}{4r}(h^4-h_1^4)}{12}. \quad . \quad . \quad . \quad . \quad . \qquad (47)$$

In any of these three cases

$$(Radius\ of\ gyration)^2=\frac{I}{A}. \quad . \quad . \quad . \quad . \quad . \qquad (48)$$

True Channel Section.

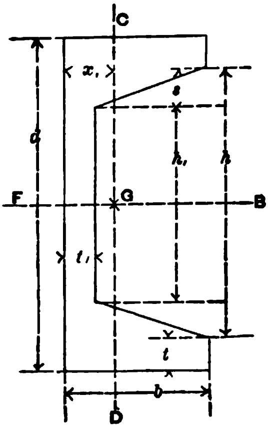


FIG. 18.

In Fig. 18 let $r=\frac{s}{b-t_1}$; as before, it is the batter or slope of the under side of the flange.

If the area of the section is not deduced from the weight,

Area of section

$$=A=2bt+ht_1+s(b-t_1). \qquad (49)$$

The centre of gravity, G, can be found by balancing a manilla, or other, pattern on a knife edge; or, analytically,

$$x_1=\frac{b^2t+\frac{1}{2}ht_1^2+\frac{1}{3}s(b-t_1)(b+2t_1)}{A}. \quad . \quad . \qquad (50)$$

The moment of inertia about *CD* is

$$I=\frac{2tb^3+h_1t_1^3+\frac{1}{2}r(b^4-t_1^4)}{3}-Ax_1^2. \quad . \quad . \quad . \quad (51)$$

If t_1 is very small compared with b, and remembering that br is then essentially equal to s, this last equation will become

$$I=\frac{(2t+\frac{1}{2}s)b^3+h_1t_1^3}{3}-Ax_1^2. \quad . \quad . \quad . \quad . \quad (52)$$

The moment of inertia about *FB* is

$$I=\frac{bd^3-\frac{1}{8r}(h^4-h_1^4)}{12}. \quad . \quad . \quad . \quad . \quad (53)$$

In any of these three cases

$$(\textit{Radius of gyration})^2=\frac{I}{A}. \quad . \quad . \quad . \quad . \quad (54)$$

Deck Section.

The head of this section will be considered circular in outline, as shown in Fig. 19. Let a be the area ot the circle C.

If the area of the section is not deduced from the weight, *Area of section*

$$=A=a+(d-h)t_1+(b-t_1)(t+\tfrac{1}{2}s). \quad . \quad . \quad (55)$$

If the centre of gravity, G, is not found by balancing a pattern on a knife edge, there will result, analytically,

$$x_1=\frac{a(2d-h)+t_1(d-h)^2+(b-t_1)(t^2+st+\frac{1}{3}s^2)}{2A}. \quad (56)$$

The moment of inertia about *FB* is

$$I=a\left\{\frac{h^2}{16}+\left(d-\frac{h}{2}\right)^2\right\}+\frac{t_1(d-h)^3}{3}+\frac{(t+s)^4-t^4}{6r}-Ax_1^2. \quad (57)$$

In which equation $r=\frac{2s}{b-t_1}$.

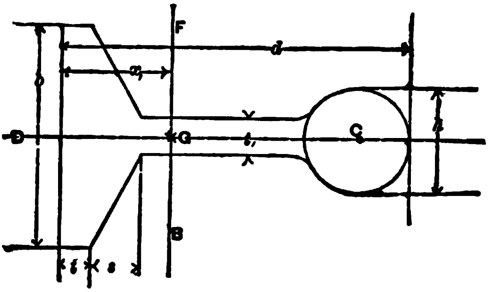


FIG. 19.

The moment of inertia about *CD* is

$$I=\frac{\frac{3}{4}ah^2+t_1^3(d-h-t-s)+tb^3+\frac{r}{8}(b^4-t_1^4)}{12}. \quad . \quad (58)$$

If t is small as compared with b, so that essentially $\frac{br}{2}=s$,

$$I=\frac{3ah^2+4t_1^3(d-h-t-s)+(4t+s)b^3}{48}. \quad . \quad . \quad (59)$$

In all cases,

$$(Radius\ of\ gyration)^2=\frac{I}{A}. \quad . \quad . \quad . \quad . \quad (60)$$

Angle Section about Oblique Axis.

The angle iron is here supposed to be equal legged, and the axis about which the moment of inertia is taken passes through the centre of gravity (before found in this article) and cuts the sides l at an angle of 45°. In Fig. 20, G is the centre of gravity and HK the axis.

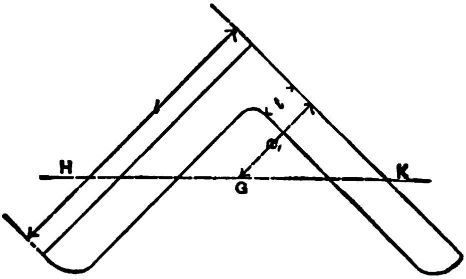


FIG. 20.

The moment of inertia about HK is

$$I=\frac{2\{x_1{}^4-(x_1-t)^4\}+t\{l-(2x_1-\frac{1}{2}t)\}^3}{3}. \quad . \quad . \qquad (61)$$

If A is the area of cross-section,

$$(\mathit{Radius\ of\ gyration})^2=\frac{I}{A}. \quad . \quad . \quad . \quad . \qquad (62)$$

If a long column has the same degree of fixedness or freedom in all directions, the *least* value of the square of the radius of gyration must be taken for insertion in Gordon's formula, because in the plane of that radius the column will offer the least resistance to flexure.

Art. 59.—Gordon's Formula for Long Columns.

Since flexure takes place if a long column is subjected to a thrust in the direction of its length, the greatest intensity of the stress in a normal section of the column may be considered as composed of two parts. In fact the condition of stress in any normal section of a long column is that of a uniformly varying system composed of a uniform stress and a stress couple. In order to determine these two parts let S represent the area of the normal cross-section; I, its moment of inertia about an axis normal to the plane in which flexure takes place; r, its radius of gyration in reference to the same axis; P, the magnitude of the imposed thrust; f, the greatest intensity of stress allowable in the column, and Δ, the deflection corresponding to f. Let p' be that part of f caused by the direct effect of P, and p'' that part due to flexure alone. Then, if h is the greatest normal distance of any element of the column from the axis about which the moment of inertia is taken, by the "common theory of flexure,"

$$c'P\Delta = \frac{p''I}{h}; \quad \therefore \quad p'' = \frac{c'P\Delta h}{I}. \quad . \quad . \quad . \quad (1)$$

If the column ends are round, $c' = 1$; but if the ends are fixed, the value of c' will depend upon the degree of fixedness.

Also

$$p' = \frac{P}{S}; \quad \therefore \quad p' + p'' = f = \frac{P}{S}\left(1 + \frac{c'S\Delta h}{I}\right). \quad . \quad (2)$$

Hence

$$P = \frac{fS}{1 + \dfrac{c'S\Delta h}{I}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Eq. (3) may be considered one form of Gordon's formula. Before deducing the more common and useful form, it will be necessary to show that $\Delta = a\frac{l^2}{h}$, in which expression a is considered constant.

Let p be the greatest intensity of bending stress in any section whose greatest value in the column is p''. By the "common theory" (taking the origin of coordinates at the centre of gravity of the cross-section at one end of the column, and the axis of x along the centre line before flexure)

$$EI\frac{d^2y}{dx^2} = \frac{pI}{h}.$$

Also

$$p = \frac{Mh}{I}, \quad \text{and} \quad p'' = \frac{M_0h}{I}; \quad . \quad . \quad . \quad . \quad (4)$$

in which equations E is the coefficient of elasticity, M the bending moment for any section, and M_0 the value of M corresponding to p''.

Hence

$$p = p''\frac{M}{M_0}, \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{p''}{Eh}\frac{M}{M_0}.$$

Consequently

$$\Delta = \int_0^{l_0}\int_{l_0}^{x} \frac{p''}{Eh} \cdot \frac{M}{M_0} \cdot dx^2. \quad . \quad . \quad . \quad (5)$$

The section located by l_0 is that at which the deflection is greatest, and for which $\frac{dy}{dx} = 0$, while $\frac{p''}{Eh}$ is considered constant. The ratio $\frac{M}{M_0}$ is numerical, though variable, being one between quantities of the same degree. M_0 is exactly the same as M, except that x, in the latter, is displaced by l_0; there are the same number of terms in each, and those terms are multiplied by the same coefficients. The quantity $\int_0^{l_0}\int_{l_0}^{x} M dx^2$ may be so arranged as to have the same number of terms as M_0, but *the coefficients of those terms will be different*, and *the exponents of l_0 in the former will be greater by 2 than the exponents of l_0 in M_0*. Hence $l_0^2 = c^2 l^2$ (c being some constant) will be a factor in *all* the terms of the definite double integral. From these considerations it follows that

$$\frac{\int_0^{l_0}\int_{l_0}^{x} M dx^2}{M_0} = a'l^2, \quad . \quad . \quad . \quad . \quad . \quad (6)$$

in which a' is some constant. Consequently

$$\Delta = \frac{a'p''}{Eh} l^2 = a_1 \frac{l^2}{h}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

It is seen, therefore, that the quantity a_1 depends upon both p'' and E, but it is ordinarily considered constant.

Since $I = Sr^2$, eqs. (1) and (7) give

$$p'' = a_1 \frac{c'Pl^2}{I} = a\frac{P}{S}\frac{l^2}{r^2}; \quad \therefore \ f = p' + p'' = \frac{P}{S}\left(1 + a\frac{l^2}{r^2}\right). \quad (8)$$

Eq. (8) shows that $a_1 c' = a$.

Hence

$$P = \frac{fS}{1 + a\frac{l^2}{r^2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

The integration by which eq. (7) is obtained, being taken between limits, causes everything to disappear which depends upon the condition of the ends of the column. Consequently eq. (9) applies to all columns, whether the ends are rounded or fixed. Let the latter condition be assumed, and let it be represented in the adjoining figure. Since the column must be bent symmetrically, there must be *at least* two points of contraflexure. Two such points only may be supposed, since such a supposition makes the distance between any two adjacent points the greatest possible and induces the most unfavorable condition of bending for the column.

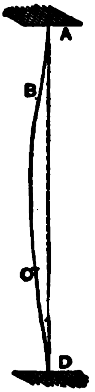


FIG. 1.

If *B* and *C* are the points of contraflexure supposed, then *BC* will be equal to a half of *AD*, for each half of *BC* must be in the same condition, so far as flexure is concerned, as either *AB* or *CD*. Also the bending moment at the section midway between *B* and *C* must be equal to that at *A* or *D*. Consequently the free- or round-end column *BC* must possess the same resistance as the fixed- or flat-end column *AD*. In eq. (9), therefore, let $l = 2BC = 2l_1$,

$$P = \frac{fS}{1 + 4a\frac{l_1^2}{r^2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Eq. (10) is, consequently, the formula for free- or round-end columns with length l_1.

The flat- or fixed-end column *AD* is also of the same resistance as the column *AC*, with one end flat and one end free or round. Hence in eq. (9) let there be put $l = \frac{4}{3}AC = \frac{4}{3}l'$, and there will result, nearly,

$$P = \frac{fS}{1 + 1.8a\frac{l'^2}{r^2}}. \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Eq. (11) is, then, the formula for a column with one end flat and the other round. A slight element of approximation will ordinarily enter eq. (11) on account of the fact that C is not found in the tangent at A just as eqs. (9) and (10) are based on the supposition that A and D lie exactly in the line of action of the imposed load.

Eqs. (10) and (11) have been and are now generally accepted as representing the resistances of columns with the end conditions to which they are intended to apply. As a matter of fact, however, tests of full-size members have demonstrated that those conditions are not realized in the actual use of columns. They have further shown that essentially but one condition of column ends need be recognized, and that is the actual pin-end condition, as realized in pin-connected structures. In that condition the end of the column is not free to turn. The compression between the pin and the metal bearing against it caused by the load carried by the column creates a considerable surface of contact over a substantial portion of which the intensity of pressure is high. This produces a condition of great frictional resistance to any motion between the pin and the end of the column, but not sufficient probably to induce a fixed-end condition. It has been found by test that flat-end columns, as a rule, give less ultimate resisting capacity than pin-end columns of the same length and same radius of gyration of cross-section. This is doubtless due to the practical impossibility to secure a central application of loading when flat ends are employed, the resulting eccentricity reducing the ultimate carrying capacity of the members. While, therefore, the classes of columns represented by eqs. (10) and (11) are still recognized, it would be more rational and more in accordance with experience to use only the general form of eq. (9) with a determined from actual pin-end tests.

In the early days of pin-connected bridge building it was thought that increased column resistance could be secured by swelling the column, i.e., by making its central cross-section of larger exterior dimension than at the ends, the actual sectional area of metal remaining the same. This meant the bending of the component parts of the column in order to attain the swelled form. It was found by actual test with the component parts usually sprung or bent cold into the desired shape, that the swelling resulted in decreased resistance in consequence of the initial bending stresses induced. The swelling of columns was, therefore, abandoned. Latterly, in some cases where special columns of great length and size are required, swelling is again resorted to, but under such circumstances that the initial bending stresses are entirely eliminated. Such swelling of form is legitimate and will give the increased resistance due to the further removal of the metal from the neutral axis of the section.

If the column should be swelled the resulting moments of inertia of section will obviously vary throughout the length, and the expression for the deflection given in eq. (5) will become

$$\Delta = \int_0^l \int_{l_0}^x \frac{p''}{Eh_0} \frac{I_0}{M_0} \frac{M}{I} dx^2. \quad . \quad . \quad . \quad . \qquad (12)$$

By using this value of the deflection, the preceding general demonstration will be in no way changed in form and eq. (9) will still represent Gordon's formula for such columns, but the quantity a would have to be determined for swelled columns if great accuracy were required.

Although the quantities f and a, in eqs. (9), (10), and (11) are usually considered constant, they are strictly variable. Eq. (7) shows that a is a function of $p'' \div E$. It is by no means certain that p'' is the same for different forms of cross-section, or even for different sections of the same form, and the coefficient of elasticity is known not to be perfectly constant. It (the latter) is known not only to vary slightly with the products of different steel mills, but even with the different products of the same mill.

Again, the greatest intensity of stress, f, which can exist in the column varies not only with different grades of material, but there is some reason to believe that it must also be considered as varying with the length of the column. The law governing this last kind of variation, for many sections, still needs empirical determination. It is clear, therefore, that both f and a must be considered empirical variables.

The expense necessarily attending experimental researches on the ultimate resistance of long columns built of structural steel or wrought iron, has prevented the attainment of many desirable results. Yet much valuable work of this kind has been done.

The earlier adaptation of the preceding formulæ to bridge or other columns took place when wrought iron was the main structural material and consequently the resulting forms of Gordon's formula were so employed in American practice as to apply to wrought iron. The latter metal has for some time been displaced by structural steel, but the great mass of full-size column tests have been made with wrought iron and a considerable number of those results must be given in this and succeeding articles. As will be indicated at another point, these formulæ for wrought iron may be adapted to structural steel, with at least a close degree of approximation, by a suitable change of the empirical quantities which enter them and whose values may be approximately deduced from the consideration of the tensile and compressive resistances of the grades of steel involved.

Among the earliest tests of full-size wrought iron columns were those made by Thomas D. Lovett, in connection with bridge construction on the Cincinnati Southern Railway. These results were reported in November, 1875, and they were used for the determinations of the quantities f and a in eqs. (9), (10), and (11) in the following manner:

If the number of experiments were sufficiently great, the results would be combined by the "Method of Least Squares." In the present instance, however, the use of the method is altogether impracticable in consequence of the small number of experiments of any given class. It will be seen, however, that the combination of the experimental results is not altogether of a random nature.

Since f and a are to be considered variable quantities, let y take the place of f and x that of a; also, let $p=\frac{P}{S}$ represent the mean intensity of stress. Eq. (9) then takes the form

$$p=\frac{y}{1+cx}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

in which $c=l^2 \div r^2$. For round- or free-end columns x will take the place of $4a$, and of $1.8a$ for columns with one end round and one end flat.

In eq. (13) there are two unknown quantities, y and x, consequently two equations are required for their determination. If two columns of different ultimate resistances per unit of section, and with different values of c, are broken in a testing machine, and the two sets of data thus established separately inserted in eq. (13), two equations will result which will be sufficient to completely give y and x. Those two equations may be written as follows:

$$y = p'(1 + c'x), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

$$y = p''(1 + c''x). \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

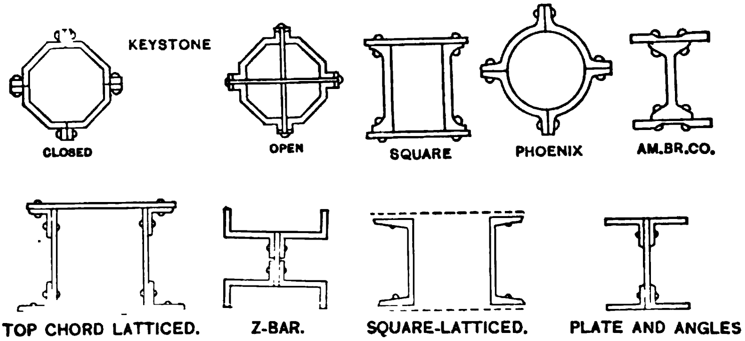
The simple elimination of y gives

$$x = \frac{p'' - p'}{c'p' - c''p''}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Either eq. (14) or (15) will then give y.

In selecting experimental results for insertion in eq. (16), care should be taken to make the differences $p'' - p'$ and $c' - c''$ as large numerically as possible, in order that the errors of experiment may form the smallest possible proportion of the first.

By employing eqs. (14), (15), and (16) in the manner indicated in connection chiefly with Lovett's tests of wrought-iron columns, the following set of formulæ, eqs. (17) to (21), inclusive, were established. They represent very closely long column practice in wrought iron when that material was used for bridge and other structures.



Keystone Columns.

Flat ends—swelled ....$p = \dfrac{36000}{1 + \dfrac{1}{18300}\dfrac{l^2}{r^2}}$; . . . (17)

Flat ends— { Open / Closed } Straight.. .....$p = \dfrac{39500}{1 + \dfrac{1}{18300}\dfrac{l^2}{r^2}}$; . . . (18)

Pin ends—swelled ....$p = \dfrac{36000}{1 + \dfrac{1}{15000}\dfrac{l^2}{r^2}}$. . . . (19)

Square Columns.

Flat ends ....$p = \dfrac{39000}{1 + \dfrac{1}{35000}\dfrac{l^2}{r^2}}$; . . . (20)

Pin ends ....$p = \dfrac{39000}{1 + \dfrac{1}{17000}\dfrac{l^2}{r^2}}$. . . . (21)

All values of x for round-end columns are found by multiplying the corresponding flat-end quantities by 4, according to eq. (10).

Eqs. (17) to (21), inclusive, give the ultimate resistances of the various classes of columns. With great variations of stress a safety factor as high as six or eight may be used, or it may be as low as three or four if the condition of stress is uniform or essentially so.

A glance at these formulæ shows that the quantity f in eqs. (9), (10), and (11) is considerably less than the ultimate resistance of wrought iron. It is in reality the average working intensity of stress on the normal cross-section of the column, and is or may be considerably less than the maximum intensity on that section existing in the extreme fibre on that side or part of the column subjected to compression by bending. This average intensity is seen to vary materially with the form of cross-section, it being substantially affected, doubtless, by the influence of the form of cross-section, those sections which are able to sustain the greatest intensity of compression without buckling giving the greatest value of f.

A consideration of these experiments would also seem to indicate that a closed column is somewhat stronger than an open one. This is undoubtedly due to the fact that the edges of the segments are mutually supporting if they are brought in contact and held so by complete closure, but not otherwise. Thus the crippling or buckling of the individual parts of the column is delayed and the ultimate resistance increased.

The general principles which govern the *resistance* of built columns may, then, be summed up as follows:

The material should be disposed as far as possible from the neutral axis of the cross-section, thereby increasing r;

There should be no initial internal stress;

The individual portions of the column should be mutually supporting;

The individual portions of the column should be so firmly secured to each other that no relative motion can take place, in order that the column may fail as a whole, thus maintaining the original value of r.

These considerations, it is to be borne in mind, affect the *resistance* of the column only; it may be advisable to sacrifice some elements of resistance, in order to attain accessibility to the interior of the compression member, for the purpose of painting. This point may be a very important one, and should never be neglected in designing compression members. It may be observed, however, that the sole object is to prevent oxidation in the interior of the column, and if the column is *perfectly* closed this object is attained. Phœnix columns which have been in the most exposed situations (in one case submerged in water at one time for several hours) during periods varying from twelve to twenty years, without the slightest oxidation in the interior of the columns, have come within the observation of the writer. Different results, however, in other cases have been found.

Art. 60.—Experiments on Phœnix Columns, Latticed Channel Columns, and Channels.

There are given in this article in full detail the results of many tests of wrought-iron columns, in spite of the fact that structural steel has completely displaced that metal. This is due to the fact that no series of tests of steel riveted columns has yet been made. A few isolated tests of the latter class are available, quite insufficient for the basis of a law or formula. Again, the general features of ultimate resistance of steel columns, including those varying with the ratio $l \div r$, are practically the same as those of wrought-iron columns. Hence, in the absence of tests of full-size steel members, it is reasonable to use for such members the formulæ for wrought iron with the empirical factors in them so increased (or decreased) as to add about 20 per cent. to the computed ultimate resistance for steel having an ultimate tensile resistance of 60,000 to 65,000 pounds per square inch, or about 25 per cent. for steel of 65,000 to 70,000 pounds ultimate tensile resistance. The following results for wrought-iron columns, therefore, possess distinct value, not only as disclosing essential features of resistance of all metal columns, but also as forming a basis of reasonable inference for formulæ applicable to steel columns.

The modern straight-line, long-column formula, eq. (3), is given as originally written. This form of formula was first proposed and used independently by Mr. Theodore Cooper, C.E., and the author at the Washington convention of the American Society of Civil Engineers in 1881. It has since found its way into extended use. It is simpler and more nearly represents the results of tests of full-size columns within the limits of use in engineering structures than Gordon's formula.

In May and July, 1873, some experiments were made at Phœnixville, Penn., on full-size Phœnix columns by the Phœnix Iron Co. The results of these experiments are given in Table I, in column headed "Experiment," while the column headed "*p*" contains the results of the following formula established by the methods of the preceding article:

$$p = \frac{42000}{1 + \frac{1}{50000}\frac{l^2}{r^2}}, \quad \text{or} \quad = \frac{42000}{1 + \frac{4}{50000}\frac{l^2}{r^2}}, \quad . \quad . \quad (1)$$

according as the ends are "flat" or "round." All columns are "4-segment" ones.

TABLE I.

Date.	Ends.	Area.	Length.	r^2.	l^2+r^2.	Experiment	p.
		Sq. Ins.	Feet.			Lbs.	Lbs.
May 3, 1873...	Flat...	5.84	23.81	4.10	19,950	30,274	30,000
May 3, 1873...	Round.	5.95	24.00	4.10	20,230	16,387	16,040
May 3, 1873...	Flat...	10.21	23.38	8.68	9,065	36,419	35,600
May 3, 1873...	Flat...	8.50	22.71	8.00	9,282	38,235	35,430
July 19, 1873..	Flat...	13.31	23.20	8.47	9,151	32,742	35,500
July 19, 1873..	Flat...	12.85	23.20	8.47	9,151	35,408	35,500

In applying the formula the length was reduced to inches, in order to bring it to the same unit as that in which the radius of gyration, r, is expressed.

The columns "Experiment" and "p" are each, of course, per square inch.

It is seen that the experimental results, and those by Gordon's formula, give a very close and satisfactory agreement. It is also seen that the analytical relation between flat and round ends is partially confirmed.

The square of the radius of gyration, 4.10, was taken the same for the first and second columns because their normal sectional areas are so nearly the same. The value 4.10 belongs to a 4-segment column, whose area is 5.88 square inches.

The same observation applies to the last two columns. The value 8.47 belongs to a column whose area of cross-section is 13.08 square inches.

A most valuable and instructive set of experiments on Phœnix columns was also made in the large testing machine at the U. S. arsenal at Watertown, Mass., under the direction of Messrs. Clark, Reeves & Co., the results of which were presented to the American Society of Civil Engineers at the thirteenth annual convention, June 15, 1881. The value of these experiments is enhanced by the fact that they were made on full-size columns, such in reality as are used in ordinary bridge construction.

In the following table are given the results of these experiments, as well as those of several formulæ presently to be explained.

The following is a portion of the notation:

l = length in inches;

r = radius of gyration in inches;

$E. L.$ = elastic limit in pounds per square inch;

$Exp.$ = ultimate resistance in pounds per square inch.

In determining r^2 for Nos. 1 to 20, inclusive, a column whose area of cross-section was 12.23 square inches was taken. The areas of the actual cross-sections varied so little from this quantity that the resulting value of r^2 was assumed to belong to all of the first 20 columns. All the columns were tested with flat ends.

TABLE II.

No.	Length.	Area.	r^2.	$l \div r$.	$l^2 \div r^2$.	*E. L.*	*Exp.*	p_1.	p'.	p''.
	Feet.	Sq. In.	Ins.			Lbs.	Lbs.	Lbs.	Lbs.	Lbs.
1	28	12.062	8.94	112	12,544	——	35,150	32,550	34,488	——
2	28	12.181	8.94	112	12,544	——	34,150	32,550	34,488	——
3	25	12.233	8.94	100	10,000	27,960	35,270	34,000	35,040	——
4	25	12.100	8.94	100	10,000	——	35,040	34,000	35,040	——
5	22	12.371	8.94	88	7,744	——	35,570	35,420	35,592	——
6	22	12.311	8.94	88	7,744	——	34,360	35,420	35,592	——
7	19	12.023	8.94	76	5,776	——	35,365	36,800	36,144	——
8	19	12.087	8.94	76	5,776	29,290	36,900	36,800	36,144	——
9	16	12.000	8.94	64	4,096	——	36,580	38,130	36,696	——
10	16	12.000	8.94	64	4,096	——	36,580	38,130	36,696	——
11	13	12.185	8.94	52	2,704	28,890	36,857	39,400	37,248	——
12	13	12.069	8.94	52	2,704	——	37,200	39,400	37,248	——
13	10	12.248	8.94	40	1,600	26,940	36,480	40,700	37,800	——
14	10	12.339	8.94	40	1,600	28,360	36,397	40,700	37,800	——
15	7	12.265	8.94	28	784	29,350	38,157	42,200	38,352	40,360
16	7	11.962	8.94	28	784	29,590	43,300	42,200	38,352	40,360
17	4	12.081	8.94	16	256	——	49,500	44,770	——	46,300
18	4	12.119	8.94	16	256	28,050	51,240	44,770	——	46,300
19	8 ins.	11.903	8.94	2.7	7.29	——	57,130	69,600	——	57,140
20	8 ins.	11.903	8.94	2.7	7.29	——	57,300	69,600	——	57,140
21	25′ 2.65″	18.300	19.37	68.8	4,733	——	36,010	37,600	36,666	——
22	8′ 9″	18.300	19.37	24	576	29,510	42,180	42,840	——	42,160

An application of eq. (1) to these columns reveals considerable discrepancies between the results of that formula and the quantities given in the column "*Exp.*" of the table, when the values of $l \div r$ become comparatively small, as was anticipated in the preceding article. Instead of the constant 42,000 in the numerator of Gordon's formula, these experiments show that a variable quantity must be used which shall increase as $l \div r$ decreases, or as $r \div l$ increases.

After several trials it was found that the following modified form of Gordon's formula would give reasonable results throughout the entire range of the experiments:

$$p_1 = \frac{40000\left(1 + \frac{2r}{l}\right)}{1 + \frac{1}{50000} \cdot \frac{l^2}{r^2}} \cdot \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The results of eq. (2) are given in the column of the table headed p_1. The agreement between the two columns is not as close as could be desired, yet the discrepancies are not sufficiently great to vitiate the safe use of the formula.

In the following figure the Watertown experiments, as well as those of Mr. Bouscaren and the Phœnix Iron Co. (given in this and the preceding article), are shown by

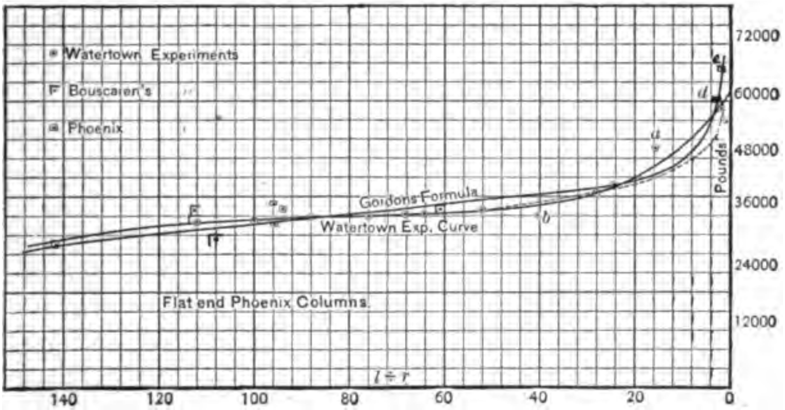


diagram. The different classes of experiments are indicated as shown. The experimental curve is drawn with particular reference to the Watertown experiments, for it is then found to be properly located in reference to the others. The other curve expresses Gordon's formula according to eq. (2). It would not be difficult to find an equation which would fit the experimental curve very closely throughout the range of the experiments, but it would not be as simple as eq. (2), or as two others to be shortly given.

It is important to observe that each experimental value in the diagram (which is a mean of two, belonging to columns of the same length, in the table) lies on or exceedingly close to the curve, with the exceptions of those shown at *a* and *b*. *a* corresponds to a mean of Nos. 17 and 18, and is abnormally high; *b* shows the mean of Nos. 13 and 14, and is abnormally low.

It may be observed that the experimental curve is nearly a straight line from a point just above *b* to the extreme left of the diagram. For that portion of the curve, therefore, the following formula applies very closely:

$$p' = 39{,}640 - 46\frac{l}{r}. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The results of this formula are given in the column headed "p'." The table, in connection with the diagram, shows that this formula may be used with accuracy for values of $l \div r$ lying between 30 and 140, and further experiments may possibly show that it is applicable above the latter limit.

For values of $l \div r$ less than 30, the following formula will be found to give results approximating very closely to the experimental curve:

$$p'' = 64{,}700 - 4{,}600\sqrt{\frac{l}{r}}. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The results of the application of this formula are given in the column headed "p''."

The extreme simplicity of eqs. (3) and (4) makes it a matter of great interest to determine, by other experiments covering extended ranges of $l \div r$, whether those forms, with different constants, may not apply to shapes other than that of the Phœnix column.

The inapplicability of the true long-column formulæ, when $\frac{l}{r}$ is found below certain limits, which is shown in Art. 24, furnishes a proper foundation for thoroughly empirical formulæ, such as those expressed in eqs. (3) and (4).

By eq. (4), the ultimate resistance of Phœnix wrought iron to pure compression would be about 60,000 pounds per square inch.

The results of the application of eqs. (3) and (4) to Bouscaren's and the Phœnix experiments are not given, but the diagram shows clearly that they would be satisfactory. Data sufficient for the application are given in this and the preceding article.

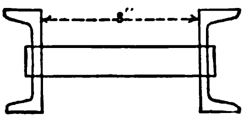
The following is the record of the Phœnix tests of the very short columns shown at *c*, *d*, and *e* in the diagram. It is a question whether the degree of distortion which accompanied the extremely high result of 65,867 pounds per square inch was not considerably greater than that which would characterize the condition of "failure" in an actual structure. This important point cannot receive too

No.	l.	Area.	r^2.	$l+r$.	l^2+r^2.	*Exp.*	p_1.	p''.
	Ins.	Sq. Ins.						
1	8	6.98	4.11	3.95	15.6	60,573	51,500	55,500
2	8	6.98	4.11	3.95	15.6	60,387	51,500	55,500
3	4	5.63	2.37	2.6	6.76	65,867	55,800	57,300
4	4	5.63	2.37	2.6	6.76	65,867	55,800	57,300
5	4	2.93	2.25	2.67	7.13	56,889	55,500	57,200
6	4	2.93	2.25	2.67	7.13	55,555	55,500	57,200

much attention in connection with short-column tests, where the *relative* distortion, in the condition of "failure," is far greater than that in long columns.

Latticed Columns and Channels.

During 1880 and 1881 Col. T. T. S. Laidley, U.S.A., tested a large number of long columns composed of two channel bars latticed in the ordinary manner (Ex. Doc. No. 12, 47th Cong., 1st Session). These columns were furnished with $3\frac{1}{2}$-inch pin ends, and were tested at Watertown, Mass., in the large government machine. The adjoining figure shows the relative positions of the channels and pin. 6″, 8″, 10″, and 12″ ⊏s were employed, and all the columns, the results of whose tests are given in Table III, preserved the uniform distance of 8 inches between the channels.



The radius of gyration, r, of the cross-section, given in that Table, is in reference to the axis of the pin.

All the posts were single latticed, and the pitch of the latticing (the distance apart of rivets in the same flange of a ⊏) was 18 inches for the 6- and 8-inch channels, and 22 inches for the 10- and 12-inch. $2'' \times \frac{1}{4}''$ latticing was used for the 6-inch ⊏s: $2'' \times \frac{3}{8}''$ for the 8- and 10-inch, and $2\frac{1}{4}'' \times \frac{3}{8}''$ for the 12-inch.

The area of cross-section for the ⊏s of the same depth in different columns varied slightly, consequently about an average area was taken.

All these columns failed as *wholes*, and each result is a mean of two. Other columns of the same set, and tested at the same time, failed by buckling of the channels; they cannot, consequently, be classed among long columns which are so constructed as to fail as *wholes*.

The values of p in Table III are shown graphically in Plate I. The ratio $l \div r$ is laid off along the horizontal line and the ultimate intensity p on the vertical line, as shown. The full curved line is then the experimental curve and possesses great value of a practical nature. Within the limits of the diagram, when the ratio

$$l \div r$$

is known, the ultimate resistance of the column per square inch (p) can be at once accurately read from the plate without calculation or scale.

The following equation,

$$p = \frac{39000}{1 + \frac{1}{30000}\frac{l^2}{r^2}}, \quad (5)$$

probably gives as accurate results as any form of Gordon's

Plate I.

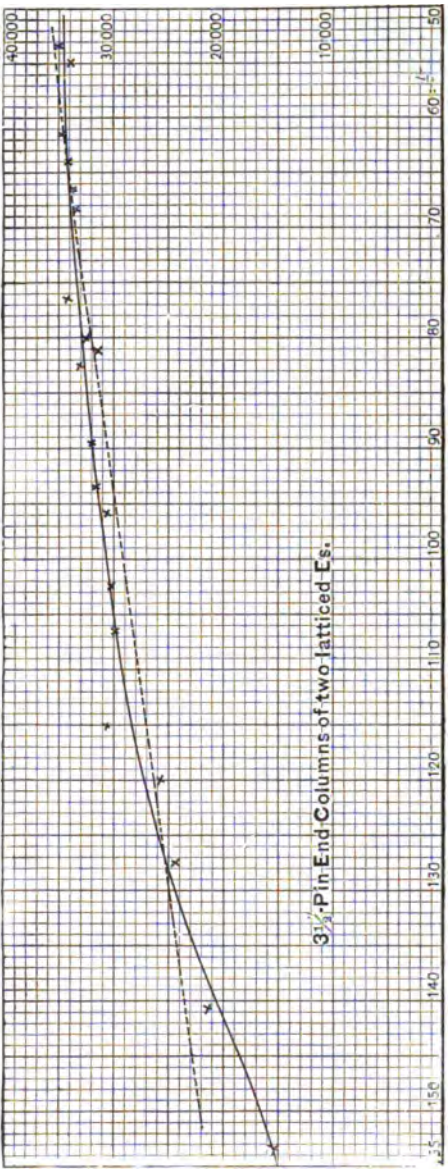


TABLE III.

PIN ENDS.—3½″ PIN.

No.	[.	Area of Section in Sq. Inches (2 [s).	Length in Inches.	Radius of Gyration, Ins.	Length over Radius; or $l+r$.	p.
	Inches.					Pounds.
1	8	7.65	160	3.00	53.3	35,025
2	10	9.70	200	3.65	54.8	33,920
3	6	4.65	144	2.35	61.3	34,450
4	6	4.65	150	2.35	63.9	34,130
5	8	7.65	200	3.00	66.7	33,790
6	10	9.70	250	3.65	68.5	33,770
7	6	4.65	180	2.35	76.7	34,180
8	8	7.65	240	3.00	80.0	32,375
9	12	12.00	360	4.44	81.0	31,475
10	10	9.70	300	3.65	82.2	33,015
11	6	4.65	210	2.35	89.5	31,935
12	8	7.65	280	3.00	93.3	31,800
13	10	9.70	350	3.65	95.9	30,780
14	6	4.65	240	2.35	102.2	30,085
15	8	7.65	320	3.00	106.7	29,600
16	6	4.65	270	2.35	115.0	30,820
17	8	7.65	360	3.00	120.0	25,885
18	6	4.65	300	2.35	127.8	24,355
19	6	4.65	330	2.35	140.6	21,330
20	6	4.65	360	2.35	153.4	15,320

"p" is the ultimate resistance per square inch, in pounds.

formula. The dotted curve is constructed from it. Its results are seen to be only tolerably approximate between the limits $\frac{l}{r} = 50$ and 135. It possesses little value when compared with the plate.

Table IV contains results for columns of the same set which failed by buckling of the individual channels of which they were composed.

If r' is the radius of gyration in reference to an axis through the centre of gravity of a single channel section, and *parallel to the web*, the following values will hold for the present cases:

6″ [; $r' = 0.58$ inch.
8″ [; $r' = 0.48$ inch.
10″ [; $r' = 0.69$ inch.
12″ [; $r' = 0.87$ inch.

TABLE IV.

No.	[.	Length, Inches, l.	Radius of Gyration in Inches, r.	$\frac{l}{r}$.	Ultimate p in Pounds per Sq. Inch.	Condition of Ends.
	Inches.					
1	6	120	2.35	51.1	36,025	Flat.
2	6	120	2.35	51.1	33,740	One flat; one pin.
3	10	126	3.65	34.5	35,450	Pin.
4	12	120	4.44	27.0	34,245	Pin.
5	12	180	4.44	40.5	34,660	Pin.
6	12	240	4.44	54.0	33,985	Pin.
7	12	300	4.44	67.5	33,590	Pin.

Although the lattice rivets were alternate in the same channel, each flange was unsupported for a distance equal to the pitch, i.e., 18″ for the 6″ and 8″ [s, and 22″ for the 10″ and 12″ [s. Hence

For 6″ [; $18 \div r' = 31.0.$
For 8″ [; $18 \div r' = 37.4.$
For 10″ [; $22 \div r' = 31.9.$
For 12″ [; $22 \div r' = 25.3.$

Table IV shows that the column of 10″ [s commenced to fail by buckling of the [s when

$$l \div r = 34.5,$$

and also when

$$22 \div r' = 31.9;$$

that the column of 12″ [s commenced to fail similarly when the length became so small that

$$l \div r = 27.0,$$

while

$$22 \div r' = 25.3.$$

These results would seem to show that pin-end columns with single but alternate latticing will begin to fail by buckling of the channels, when $l \div r$, for the column as a whole, becomes so small that it is about equal to the same ratio for a single channel between two adjacent rivets in the same flange.

Nos. 1 and 2 of Table IV show that if the ends possess a greater degree of fixedness, the value of $l \div r$ is much greater when buckling begins to take place, but the number of experiments is not sufficient to indicate the exact amount.

As would be anticipated under the circumstances, p maintains about the same value for all the columns in Table IV. Hence when $l \div r$ becomes so small that buckling takes place, the ultimate resistance of the column is independent of the length.

The graphical representations of the results given in this article show that the curve of ultimate resistances has a very sharp declivity for small values of $l \div r$, but that it becomes nearly straight and horizontal for larger values, and that it again increases in declivity with a still farther increase in that ratio. These phenomena seem to be much more pronounced in the tubular variety of columns. They find a simple and obvious explanation in the fact that in columns of moderate length the deflection at the centre of the column about keeps pace (in the same direction) with the movement of the centre of pressure at the ends.

Plate I shows (what was to be anticipated) that this effect is also much less pronounced with pin ends than with flat ones, it being borne in mind that the phenomena here considered do not produce the horizontal straight line which would be seen if Plate I included less values of $l \div r$ than 50. The latter represents the buckling of the individual parts of the column, and not the failure of the column as a whole.

A few experiments by Col. Laidley with columns of the same ⊏s as the above, but with pins only three inches in diameter, gave uniformly less ultimate resistance than those with three and a half inch pins. Although this result was to be expected, the number of experiments was not sufficient to justify any quantitative conclusions; it can only be stated that the smaller the pin the less will be the ultimate resistance.

TABLE V.

FLAT-END [S.

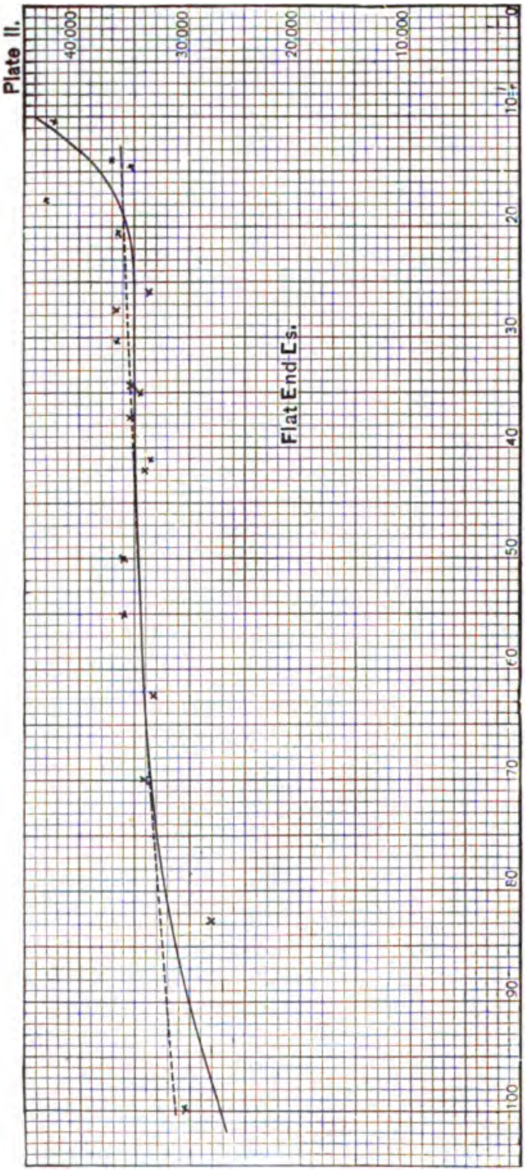
No.	[.	Area of Section in Sq. Inches.	l.	r'.	$\frac{l}{r'}$.	Ultimate Resistance, in Lbs. per Sq Inch, $= p$.
	Inches.		Inches.	Inches.		
1	6	2.33	6.00	0.58	10.3	42,293
2	6	2.33	17.60	0.58	30.3	36,835
3	6	2.33	23.90	0.58	41.1	33,910
4	6	2.33	48.00	0.58	82.6	28,140
5	8	3.80	8.00	0.48	16.6	43,295
6	8	3.80	17.90	0.48	37.2	35,280
7	8	3.80	23.90	0.48	49.7	35,975
8	8	3.80	29.90	0.48	62.2	33,400
9	8	3.80	48.00	0.48	99.8	30,620
10	10	4.85	10.00	0.69	14.5	35,080
11	10	4.85	17.90	0.69	26.0	33,820
12	10	4.85	23.90	0.69	34.7	34,355
13	10	4.85	29.90	0.69	43.4	34,050
14	10	4.85	48.00	0.69	69.6	34,080
15	12	6.00	12.00	0.87	13.8	37,240
16	12	6.00	17.80	0.87	20.5	36,590
17	12	6.00	23.90	0.87	27.5	36,695
18	12	6.00	29.90	0.87	34.4	35,150
19	12	6.00	48.00	0.87	55.2	36,040

Table V contains the results of Col. Laidley's tests of portions of the [s used in the columns which have just been treated. These portions had flat ends.

The moment of inertia of the section, from which the radius of gyration r' was computed, was taken about an axis parallel to the web of the channel and passing through its centre of gravity.

Many of the results are means of two tests each.

The results given in Table V are shown graphically in Plate II. The values of the ratio $l \div r$ are laid off on the horizontal base line to the left from O, while the values of p in pounds per square inch are laid off vertically from O,



as shown. The full curve then represents with great accuracy the experimental results.

The dotted curve represents the following form of Gordon's formula for the ultimate resistance in pounds per square inch:

$$p = \frac{36000}{1 + \frac{1}{63000} \frac{l^2}{r'^2}} \quad (6)$$

This formula is sufficiently accurate for all ordinary purposes between the limits

$$l \div r' = 15$$

and

$$l \div r' = 90,$$

but does not compare in value with the experimental (full) curve.

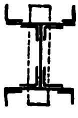
Addendum to Art. 60.

TESTS OF FULL-SIZE WROUGHT-IRON COLUMNS—TESTS OF STEEL ANGLES, TEES AND OTHER FORMS OF COLUMNS.

The observations made at the beginning of Art. 60 are applicable to the results of tests of wrought-iron columns found in this Addendum, while the results of tests of steel struts by Mr. James Christie are of exceptional value in the most advanced structural practice.

Since the issue of the first edition of this book, the series of tests of full-sized columns, of which Table III gives the results of the first 20, has been continued at Watertown, Mass., and the test records are given in "Ex. Doc. No. 5, Senate, 48th Congress, 1st Session," and "Ex. Doc. No. 35, Senate, 49th Congress, 1st Session." Table VI shows the digested records put in shape to be of some value to engineers. These columns had 3½-inch pin ends, and the results belong to failures in the plane normal to the pin-axes.

Columns 21, 22, 23, 24, 25, 26, and 37 to 48 inclusive were closed box columns composed of two channels and two plates; the remaining columns, except those of the Wilson section, were composed of two channels latticed together in the usual manner. The word "built" in the Table indicates that the channels were built of plates and angles; otherwise they were rolled. The Wilson column is that formerly used so frequently by Jas. M. Wilson, C.E., while Eng'r of Bridges and Buildings, Penna R. R. It has the section shown by the sketch in the margin. In these columns the pin was always placed parallel to the plate between the channels, i.e., normal to the webs of the channels and as shown by the broken lines.



In columns 25, 26, 39, 40, 41, 42, 47, and 48, the pins were placed through (i.e., normal to) the webs of the channels, as shown in the figure on page 486; in all the other channel columns the pins were placed parallel to the webs of the channels.

The results given in Tables III and VI are shown graphically on Plate A. All results are brought together

TABLE VI.

3½-INCH PIN-END COLUMNS.

No.	⊏ Inches.	Section, Sq. Ins.	Length in Inches.	Radius of Gyration, Inches.	Length over Radius, $l+r$.	Ultimate in Lbs. per Sq. In., p.	Remarks.
1	8	7.6	160	4.5	36	33,910	
2	8	8.1	160	4.5	36	36,580	
3	8	7.6	160	5.23	31	34,340	Swelled.
4	8	7.6	160	5.23	31	33,530	"
5	10	11.9	200	4.6	44	33,740	
6	10	12.3	200	4.6	44	34,670	
7	10	12.4	200	5.98	34	31,130	Swelled.
8	10	12.7	200	5.98	34	31,990	"
9	8	7.5	240	5.23	46	33,390	"
10	8	7.5	240	5.23	46	34,390	"
11	8	7.6	240	4.5	53	34,120	
12	8	7.6	240	4.5	53	33,410	
13	10	12.1	300	4.6	65	33,630	
14	10	12.2	300	4.6	65	32,440	
15	10	11.9	300	5.98	50	32,830	Swelled.
16	10	11.9	300	5.98	50	32,740	"
17	8	7.7	320	4.5	71	31,610	
18	8	7.7	320	4.5	71	29,870	
19	8	7.7	320	5.23	61	30,840	Swelled.
20	8	7.7	320	5.23	61	30,770	"
21	8	16.2	320	3.78	85	28,020	Built.
22	8	16.3	320	3.78	85	27,910	"
23	8	21.0	320	3.8	84	25,770	"
24	8	20.6	320	3.8	84	25,950	"
25	8	17.9	320	2.7	119	26,480	
26	8	19.4	320	2.7	119	25,290	
27	6	9.8	120	1.87	64	30,220	Wilson column.
28	6	10.2	120	1.87	64	31,380	
29	6	10.0	180	1.87	96	25,160	
30	6	10.0	180	1.87	96	21,050	
31	8	16.1	240	2.44	98	26,430	
32	8	16.3	240	2.44	98	22,540	
33	6	9.7	240	1.87	128	19,380	
34	6	9.8	240	1.87	128	16,220	
35	8	16.2	320	2.44	131	19,700	
36	8	16.1	320	2.44	131	17,570	
37	5.6	9.21	180	2.9	62	31,650	Built.
38	5.6	9.44	180	2.9	62	30,720	"
39	6.0	11.42	180	2.73	66	33,205	
40	6.0	11.42	180	2.73	66	32,329	
41	8.0	17.8	240	3.6	67	32,077	
42	8.0	17.2	240	3.6	67	32,253	
43	8.0	12.65	240	3.6	76	37,668	
44	8.0	12.76	240	3.6	76	30,596	
45	5.6	9.24	240	2.9	83	28,950	Built.
46	5.6	9.36	240	2.9	83	29,879	"
47	6.0	11.42	240	2.73	88	29,947	
48	6.0	11.31	240	2.73	88	29,186	
49	8.0	15.34	160	4.2	38	30,965	Wilson column.
50	8.0	15.40	160	2.5	64	31,494	

TABLE VI*a*.
FLAT-END CHANNEL COLUMNS.

No.	[Ins.	Section, Sq. Ins.	Length in Inches.	Radius of Gyration, Inches.	Length over Radius, $l \div r$.	Ultimate in Lbs. per Sq. In., p.	Remarks.
1	8	17.0	168	3.6	47	34,950	
2	8	17.8	168	3.6	47	35,595	
3	7.2	21.0	168	3.26	51.5	33,682	Built channels.
4	7.2	21.5	168	3.26	51.5	33,061	
5	7.5	15.7	248	3.32	75	33,003	
6	7.5	15.6	248	3.32	75	34,505	
7	7.2	21.2	248	3.26	76	33,019	
8	7.2	21.5	248	3.26	76	33,943	
9	10	17	308	4.02	76.6	34,279	These columns latticed on one side. All above are complete box columns.
10	10	17.4	308	4.02	76.6	33,333	
11	8	12.6	248	3.16	78.5	32,666	
12	8	12.7	248	3.16	78.5	33,862	
13	6	4.8	121	2.3	53	36,720	Rolled channels latticed both sides.
14	6	4.7	121	2.3	53	35,330	

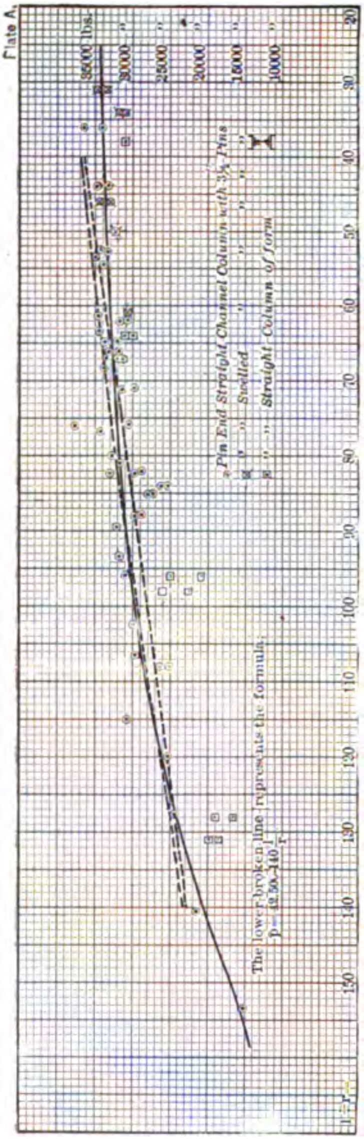
The results given in this table are a digest of the records of flat-end column tests given in "Senate Ex. Doc. No. 35, 49th Congress, 1st Session." Within the limits of these tests, a comparison with the 3½-inch pin-end results shows that the difference in the end conditions exerts no influence on the ultimate compressive resistance per square inch up to at least $l \div r = 80$. The number of tests in this table is quite insufficient to establish any law between pin- and flat-end columns of this character. Hence no diagram is drawn or formula given.

on one plate in order to obtain the most probable curve for ordinary wrought-iron columns with 3½-inch pin ends.

The various kinds of columns covered by these experiments are seen to possess about the same resistance, except the Wilson column, which falls from 10 to 20 per cent. below the others. This is due to the fact that in this section the greater portion of the metal is but slightly supported.

The full line on the Plate is drawn as a mean of the channel columns only, and is of great practical value. The upper broken line expresses eq. (5) of page 487, which is probably as good a pin-end formula for channel columns as can be devised.

Results of experiments will be given below which show that the resistance per square inch of a pin-end column increases with the diameter of the pin, and inasmuch as pins ordinarily used in columns of the dimensions of those tested usually considerably exceed 3½ inches, the mean value of these tests may probably be a little too low for ordinary bridge practice.



Tables VII and VIII contain the mean of a large number of most valuable tests of full-size iron and steel angle, tee, channel, and beam struts, with the various end conditions indicated, by James Christie, Esq., Supt. of the Pencoyd Iron Co. The detailed account of this complete series of tests should be carefully consulted; it may be found in the "Trans. of the Am. Soc. of C. E.," Vol. XIII, 1884. All sizes of angles and tees up to 4 inches by 4 inches by ⅜ inch and over 15 feet in length were used in these tests. The "hinged ends" were either one-inch or two-inch pins in semi-cylindrical bearings or one-inch or two-inch balls in sockets. The "round ends" were the above-described balls resting on flat or plane surfaces.

The "flat ends" were se-

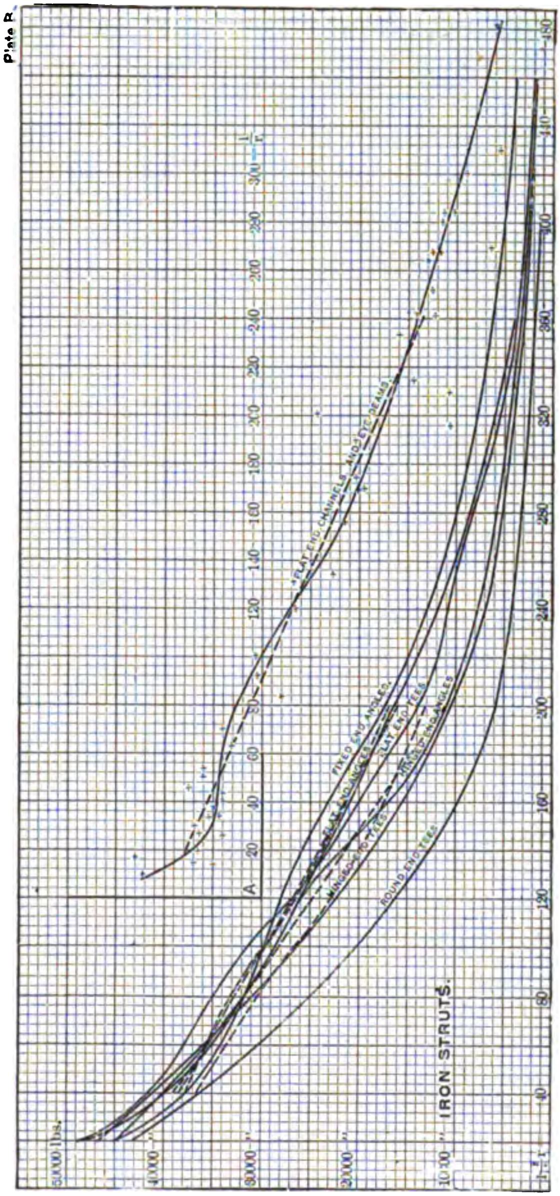


Plate B
IRON STRUTS.
FLAT END CHANNELS AND EYE BEAMS.
FIXED END ANGLES.
FLAT END ANGLES
FLAT END TEES
HINGED END ANGLES
HINGED END TEES
ROUND END TEES
A
50000 lbs.
40000 "
30000 "
20000 "
10000 "
$\frac{l}{r}$ =
$300 = \frac{l}{r}$
20
40
60
80
120
140
160
180
200
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360
380
400
420
440
460
480

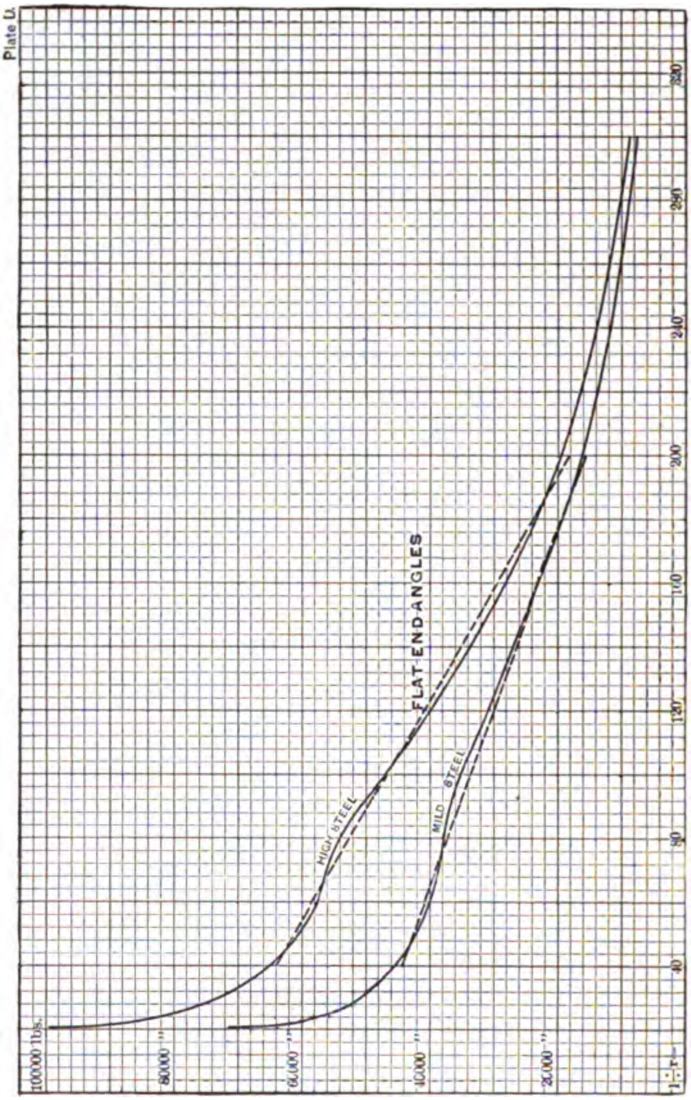


Plate U.
FLAT-END-ANGLES
HIGH STEEL
MILD STEEL
100000 lbs.
80000 "
60,000 "
40,000 "
20,000 "
l÷r
40
80
120
160
200
240
280
320

cured by simply resting the carefully faced ends of the struts on the plane bearing surfaces of the testing machine, while the "fixed" strut "ends" were obtained by clamping the ends of the struts rigidly to those bearing surfaces.

Every imaginable means was taken by Mr. Christie to secure the utmost accuracy in all details of these tests.

In Tables VII and VIII, l is the length of strut and r the *least* radius of gyration of its normal cross-section. In order to get the least radius of the angle sections, the moment of inertia was taken about an axis through the centre of gravity of the cross-section and parallel to a line through the extremities of the legs.

TABLE VII.

MEAN RESULTS OF WROUGHT-IRON STRUT TESTS.

$\frac{l}{r}$.	Flat-end Angles.	Hinged-end Angles.	Fixed-end Angles.	Flat-end Tees.	Hinged-end Tees.	Round-end Tees.	Flat-end Channels and Beams.
20	49,000	45,000	45,000	49,000	47,000	44,000	38,000
40	40,000	40,000	38,000	42,000	41,000	36,500	35,000
60	35,000	36,000	34,000	38,000	36,000	30,500	34,000
80	32,000	32,000	32,000	35,000	31,000	25,000	31,500
100	29,000	29,000	30,000	31,500	27,000	20,500	29,000
120	26,000	26,000	28,000	27,000	22,500	16,500	26,000
140	23,500	22,000	25,500	23,000	18,500	12,800	24,000
160	21,000	17,000	23,000	20,000	15,500	9,500	21,000
180	19,000	13,000	20,000	17,000	12,500	7,500	18,000
200	16,500	11,000	17,500	14,000	10,500	6,000	15,000
220	14,000	9,000	15,000	12,000	8,500	5,000	12,500
240	12,000	8,000	13,000	11,000	7,000	4,300	11,000
260	10,500	7,000	11,000	10,000	6,000	3,800	10,000
280	9,000	6,000	10,000	8,500	5,500	3,200	9,000
300	7,500	5,000	9,000	7,000	5,000	2,800	7,500
320	6,000	4,500	8,000	5,500	4,500	2,500	6,000
340	4,800	4,000	7,000	4,500	4,000	2,100	5,000
360	3,800	3,500	6,500	4,000	3,500	1,900	4,000
380	3,200	3,000	5,800	3,500	3,000	1,700	——
400	2,900	2,500	5,200	3,000	2,500	1,500	——
420	2,500	2,300	4,800	2,500	2,200	1,300	——
440	2,200	2,100	4,300	——	——	——	——
460	2,000	1,900	3,800	——	——	——	——
480	1,900	1,700	——	——	——	——	——

All results in Table VII belong to wrought-iron struts, while Table VIII belongs to struts of Bessemer steel. The "mild" steel contained from 0.11 to 0.15 per cent. carbon, but 0.36 per cent. carbon was found in the "high steel." The ultimate tensile resistance of the former varied from 60,000 to 66,000 pounds per square inch with 26 to 24 per cent. stretch in 8 inches, while the high steel possessed an ultimate tensile resistance of about 100,000 pounds per square inch and a stretch of about 16 per cent. in 8 inches.

TABLE VIII.

FLAT-END STEEL ANGLE STRUTS.

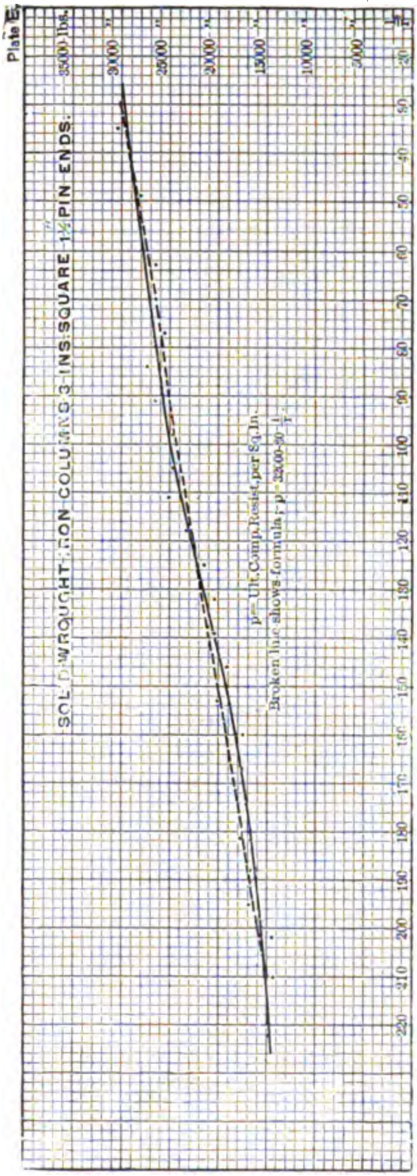
$\frac{l}{r}$.	Ultimate Resistance, Pounds per Square Inch.		$\frac{l}{r}$.	Ultimate Resistance, Pounds per Square Inch.	
	Mild Steel.	High Steel.		Mild Steel.	High Steel.
20	72,000	100,000	170	21,000	26,000
30	51,000	74,000	180	19,500	23,800
40	46,000	65,000	190	18,000	21,800
50	43,000	61,000	200	16,500	20,000
60	41,000	58,000	210	15,200	18,400
70	39,000	56,000	220	14,000	16.900
80	38,000	54,000	230	13,000	15,400
90	36,500	51,000	240	12,000	14,000
100	35,000	47,000	250	11,100	12,800
110	33,500	43,500	260	10,300	11,800
120	31,500	40,000	270	9,600	11,000
130	29,000	36,500	280	9,000	10,200
140	27,000	33,500	290	8,400	9,500
150	25,000	30,800	300	7,900	9,000
160	23,000	28,300			

It is to be observed that up to 80 radii of gyration the resistance of the fixed-end angles falls below that of both the hinged- and flat-end struts, but beyond that limit it exceeds them both until it reaches over double their values at and about 400 radii of gyration.

The flat- and hinged-end conditions approach each other in their resistances until they become nearly equal at the highest values of $l \div r$.

Plates B and D represent graphically the results given in Tables VII, VIII, and IX; Plate D being devoted wholly to Table VIII. The curve for flat-end channels and beams has been moved to the right in order to separate it from the others. This curve includes not only Mr. Christie's data, but that of Table V and results of later tests found in "Senate Ex. Doc. No. 1, 47th Congress, 2d Session," and given in Table IX.

Plates B and D and the preceding Tables show that at and above 200 radii of gyration the iron and mild steel angle struts possess the same ultimate resistance per square inch. The iron and high steel continually approach each other, and undoubtedly become equal in unit resistance at a length a little above 300 radii of gyration. This is due to the fact that the coefficients of



elasticity for the three metals are essentially identical, since it has been shown in Art. 24 that long-column resistance varies directly with the coefficient of elasticity.

TABLE IX.

FLAT-END I-BEAM STRUTS.

No.	Beam.	Area of Section, Square Inches.	Length in Inches.	r Least Radius Gyration, Inches.	$\frac{l}{r}$.	Ultimate Resistance in Lbs. per Sq. Inch = p.
1	6″	4.18	120	0.6	200	24,210
2	7″	6.05	180	0.75	240	13,990
3	8″	6.65	192	0.8	241	12,540
4	8″	6.59	193	0.8	242	14,000
5	9″	14.4	57	1.24	46	37,850
6	9″	6.85	192	0.72	267	12,460
7	9″	7.15	192	0.72	267	11,920
8	10½″	10.26	155	0.92	169	20,170
9	10½″	9.3	216	0.92	234	16,020(?)
10	10½″	10.19	264	0.93	284	11,100
11	10½″	10.46	264	0.93	284	10,300
12	15″	14.8	264	1.00	264	12,400
13	15″	14.74	264	1.00	264	12,690

Table VII also shows that, from 40 to 120 radii of gyration, the resistance per square inch of hinged- and flat-end angle struts are identical.

Although no formulæ can be found that will exactly fit the curves of Plates B and D, those of the form of eq. (3), on page 484, most nearly accomplish that result. Inasmuch as it is generally impossible, in engineering design, to separate the conditions of flat and fixed ends, one formula only is given for these two conditions, the influence of the former predominating. Round-end members are seldom or never found in engineering structures, hence a formula is given for pin- or hinged-end angles and tees. If round-end members should be used, the Table and Plate will show how much the pin-end resistance must be reduced for a given value of $l \div r$, in order to get the round-end resistance.

The straight broken lines on Plates B and D represent the following formulæ:*

Flat- and fixed-end iron angles and tees:

$$p = 44{,}000 - 140\frac{l}{r}. \qquad (7)$$

Hinged-end iron angles and tees:

$$p = 46{,}000 - 175\frac{l}{r}. \qquad (8)$$

Eqs. (7) and (8) are to be used only between the limits of $l \div r = 40$ and $l \div r = 200$.

Flat-end iron channels and I beams:

$$p = 40{,}000 - 110\frac{l}{r}. \qquad (9)$$

Eq. (9) is to be used only between the limits of $l \div r = 20$ and $l \div r = 240$.

Flat-end mild steel angles: Use for any shape of column –

$$p = 52{,}000 - 180\frac{l}{r}. \qquad (10)$$ [illegible]

---

* Although the above formulæ possess great advantages, both in accuracy and simplicity, over the old Gordon or Tredgold forms, it is not amiss to state that the curved broken lines on Plate B represent the following formulæ:

Flat- and fixed-end iron angles and tees:

$$p = \frac{40000}{1 + \frac{l^2}{r^2}\frac{1}{30000}}. \qquad (a)$$

Hinged-end iron angles and tees:

$$p = \frac{40000}{1 + \frac{l^2}{r^2}\frac{1}{20000}}. \qquad (b)$$

These formulæ can be used with fairly good results between the limits of $l \div r = 40$ and $l \div r = 180$. They are given simply in deference to an old usage, with the decided opinion that they should be abandoned.

Flat-end high steel angles:

$$p=76,000-290\frac{l}{r}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Eqs. (10) and (11) are to be used only between the limits of $l\div r=40$ and $l\div r=200$.

Table X.

SOLID 3-INCH SQUARE COLUMNS—1½-INCH PIN ENDS.

Length, Inches.	$\frac{l}{d}$.	$\frac{l}{r}$.	Ultimate Compressive Resistance, Lbs. per Sq. In.	Length, Inches.	$\frac{l}{d}$.	$\frac{l}{r}$.	Ultimate Compressive Resistance, Lbs. per Sq. In.
30	10	35	30,125	137.6	46	160	17,780
42	14	49	28,160	143.8	48	167	17,600
54	18	63	26,515	149.8	50	174	17,180
60	20	70	26,475	155.7	52	181	17,670
66	22	77	25,430	161.8	54	188	16,725
72	24	84	27,245	167.8	56	195	16,900
78	26	91	26,800	173.6	58	202	14,525
84	28	98	24,630	179.5	60	210	14,355
90	30	105	24,705				
95.5	32	111	25,050	*Flat ends.*			
101.7	34	118	23,365	89.6	30	104	26,180
107.6	36	125	21,415	119.4	40	139	22,730
113.6	38	132	20,395				
119.6	40	139	20,430	*One flat and one pin end.*			
125.6	42	146	19,085	89.7	30	104	25,155
131.7	44	153	20,150	119.5	40	139	22,160

Plate E shows the results of tests of solid 3-inch square wrought-iron columns with ends bearing on pins 1.5 inches in diameter, as given in Table X, which has been digested from the records of tests found in "Senate Ex. Doc. No. 5, 48th Congress, 1st Session." According to the usual notation, l in the Table is the length in inches; r the radius of gyration (in inches) of a normal section, and d the length of a side (3 inches). As all bars are here 3 inches square, there is a constant ratio between d and r.

The formula shown by the broken line on Plate E is as follows, for *pin-end solid wrought-iron columns:*

$$\left.\begin{aligned} p &= 32{,}000 - 80\frac{l}{r} \\ p &= 32{,}000 - 277\frac{l}{d} \end{aligned}\right\} \quad \ldots\ldots \quad (12)$$

Eq. (12) is to be used only between the limits of $l \div r = 20$ and $l \div r = 220$, or $l \div d = 6$ and $l \div d = 65$.

TABLE XI.

THREE-INCH SQUARE SOLID COLUMNS.

Pin Diameter, Inches.	Length, Inches.	$\frac{l}{d}$.	$\frac{l}{r}$.	Ultimate Resistance, Lbs. per Sq. In.
——	120	40	139	16,285
1⅛	120	40	139	18,335
1½	120	40	139	20,430
1⅞	120	40	139	21 440
2¼	120	40	139	22,250

The "flat-end" and "one pin-and one flat-end" results in Table X are both interesting and important—as showing that the resistance of the latter end condition is essentially a mean between those for pin and flat ends.

Table XI, taken from the same source as Table X, also possesses no little importance as showing the influence of pin diameter. An increase of ⅜ inch in pin diameter below 1½ inches increases the column resistance over 2000 pounds per sq. in. Above that limit the increment of resistance for the same increase in pin diameter is continually less, although very material. As a general principle, it may be said that an increase in pin diameter will produce a corresponding increase in column resistance.

Formulæ for Engineering Practice.

If the greatest allowed working stresses in columns be taken at one one-fifth the ultimate resistance, as is usual for railway structures, the following formulæ will result:

Flat-end latticed channel columns:

$$p=\frac{8000}{1+\frac{1}{40000}\frac{l^2}{r^2}}. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (13)$$

Pin-end latticed channel columns:

$$p=\frac{8000}{1+\frac{1}{30000}\frac{l^2}{r^2}}. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (14)$$

Or

$$p=8500-28\frac{l}{r}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (15)$$

Eqs. (13), (14), and (15) should be used only between the limits of $l \div r = 40$ and $l \div r = 140$; and eq. (13) is given only as a formula which is quite generally used among engineers, but which as yet has no foundation on a series of tests of full-size columns; it gives results which are probably too high.

Flat- and fixed-end iron angles and tees:

$$p=8800-28\frac{l}{r}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (16)$$

Hinged-end iron angles and tees:

$$p=9200-35\frac{l}{r}. \qquad (17)$$

Eqs. (16) and (17) are to be used only between the limits of $l \div r = 40$ and $l \div r = 200$.

Flat-end iron channels and I beams:

$$p=8000-22\frac{l}{r}. \qquad (18)$$

Eq. (18) is to be used only between the limits of $l \div r = 20$ and $l \div r = 240$.

Pin-end solid wrought-iron square columns:

$$\left.\begin{aligned} p&=6400-16\frac{l}{r} \\ p&=6400-55\frac{l}{d} \end{aligned}\right\}. \qquad (19)$$

Eq. (19) is to be used only between the limits of $l \div r = 20$ and $l \div r = 220$, or $l \div d = 6$ and $l \div d = 65$.

Flat-end mild-steel angles:

$$p=10{,}400-36\frac{l}{r}. \qquad (20)$$

Flat-end high-steel angles:

$$p=15{,}200-58\frac{l}{r}. \qquad (21)$$

Eqs. (20) and (21) are to be used only between the limits of $l \div r = 40$ and $l \div r = 200$.

For columns with one flat and one pin end, in all cases use a mean of two pin ends and two flat ends.

For hinged-end angles of steel, in the absence of experimental data, the proper reduction from flat-end angles of the same material may be assumed to be the same percentage or ratio as that between flat- and hinged-end iron angle columns with an equal value of $\frac{l}{r}$.

It is important to observe that the new form of column formula (eqs. (15) to (21) inclusive) is better adapted to forms of section in which the metal is near the neutral axis than to those in which the metal is placed at the greatest possible distance from that axis. It is yet a question whether the old Tredgold form (eqs. (13) and (14)) is not the best for channel columns and those of similar section. The new form, on the other hand, is much the best for angles, tees, I beams, solid sections, etc. No formula, however, which can be devised, is to be compared in value with the experimental diagram, like Plates A to E.

Steel Latticed Channel Columns.

Although some interesting tests of full-size pin-end channel columns of Bessemer steel have been published by Mr. James Dagron in the "Trans. of the Am. Soc. of C. E. for 1887," yet the number was only 8, and the range of $\frac{l}{r}$ too limited for the deduction of any law or formula, had the design of the columns been satisfactory. Again, it is not stated whether the rivet holes were drilled or punched, or punched and reamed, while the resistance of the columns would probably be materially affected by those processes. Tests of full-size steel latticed columns are therefore still needed in order positively to fix their resistance.

Such tests as have been made, however, indicate that properly designed and fabricated steel columns, of metal ranging in tensile strength, in specimens, from 65,000 to 73,000 pounds per sq. in., will give a resistance from 20 to 25 per cent. in excess of that of wrought-iron columns with the same value of $\frac{l}{r}$, provided that ratio does not exceed 135 to 140. The working stresses for such columns, therefore, can be found by increasing those given for wrought iron 20 per cent. for ordinary railway practice and usual lengths of span, or 25 per cent. for spans of, say, 300 feet and over.

Details of Columns.

In addition to the data already given in another portion of this article, the tests cited in this addendum show that the unsupported width of no plate in a compression member should exceed 30 to 35 times its thickness. These tests have usually been made with plates or metal ¼ to ½ inch in thickness, and it is altogether probable that the above ratio of width over thickness would be increased with greater thicknesses.

In built columns, however, *the transverse distance between centre lines of rivets securing plates to angles or channels, etc., should not exceed* 35 *times the plate thickness.* If this width is exceeded, longitudinal buckling of the plate takes place, and the column ceases to fail as a whole, but yields in detail.

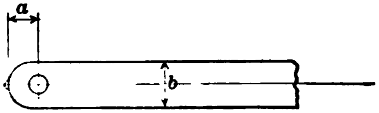
The same tests show that *the thickness of the leg of an angle to which latticing is riveted should not be less than* $\frac{1}{9}$ *of the length of that leg or side*, if the column is purely and wholly a compression member. The above limit may be passed somewhat in stiff ties and compression members designed to carry transverse loads.

The panel points of latticing should not be separated by a greater distance than 60 *times the thickness of the angle leg to which the latticing is riveted*, if the column is wholly a compression member.

The rivet pitch should never exceed 16 *times the thickness of the outside thinnest metal pierced by the rivet,* and if the plates are very thick it should never nearly equal that value.

Art. 61.—Complete Design of Pin-end Steel Columns.

In actual design it is necessary not only to make application of the preceding formulæ for ultimate resistance of columns, but also to proportion a considerable number of details as matters largely of judgment and experience. If the column, like the section shown as the latticed channel or latticed upper chord in the preceding article, has two open sides as in the former or one open side as in the latter latticed, i.e., has small bars of iron running diagonally across those open sides in order to hold the parts of the column in their proper relative positions, those lattice bars vary in size with the size of column. While the dimensions vary somewhat among engineers, the following table, which has been largely used, illustrates effectively sizes that may properly be employed.



For 6	inch rolled or built channels	$1\frac{3}{4}'' \times \frac{5}{16}''$
“ 7	“ “ “ “ “	$1\frac{3}{4} \times \frac{5}{16}$
“ 8	“ “ “ “ “	$1\frac{3}{4} \times \frac{5}{16}$
“ 9	“ “ “ “ “	$1\frac{3}{4}$ and $2 \times \frac{3}{8}$
“ 10	“ “ “ “ “	$1\frac{3}{4}$ “ $2 \times \frac{3}{8}$
“ 11	“ “ “ “ “	$2 \times \frac{3}{8}$
“ 12	“ “ “ “ “	$2 \times \frac{3}{8}$
“ 13	“ “ “ “ “	$2\frac{1}{4} \times \frac{3}{8}$
“ 14	“ “ “ “ “	$2\frac{1}{4} \times \frac{3}{8}$
“ 15	“ “ “ “ “	$2\frac{1}{4} \times \frac{3}{8}$
“ 16	“ “ “ “ “	$2\frac{1}{2} \times \frac{3}{8}$
“ 18	“ “ “ “ “	$2\frac{1}{2} \times \frac{3}{8}$
“ 19–23	“ “ “ “ “	$2\frac{1}{2} \times \frac{7}{16}$
“ 24–29	“ “ “ “ “	$2\frac{1}{2} \times \frac{1}{2}$
“ 30	“ “ “ “ “	$3 \times \frac{1}{2}$

$$a = \frac{b}{2} + \frac{1}{4}''$$

	b	a
for	$1\frac{1}{2}''$	$1''$
“	$1\frac{3}{4}$	$1\frac{1}{8}$
“	2	$1\frac{1}{4}$
“	$2\frac{1}{2}$	$1\frac{1}{2}$

These bars or lattices may be used in single system, in which case each one should make an angle of about 60° with the centre line of the side of the column on which they are placed. If they are used in double system each pair of bars will intersect at their mid-points, and in this case the bars may make angles of 45° with the centre line of the side of the column on which they are employed. In the case of double latticing the intersecting pairs of bars are riveted at their intersections. Lattice bars are held at their ends by one rivet or by two rivets according to the size of the column, as shown in the next table.

Figs. 1, 2, and 3 illustrate different modes of riveting the ends of lattice bars. The size and number of rivets will obviously depend upon the size of the lattice bars employed and to some extent upon the manner in which their ends are held.

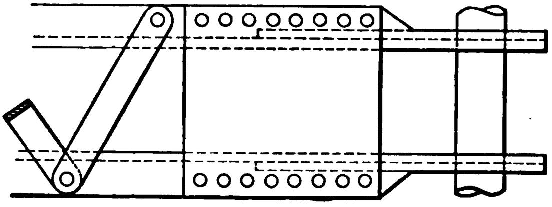


FIG. 1.

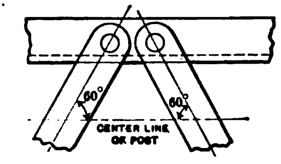


FIG. 2.

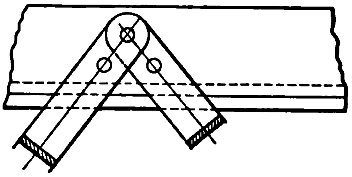


FIG. 3.

The following table has been used in actual structural practice and exhibits good practice in the design of single latticing. It is based on the supposition that the lattice bars are flats. In very large columns or in some exposed situations it is necessary to use steel angles for latticing, the ends of which must be secured by rivets proportionate in number and diameter to the size of angle.

Size of Lattice.	Rivets: Number and Size.	Number of Rivets at Lattice Point.	Limiting Length of Lattice Centre to Centre of Inner Rivets.
$1\frac{3}{4}\times\frac{5}{16}$ and $\frac{3}{8}$	1 . . . $\frac{5}{8}''$	1	13 inches
$2\times\frac{5}{16}$	1 . . . $\frac{5}{8}$	1	16 "
$2\times\frac{5}{16}$	1 . . . $\frac{3}{4}$	1	10 "
$2\times\frac{3}{8}$	1 . . . $\frac{3}{4}$	1	23 "
$2\times\frac{3}{8}$	1 . . . $\frac{3}{4}$	1	16 "
$2\frac{1}{2}\times\frac{3}{8}$	1 . . . $\frac{3}{4}$	1 or 2	20 "
$2\frac{1}{2}\times\frac{3}{8}$	1 . . . $\frac{7}{8}$	1 " 2	15 "
$2\frac{1}{2}\times\frac{7}{16}$	1 . . . $\frac{7}{8}$	1 " 2	20 "
$2\frac{1}{2}\times\frac{7}{16}$	1 . . . $\frac{15}{16}$	1 " 2	17 "
$2\frac{1}{2}\times\frac{1}{2}$	1 . . . $\frac{7}{8}$	1 " 2	26 "
$2\frac{1}{2}\times\frac{1}{2}$	1 . . . $\frac{15}{16}$	1 " 2	24 "
$2\frac{1}{2}\times\frac{1}{2}$	2 . . . $\frac{3}{4}$	4	15 "
$3\times\frac{3}{8}$	1 . . . $\frac{7}{8}$	1 or 2	18 "
$3\times\frac{3}{8}$	1 . . . $\frac{15}{16}$	1 " 2	16 "
$3\times\frac{3}{8}$	2 . . . $\frac{3}{4}$	4	9 "
$\times\frac{7}{16}$	1 . . . $\frac{7}{8}$	1 or 2	25 "
$\times\frac{7}{16}$	1 . . . $\frac{15}{16}$	1 " 2	22 "
$3\times\frac{7}{16}$	2 . . . $\frac{3}{4}$	4	15 "
$3\times\frac{1}{2}$	1 . . $\frac{7}{8}$	1 or 2	32 "
$3\times\frac{1}{2}$	1 . . . $\frac{15}{16}$	1 " 2	29 "
$3\times\frac{1}{2}$	2 . . . $\frac{3}{4}$	4	21 "
$3\times\frac{1}{2}$	2 . . . $\frac{7}{8}$	4	11 "
$4\times\frac{7}{16}$	1 . . . $\frac{15}{16}$	1 or 2	28 "
$4\times\frac{7}{16}$	2 . . . $\frac{3}{4}$	4	22 "
$4\times\frac{7}{16}$	2 . . . $\frac{7}{8}$	4	15 "

At each end of the open or latticed sides of the column are placed batten plates which limit the latticing. The width of these batten plates is determined evidently by the width of the column, but the lengths vary somewhat under different specifications. A good and convenient rule is to make the length of a batten plate at least equal to its width. The thickness of a batten plate will depend upon the size of column; it is seldom made less than $\frac{3}{8}$ in. and usually not more than $\frac{5}{8}$ in. for large columns. The size of rivet will also depend upon the size of columns. Rivets less than $\frac{3}{4}$ in. in diameter are seldom used in railroad work and rarely more than 1 in., the prevailing diameter being ⅞ in.

One of the most important details of a column is the jaw or extension of one side at the end. The two jaws contain the pin holes through which are transferred to the pin the total load carried by the column. These jaws or extensions are formed so as to fit in between the parts of intersecting members, usually the upper or lower chords and eye-bars. It is, therefore, imperative to make them as thin as the bearing upon the pins and the carrying capacity of the jaws themselves acting as short columns will permit. Figs. 4, 5, 6, and 7 exhibit some types of these post jaws as they commonly occur. As the figures show, they are formed by cutting away the flanges of the angles or channels forming parts of the posts and riveting on the pin or thickening plates required to strengthen the detail. The jaws form short columns whose lengths should be taken from the centre of the pin hole to the last centre line of rivets in the body of the column back of the

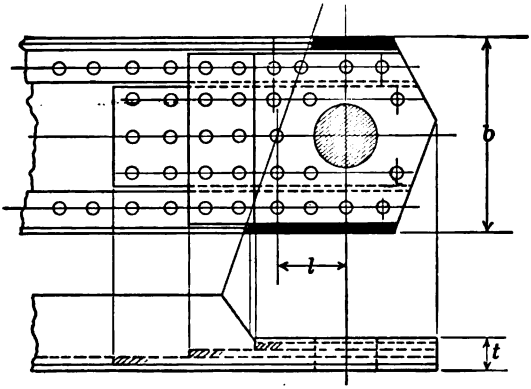


Fig. 4.

cut in the angle or in the flange of the channel. This length indicated by l is shown in each of the figures.

There have been but few tests made to determine the resisting capacities of this particular detail, but those which have been made form the basis of the following formula for medium steel columns. Obviously there will usually

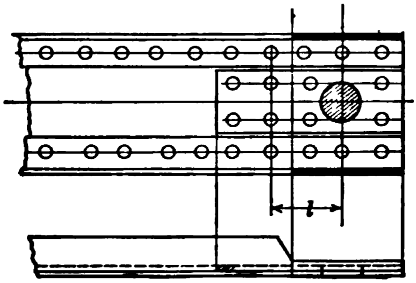


Fig. 5.

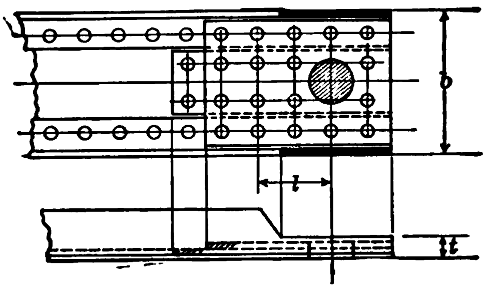


Fig. 6.

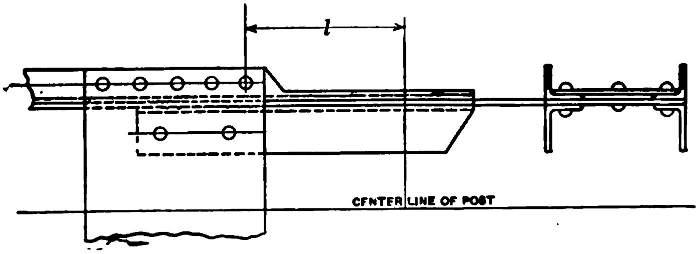


Fig. 7.

be at least two jaws at the end of each column. The width of the side of the column will be represented by b, as shown in Figs. 4 and 6, and t will represent the total thickness of metal whose width is b, also as indicated in the same figures. If P represents the total load on one jaw of the post, usually one half the total load carried by the post or column, the average working intensity of pressure on the section of metal bt may be written

$$\frac{P}{bt} = 8000 - 300\frac{l}{t}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

The thickness t of metal is usually the quantity desired, and eq. (1) gives

$$t = \frac{P}{8000b} + \frac{l}{27}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

In these equations P should be taken in pounds, with b, t, and l in inches.

Eq. (2) has been used to a considerable extent in the design of steel railroad bridges, and it is probably as reasonable and safe a value of the thickness t as can be written with the experimental data and experience now available. It is applicable to steel with ultimate tensile resistance running from 60,000 to 68,000 pounds per square inch. For higher steel or for highway bridges, or for other structures where less margin of safety may be justifiable, the value of t may be made correspondingly less than that given in eq. (2).

Prob. 1. It is required to design a mild-steel pin-end column 45 feet long between centres of pins to carry a load of 232,000 pounds. The column formula to be used is essentially that given as eq. (20) of Art. 60:

$$p = 10{,}000 - 40\frac{l}{r}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

This equation gives the greatest mean intensity allowed on the column, so that p multiplied by the area of cross-section to be determined must be equal or nearly equal to 232,000.

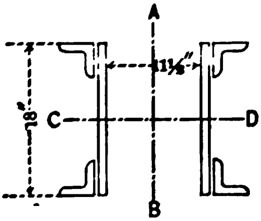


FIG. 8.

The least diameter or width of a built column should not exceed about one thirty-fifth of its length, except where posts or columns are used as lateral members, when the length may reach as much as 40 times the least diameter or width of cross-section. In this case the column is to be built of two plates and four angles, as shown in Fig. 8, and the width of plate *FG* must, therefore, not be less than about 16 inches. A width of 18 inches will make a well-proportioned column and that dimension will be assumed. The separation of the plates is preferably made such that the moment of inertia of the section about the axis *AB* will be a little larger than the moment about the axis *CD*. The pin will pierce the two plates so that its axis will be parallel to *CD*. Under these conditions, if the column is designed so as to be strong enough with the moment of inertia of section taken about *CD*, it will be still stronger in reference to the axis *AB*, and no further attention need be given to possible failure about the latter axis.

If columns of this type are proportioned in the general manner indicated, the radius of gyration of the section about the axis *CD* will be approximately .35 of the width. In this case that trial radius will, therefore, equal 6.3 inches. Hence, inserting the values of $l = 540$ inches and r 6.3 inches in eq. (3), there will result $p = 6572$ pounds per square inch. The total area of section required, therefore, will be closely $232,000 \div 6572 = 35.3$ sq. ins. The distribution of this metal between the plates and angles is largely a matter of judgment. Let there be assumed

Two	18″×⅝″ plates	= 22.5 sq. ins.
Four	3½″×3½″×11-pound angles	= 13 " "
	Total	= 35.5 sq. ins.

This is a tentative composition of section which must be tested by eq. (3) to determine whether it is as nearly accurate as it should be. In order to do this, the moments of inertia of the section, as indicated, must be taken about the two axes *AB* and *CD*.

MOMENT OF INERTIA ABOUT *CD*:

Two	18″×⅝″ plates	$= 2\times\frac{5}{8}\times\frac{18^3}{12}$	= 607.50
Four	3½″×3½″×11-lb angles about own axis		= 14.20
Four	3½″×3½″×11-lb. angles about *CD*	$= 4\times 3.25\times(7.99)^2$	= 829.92
	Moment of inertia		= 1451.62

MOMENT OF INERTIA ABOUT *AB*:

Two	18″×⅝″ plates about own axis	$2\times\frac{18(\frac{5}{8})^3}{12}$	= .74
Two	18″×⅝″ plates about *AB*	$2\times 11.25\times(6.06)^2$	= 758.70
Four	3½″×3½″×11-lb. angles about own axis		= 14.20
Four	3½″×3½″×11-lb. angles about *AB*	$= 4\times 3.25\times(7.38)^2$	= 708.38
	Moment of inertia		= 1482.02

These computations show, first, that the moment of inertia about *AB* is a little larger than that about *CD*, which is as it should be. They also show that the radius of gyration r is 6.39 inches. The approximate rule gives $r = 6.3$ inches. These two values are sufficiently near to accept the former. The trial composition of section may, therefore, be considered satisfactory and final. The thickness of the side plates, .625 inch, is sufficient to insure no buckling in the unsupported width between rivets. Similarly the length of leg of the 3½-inch angles is also far within safe or proper limits. All features of the cross-section are, therefore, so arranged as to meet all the requirements of suitable resistance in detail.

The details of the ends of the columns where they are formed into jaws, as shown by Figs. 9 and 10, still remain

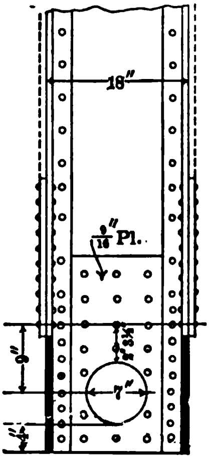


FIG. 9.

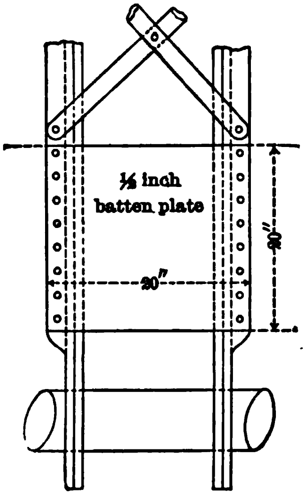


FIG. 10.

to be designed. The diameter of pin will be taken at 7 inches, as shown in Fig. 9. The permissible intensities of shearing and of the bearing on the walls of rivet and pin holes will be taken as follows:

Shearing on rivets = 9000 pounds per sq. in.
Bearing on rivets and pins = 16,000 pounds per sq. in.

The total thickness of metal in the two post jaws will, therefore, be

$$\text{Thickness of metal} = \frac{232000}{7 \times 16000} = 2.1 \text{ inches.}$$

The thickness of metal in each jaw must therefore be at least $1\frac{1}{16}$ inches. Inasmuch as the thickness of side plates of the column is $\frac{5}{8}$ inch, the pin plates to be riveted to the side plates must be at least $\frac{7}{16}$ inch thick to supply the proper bearing surface for the pin; but that thickness must be decided by the formula for the jaws, eq. (2). In that equation, $P=116{,}000$ pounds, while $b=18$ inches and l, from Fig. 9, is 9 inches. Making these substitutions in eq. (2),

$$t=1.13 \text{ inches.}$$

In order to meet the requirements of the post-jaw formula, therefore, the pin plate must be at least $\frac{1}{2}$ inch thick. It is essential however to make these details specially stiff and strong and the thickness will, therefore, be taken at $\frac{9}{16}$ inch, as shown in Fig. 9.

The number of rivets required above the pin hole would ordinarily be computed for the thickness of plate required for bearing on the pin, i.e., with the thickness of pin plate of $\frac{7}{16}$ inch. Assuming that thickness for this purpose, the rivets being taken $\frac{7}{8}$ inch in diameter, the bearing value of a single rivet will be

$$\tfrac{7}{8}\times\tfrac{7}{16}\times16{,}000=6125 \text{ lbs.}$$

The single shear of one $\frac{7}{8}$-inch rivet at 9000 pounds per square inch has a value of 5412 pounds which is less than the bearing value; the shear will, therefore, decide the number of rivets required. The bearing value of the $\frac{5}{8}$-inch side plate on the pin is $7\times\frac{5}{8}\times16{,}000=70{,}000$ pounds. Hence the number of rivets required in the pin plate on each side of the column will be

$$\frac{116000-70000}{5412}= \text{nine rivets (nearly).}$$

These nine rivets must be found above the pin. That number, however, is far too small for the pin plate acting as a part of the jaw, and it will be judicious to make the total number of rivets above the pin 12, as shown in Fig. 9.

The jaw plates will extend 5 inches beyond the pin, as shown. The two batten plates above which the latticing begins will each be taken ½ inch thick, and they will be placed as shown in both Figs. 9 and 10.

It is assumed that the ends of the column are to fit into or between other members of the truss, so as to require cutting away the legs of the steel angles, as shown, as this is a common requirement.

The length of a batten plate should not be less than its width. In the present instance the width of batten will be 19.75 inches; the length will, therefore, be taken as 20 inches.

As indicated in the tabular statement at the beginning of this article, the lattice bars, fully shown in Fig. 10, will be 2½ × ⅜ inches, and the latticing will be taken as double, although this is not always done for the size of column in this particular instance. The lattice bars will be riveted at their intersections also as shown in Fig. 10. The length of lattice bar between rivets will be about 11 inches, as the angle made by each lattice bar with the side of the column will be about 45 degrees. A single ⅞-inch rivet, therefore, at the end of each bar will be sufficient, as shown by the second table of this article. At each panel point of latticing a single ⅞-inch rivet will hold the ends of both lattice bars.

The complete bill of material for the entire column will be as follows:

Four 3½″×3½″×11-lb. angles, 46.42 ft. long. .	185.7×11 =	2,043 lbs.
Two 18″×⅝″ plates, 46.42 ft. long.	93×38.25 =	3,557 "
Four 27″×11″×$\frac{9}{16}$″ plates	9×21 =	189 "
Four 20″×20″×½″ battens	6⅝×34 =	227 "
240 lin. ft. of 2½″×⅜″ latticing.	240×3.19 =	766 "
1060 ⅞″ rivets.	10.6×54 =	572 "
Total weight of one column.	=	7,354 lbs.

Prob. 2. Let it be required to design a mild-steel column with pin ends, 36 feet long between centres of pins, to carry a load of 225,500 pounds. It is supposed that the column is a member of a railroad bridge, so that the load given includes a full allowance for impact. Gordon's formula in the form employed in the American Bridge Company's specification will be used:

$$p = \frac{17000}{1 + \dfrac{l^2}{11000r^2}}.$$

In this formula p is the greatest mean intensity of working pressure allowed on the section of the column, l the length between centres of pins in inches, and r the radius of gyration of the column section in inches. As the length of the column is but 36 ft. $=432$ inches two rolled 15-inch channels latticed may be taken as the principal parts, as shown in Fig. 11. By turning to the tables at the end of the book, it will be found that the radius of gyration of a 15-inch channel about the axis AB varies from about 5.6 inches to nearly 5.2 inches. The larger of the two values will be tentatively employed. Substituting $l=432$ and $r=5.6$ in the above formula for p,

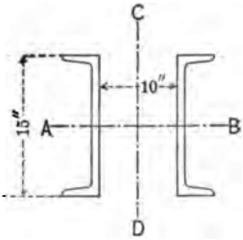


FIG. 11.

$$p = 11{,}000 \text{ pounds per sq. in.}$$

Hence the total area required is

$$\frac{225500}{11000} = 20.5 \text{ sq. ins.}$$

The table of steel channels at the end of the book shows that the combined area of two 15-inch 35-pound channels is 20.58 sq. in., and they will be accepted as correct. The same table gives the radius of gyration r about the axis AB, Fig. 11, as 5.57 inches, which is essentially equal to the trial value 5.6 inches.

As shown in Prob. 1, it is desirable to have the moment of inertia of the section about AB, Fig. 11, a little less than that about CD, the former (AB) being parallel to the axis of the pin. Let the separation of the channels be made 10 inches in the clear. By using the values of the table, the moments of inertia about the two axes may be written:

About Axis AB:

Moment of inertia $=320\times 2=640$.

Hence $$r^2=\frac{640}{20.58}=31.02;\quad \therefore r=5.57 \text{ ins.}$$

About Axis CD:

Moment of two channel sections each about axis parallel to CD and through centre of gravity .... $2\times 8.48=\ 16.96$

$2\times 10.29\times \overline{5.79}^2$ .... $=689.84$

Moment of inertia .... $=706.80$

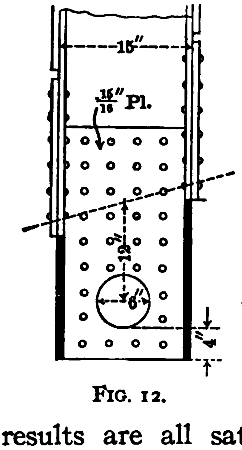


FIG. 12.

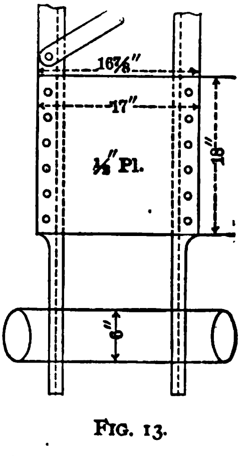
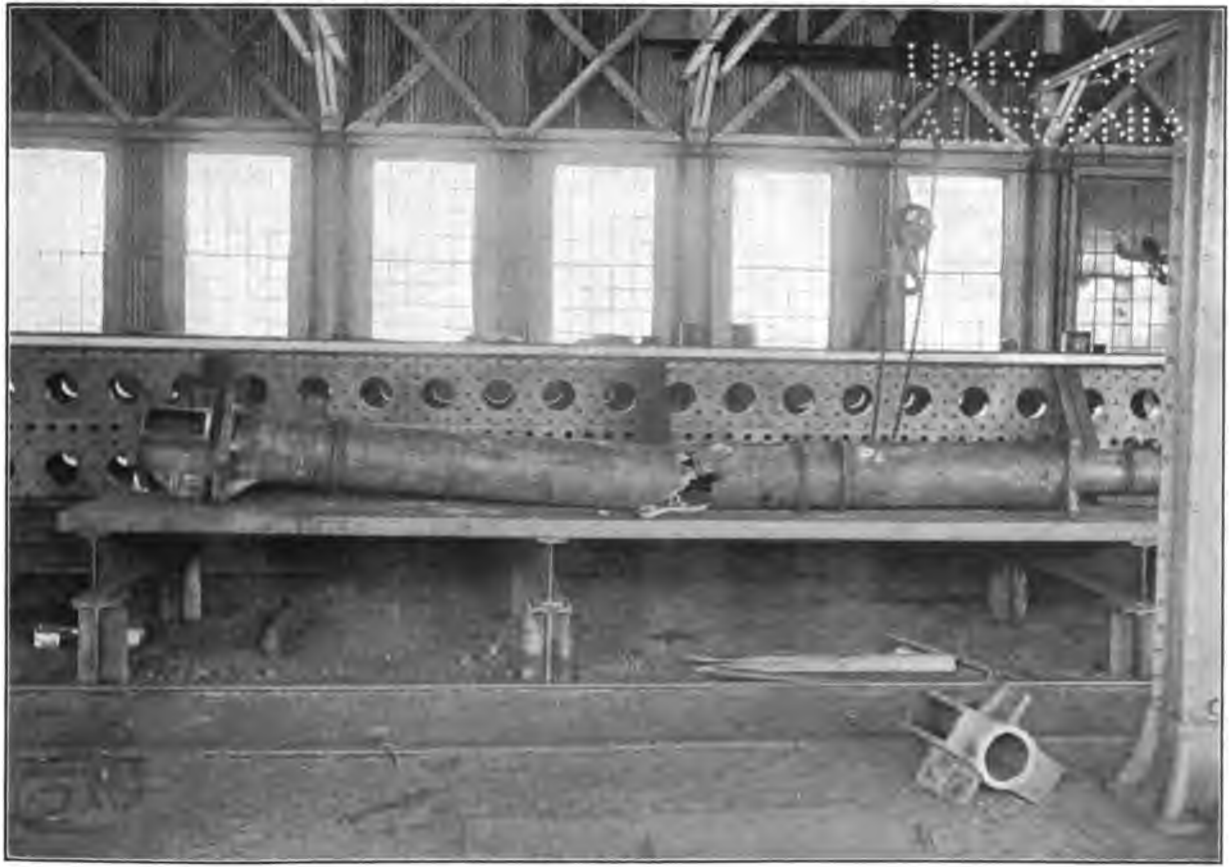


FIG. 13.

These results are all satisfactory and show that no revision of the section as given in Fig. 11 is needed.

The end details and latticing shown in Figs. 12 and 13



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View of failure of a cast-iron column tested in the 1000-ton testing-machine of the Phœnix Iron Co. at Phœnixville, Pa.

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remain to be considered. The following data will be required:

Thickness of channel web. . . . . = .43 inch.
Allowed shearing on rivets and pins. . . . . = 10,000 lbs. per sq. in.
Allowed bearing on rivets and pins. . . . . = 20,000 lbs. per sq. in.
Diameter of rivets. . . . . = $\frac{7}{8}$ inch.
Diameter of pin. . . . . = 6 inches.
Value of one $\frac{7}{8}$-inch rivet in single shear. . . . . = 6,013 lbs.
Bearing of pin on channel web. = $6 \times .43 \times 20{,}000$
= 51,600 lbs.

Bearing to be carried by pin plate $= \frac{225500}{2} - 51{,}600 = 61{,}150$ lbs.

Thickness of pin plate. . . . . $= \frac{61150}{6 \times 20000} = .51$ inch.

Bearing value of one $\frac{7}{8}$-inch rivet on $\frac{1}{2}$-inch plate =
$\frac{7}{8} \times \frac{1}{2} \times 20{,}000 =$ 8,750 lbs.

Hence one pin plate needs $\frac{61150}{6013}$ = ten $\frac{7}{8}$-inch rivets.

It is assumed that the ends of the column must be formed into the jaws shown in Figs. 12 and 13. As indicated in Fig. 12 the mean or effective length of the jaw is 12 inches. The load carried by one jaw is 112,750 pounds; hence the thickness of that jaw is by eq. (2)

$$t = \frac{112750}{8000 \times 15} + \frac{12}{27} = 1\tfrac{7}{16} \text{ inch (nearly).}$$

The thickness of the jaw or pin plate to be riveted to the jaw must therefore be $1\frac{7}{16} - .43 = \frac{15}{16}$ inch. In order that these plates may be firmly made a solid extension of the post or column they should be riveted to the webs of the channels with the rivets shown in Fig. 12. The proper design of the jaw, therefore, requires a much longer and thicker plate and more rivets than the simple consideration of the pin and rivet bearing and shearing.

The width of channel flange is 3.43 inches, hence the total width of column over these flanges, as shown in Fig.

13, is 16⅞ inches. Each batten plate is therefore taken as 17 inches by 18 inches.

The length of each lattice bar of the single, 30-degree latticing will be about 16 inches between centres of rivets at their ends. Lattice bars 2½ inches by ⅜ inch in section will, therefore, be used.

The complete bill of material for one column will then be

Two	15″ 35-lb. channels 37⅙ ft. long	$2 \times 35 \times 37\frac{1}{6}$	= 2,602 lbs.
Four	$13'' \times 30'' \times \frac{15}{16}''$ plates	10×41.44	= 415 "
Four	$17'' \times 18'' \times \frac{1}{2}''$ plates	6×28.9	= 173 "
Forty-six	$2\frac{1}{2}'' \times \frac{3}{8}'' \times 19''$ bars	72×3.19	= 230 "
Two hundred and twenty-five	⅞″ rivets	$2\frac{1}{4} \times 54$	= 122 "
	Total weight of one column		= 3,542 lbs.

Art. 62.—Cast-iron Columns.

Cast iron was the earliest form in which the metal iron was used for columns, and it is natural, therefore, that the first long-column formulæ for cast iron should have been among the earliest for that class of members. The first experimenter was Eton Hodgkinson, who published the results of his tests on small cast-iron columns, the greatest length of which was but 60.5 inches, in the "Philosophical Transactions of the Royal Society of London for 1840." He not only recognized the round- and fixed-end conditions, but he also made the distinction between long columns and short blocks, the length of the latter being from 4 to 5 times the diameter or least cross-section dimension. If d be the diameter of the column in inches and l the length in feet, and in the case of hollow round columns if D be the exterior diameter in inches and d the interior diameter in the same unit, while P is the total or ultimate load in pounds on the column, Hodgkinson established the following formulæ for long cast-iron columns:

$$P = 33{,}379 \frac{d^{3.76}}{l^{1.7}}; \text{ (for rounded ends).} \quad . \quad . \quad . \quad (1)$$

$$P = 98{,}922\frac{d^{3.55}}{l^{1.7}}\text{; (for fixed ends).} \quad . \quad . \quad . \quad . \quad (2)$$

For hollow cylindrical columns of cast iron

$$P = 29{,}120\frac{D^{3.76} - d^{3.76}}{l^{1.7}}\text{; (for rounded ends).} \quad . \quad (3)$$

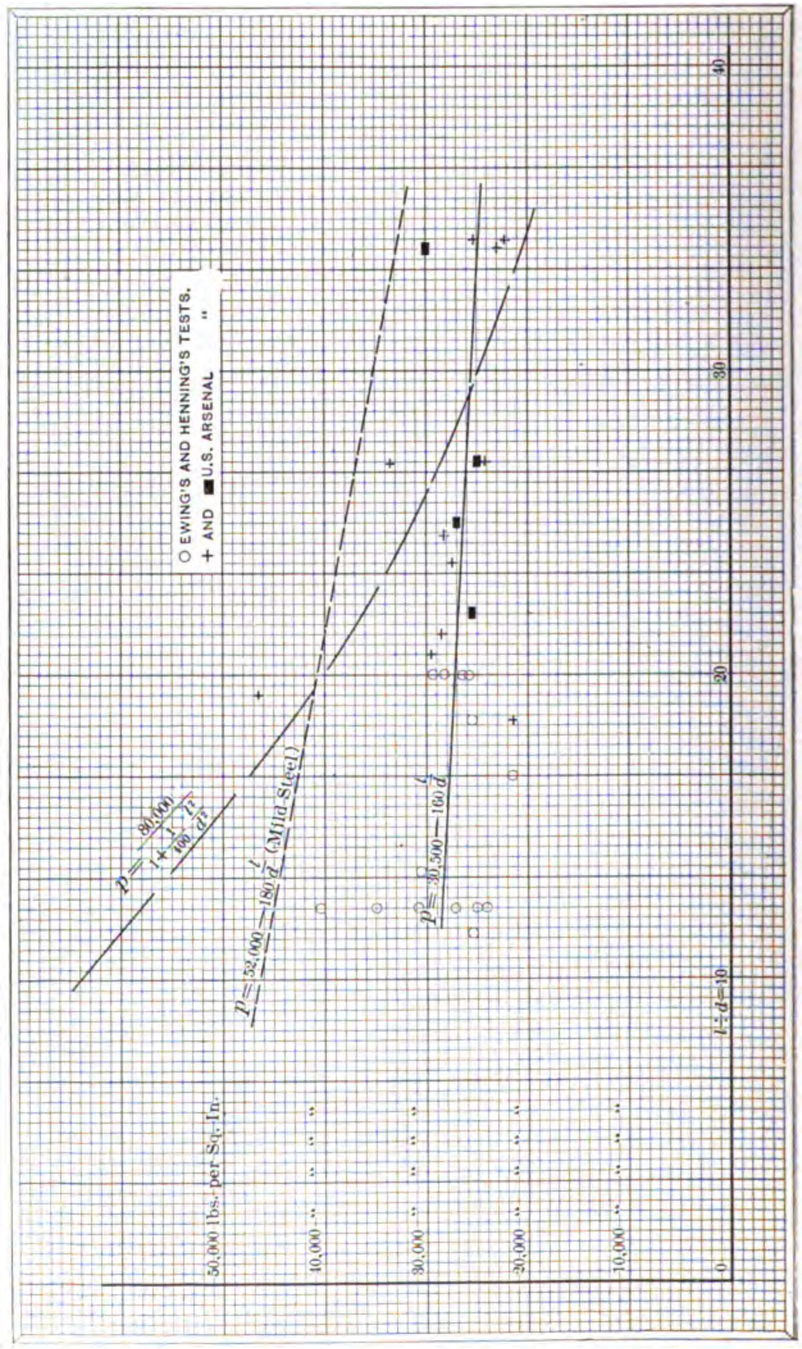
$$P = 99{,}320\frac{D^{3.55} - d^{3.55}}{l^{1.7}}\text{; (for fixed ends).} \quad . \quad . \quad (4)$$

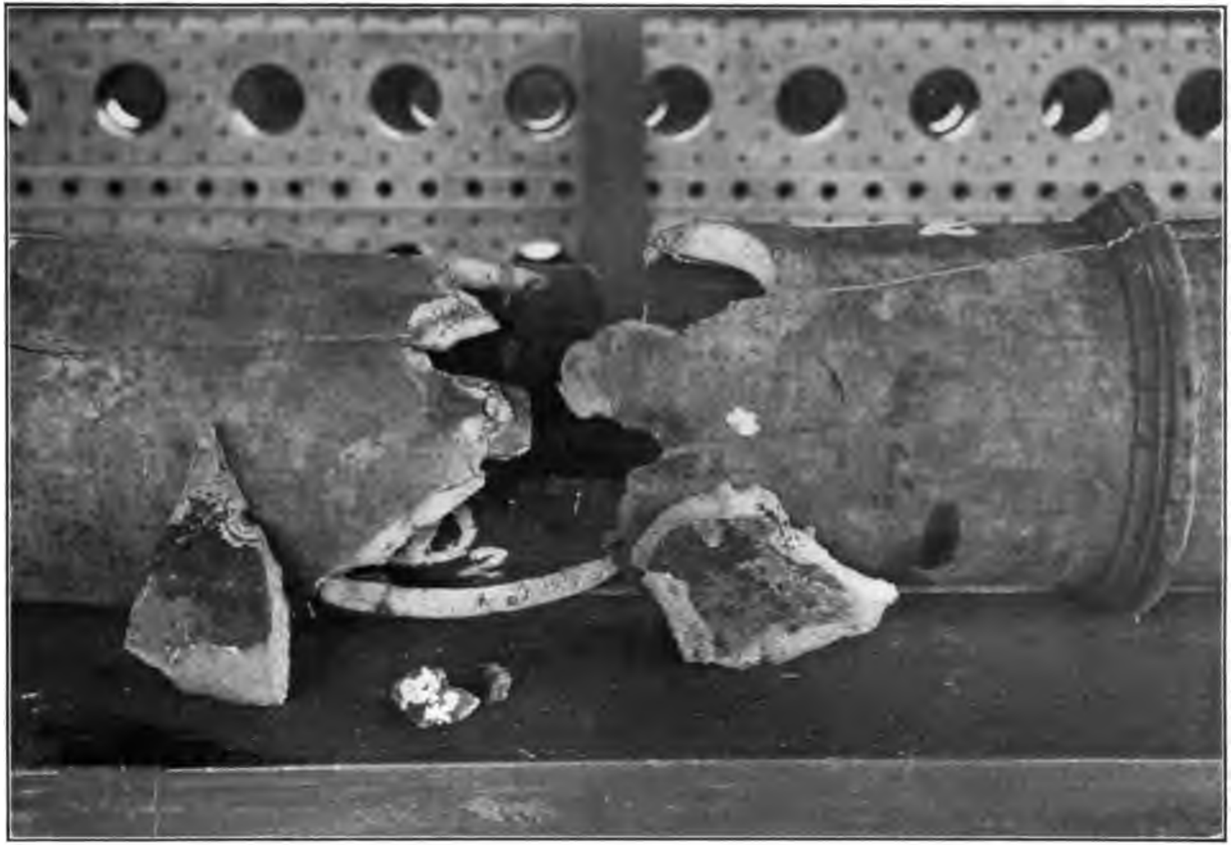
The working or maximum load allowed in any design of cast-iron columns would be found by taking one fifth to one eighth of the values given in eqs. (1) to (4) inclusive. It will be observed that Hodgkinson's formulæ expressed in the preceding equations are simply Euler's formulæ as given in eqs. (6) and (9) of Art. 24. with the introduction of an empirical coefficient and with the indices of d and l changed to harmonize with the experimental results.

As Hodgkinson's experiments were made on very small columns of different metal from that used in cast-iron columns of the present day, his formulæ cannot safely be used for practical purposes at the present time.

A correct formula for cast-iron columns must be based upon tests of full-size columns cast with the metal ordinarily employed in structural practice. Such tests have been made at the U. S. Arsenal at Watertown, Mass., and will be found reported in H. R. Ex. Doc. No. 45, 50th Congress, 2d Session, and in H. R. Ex. Doc. No. 16, 50th Congress, 1st Session. A valuable series of tests was also made at Phœnixville, Pa., at the works of the Phœnix Bridge Co., under the auspices of the Department of Buildings of New York City in 1896-97. Although the entire series, including both the tests at Watertown and Phœnixville, do not cover the variety of sectional forms and range of ratio of length to diameter that could be

PLATE I.


○ EWING'S AND HENNING'S TESTS.
+ AND ■ U.S. ARSENAL "
$p = \frac{80,000}{1 + \frac{1}{100}\frac{l^2}{d^2}}$
$p = 52,000 - 180\frac{l}{d}$ (Mild Steel)
$p = 30,500 - 160\frac{l}{d}$
50,000 lbs. per Sq. In.
40,000 " " " "
30,000 " " " "
20,000 " " " "
10,000 " " " "
0
$l \div d = 10$
20
30
40



View of failure of a cast-iron column tested in the 1000-ton testing-machine of the Phœnix Iron Co. at Phœnixville, Pa.

(*To face page* 526.)

desired, the results are sufficiently extended to show closely what may be considered the proper ultimate values for hollow round cast-iron columns of full size.

TABLE I.

No.	Length in Inches.	Diameter in Inches.				Area of Section in Square Inches.	Length over Exterior Diameter	Ultimate Resistance in Pounds per Square Inch.
		Large End.		Small End.				
		Ext.	Int.	Ext.	Int.			
1	190.25	15	13	——	——	43.98	12.7	30,830
2	"	15	12.75	——	——	49.03	12.7	27,126
3	"	15	12.75	——	——	49.03	12.7	24,434
4	"	15⅛	12.75	——	——	49.48	12.7	25,182
5	"	15	12.66	——	——	50.91	12.7	35,435
6	"	15	12.63	——	——	51.52	12.7	40,411*
7	160	8	6	——	——	21.99	20	29,604
8	160	8	5.91	——	——	22.87	20	28,229
9	120	6.06	3.78	——	——	17.64	20	25,805
10	120	6.09	3.96	——	——	17.37	20	26,205
11	147.75	8	6.5	——	——	17.08	18.5	25,973
12	150	9	7	——	——	25.14	16.7	21,183
13	162	12	10	——	——	34.55	13.5	30,813
14	159.75	14	12	——	——	40.84	11.4	25,400
15	169	5	4.54	——	——	3.5	34	29,854
16	157	7.17	4.83	——	——	21.8	22	25,470
17	157	6.35	3.9	——	——	17.28	25	27,210
18	156	5.8	4.03	——	——	13.22	27	25,100
19	142.6	7.68	5.52	5.94	4.3	17.49	26.7	29,310
20	146.8	8.01	5.58	5.9	4.35	18.65	21.3	28,520
21	150	6.17	4.85	5.09	3.48	12.08	27	33,500
22	145.5	6	4.35	4.74	2.73	12.81	37.1	24,620
23	133.6	6.02	4.36	4.84	2.88	12.87	24.6	28,060
24	129.3	6.03	4.35	4.87	2.95	12.87	23.7	27,350
25	127.6	7.47	5.97	5.72	4.62	12.13	19.3	46,660
26	118.5	3.98	1.96	2.97	1.49	7.16	34.1	23,090
27	119	3.98	1.96	2.98	1.47	7.17	34.3	22,040
28	118	3.97	1.95	2.99	1.39	7.32	34.2	25,060
29	84.6	4.88	3.03	4.27	2.08	11.25	18.5	31,190

* Not broken.

Table I shows the results of all these tests, while the Plate exhibits the same results graphically. The tests Nos 1 to 10 inclusive were made at Phœnixville in December, 1897, and Nos. 11 to 14 inclusive in 1896; the former group under the immediate direction of Mr. W. W. Ewing, and the latter under the immediate direction of Mr. Gus C. Henning. The results shown for tests 15 to 18 inclusive were taken from H. R. Ex. Doc. No. 45, 50th Congress, 2d Session, but those for Nos. 19 to 29 inclusive are either taken or digested from H. R. Ex. Doc. No. 16, 50th Congress, 1st Session, being portions of reports of tests of metals and other materials at the United States Arsenal, Watertown, Mass.

As Table I shows, the columns Nos. 19 to 29 inclusive were slightly conical, although probably not enough so to affect appreciably their resistances. The areas of section in square inches for these columns were taken at mid-distance between their ends. As the area of section varied considerably in some columns, that operation may be a source of a little error in determining the ultimate resistance per square inch from the result of the tests, but if the error exists at all it must be very small. The mid-external diameter was also taken for these columns in determining the ratio of the length over the diameter shown in the Table and in the Plate.

As will be observed both in the Table and in the Plate, the ultimate resistances per square inch determined by the tests are quite variable, even for the same ratio of length over diameter. Indeed, in a number of cases they are quite erratic. In Nos. 1 to 6, for which the ratio of length over diameter was 12.7, the ultimate resistances vary from a little over 24,000 lbs. per square inch to over 40,000 lbs. per square inch with no failure at the latter value. Again, the ultimate resistance per square inch for No. 25, which shows a ratio of length over diameter of less than 20, is nearly 47,000 lbs. per square inch, which is excessively high as compared with other ultimate resistances with the same or less ratio of length over diameter.

These erratic results are not surprising in view of the ordinary character of the metal. It should be remembered that the failures of these columns are frequently recorded with such "remarks" as the following: "Foundry dirt or honey-comb between inner and outer surfaces," "bad spots," "cinder pockets and blow holes near middle of column," "flaws and foundry dirt at point of break." In other words, it was no uncommon feature to observe that defects, flaws, or blow holes or thin metal had determined the place of failure. There is considerable uncertainty in platting the results of tests affected by these abnormal conditions, but a more or less satisfactory law for the generality of cases may be determined from a graphical representation of the results, as shown on Plate I. On that Plate the ultimate resistances in pounds per square inch, as shown in Table I, have been platted as vertical ordinates, while the ratios of length over diameter given in the same Table are represented by the horizontal abscissas, all as clearly shown. The full straight line drawn in about a mean position among the results of the tests probably represents as near as any that can be found a reasonable law of variation of ultimate resistance with the ratio of length over diameter. It is evident that within the range of these experiments a straight line will represent the ultimate resistances fully as well as any curve, if not better, although the results for the lengths of thirty-four times the diameter begin to indicate a little curvature. The formula which represents this straight line, i.e., which gives the ultimate resistance per square inch, is as follows:

$$p = 30{,}500 - 160\frac{l}{d}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

It is to be borne in mind that these columns were round and hollow, and that they were tested with flat ends in all cases. The ordinary formula, based upon Hodgkinson's tests, and frequently used in cast-iron column construction, is as follows:

$$p = \frac{80000}{1 + \frac{1}{400}\frac{l^2}{d^2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The curve corresponding to this particular form of Tredgold's formula is also shown on the Plate. It will be seen that at the ratio of length over diameter of 10 to 12 (not an uncommon ratio) the ultimate, as given by this formula, is just about double that shown by actual test. In other words, if a safety factor of 5 were required, as is the case in some building laws, the actual safety factor would be but 2½. The curve represented by eq. (6) is seen to cross the true curve at a ratio of length over diameter of about 29. A glance at the Plate will show how erroneous and dangerous is the use of the usual formula for hollow round cast-iron columns; indeed, that formula is grossly wrong, both as to the law of variation and the values of ultimate resistance.

In view of the working resistances, which have been used in the design of cast-iron columns, it is no less interesting than important to compare the ultimate resistances per square inch of mild-steel columns, as determined by actual tests, with the ultimate resistances of cast-iron columns, as shown by the tests under consideration. The broken line of short dashes represents the formula

$$p = 52{,}000 - 180\frac{l}{d} \text{ - - - - - -}. \quad . \quad . \quad . \quad . \quad . \quad (7)$$

determined by actual tests of mild-steel angles made by Mr. James Christie at the Pencoyd Bridge Works, and given in Art. 60. This line or formula shows that the ultimate resistances per square inch of mild-steel columns are from 40 to 50% greater than the corresponding quantities for cast-iron, the same ratio of length over diameter being taken in each comparison.

When the erratic and unreliable character of cast-iron columns is considered, it is no material exaggeration to state that these tests show that the working resistance per square inch may be taken twice as great for mild-steel columns as for cast-iron; indeed, this may be put as a reasonably accurate statement.

The series of tests of cast-iron columns represented in the Plate constitute a revelation of a not very assuring character in reference to cast-iron columns now standing, and which may be loaded approximately up to specification amounts. They further show that if cast-iron columns are designed with anything like a reasonable and real margin of safety the amount of metal required dissipates any supposed economy over columns of mild steel.

If the average working stress per square inch is one fourth of the ultimate resistance, eq. (5) gives

$$p = 7600 - 40\frac{l}{d}. \qquad (8)$$

If the working stress is to be taken at one fifth the ultimate, eq. (5) gives

$$p = 6100 - 32\frac{l}{d}. \qquad (9)$$

In these equations p is the average working intensity of pressure in pounds per square inch. The length l and the exterior diameter d must be taken both in the same unit, ordinarily the inch.

These formulæ may be used between the limits of $10\frac{l}{d}$ and $35\frac{l}{d}$ or even $40\frac{l}{d}$. They may also be applied to hollow rectangular columns with reasonably close approximation, *d* being taken as the smaller exterior side of the rectangular cross-section.

Art. 63.—Timber Columns.

Tests of this class of members, the results of which have been published, although of great value, have not been made with sufficiently large ratios of length to radius of gyration to produce true "long-column" failures. This renders impossible the establishment of a long-column formula or diagram for practical use in connection with the use of long timber columns.

Some valuable experiments, however, have been made with full-size columns having lengths as great as fourteen feet. The first results to be given are those of a large number of tests by Prof. Lanza, of Boston, in which he used the United States testing machine at Watertown, Mass. These tests were made during 1881, on such members as are commonly used in the construction of cotton and woollen mills.

Table I contains the results of Prof. Lanza's tests. A large majority of the columns had cores bored out of the centre, which varied in diameter from 1.5 to 2.0 inches. The absence of material did not affect in any way, so far as could be observed, the resistance per square inch.

Column 20 had the force applied 2½ inches out of centre at one end, and column 35, 1.9 inches. These tests were made in order to observe the effect of eccentricity in the application of loads. They show a marked decrease in ultimate resistance.

Although the ends of Nos. 39 and 44–51 were flat, they were not parallel.

All the columns indicated by "Round" were tapered, and they almost invariably gave way by the crushing of

TABLE I.

TIMBER MILL COLUMNS.

No.	Form.	Length, Feet.	Diameter in Inches at			Section Area in Sq. Ins. at		Ultimate Resistance, Lbs. per Sq. Inch.	Ends.
			Top.	Mid.	Base.	Top.	Base.		
			Yellow pine; partially seasoned.						
1	Round.....	12.01	9.31	10.65	10.55	65.89	85.22	4,098	Flat.
2	"	12.02	8.30	9.71	10.07	51.84	77.36	3,665	"
3	"	12.01	7.54	8.88	8.99	42.38	61.21	4,719	"
4	"	12.02	6.40	7.80	7.79	29.98	45.47	4,602	"
5	Cylindrical.	12.00	10.45	——	10.45	83.75	——	4,657	"
6	" .	11.92	8.96	——	8.96	61.19	——	4,086	"
7	" .	11.98	7.70	——	7.70	44.72	——	4,584	"
8	" .	2.01	10.46	——	10.46	85.92	——	4,422	"
9	" .	2.00	9.98	——	9.98	78.23	——	4,705	"
10	" .	2.01	8.91	——	8.91	62.35	——	4,330	"
11	" .	2.00	7.79	——	7.79	47.66	——	4,511	"
12	Square	12.44	8.43	——	8.40	68.80	——	5,451	"
13	"	12.57	8.30	——	8.30	63.10	——	3,804	"
14	Cylindrical.	11.93	9.92	——	9.92	75.45	——	3,512	"
			Yellow pine; air seasoned.						
15	Round.....	13.99	7.70	——	7.90	44.56	47.01	4,488	"
16	Cylindrical.	2.00	7.70	——	7.90	44.56	——	4,892	"
			Yellow pine; dock seasoned.						
17	Cylindrical.	11.92	8.00	——	8.00	48.26	——	4,662	One flat; one round.
18	"	1.98	7.93	——	7.98	48.00	——	3,604	Flat.
19	Square	11.93	8.08	——	8.08	63.28	——	3,477	One flat; one round.
20	"	12.94	8.75	——	8.92	76.04	——	3,682	One flat; one round.
21	"	12.85	10.05	——	10.13	99.79	——	5,111	Flat, pintle.
22	"	2.00	3.98	——	9.02	81.00	——	5,951	Flat.
23	"	2.01	10.20	——	10.07	102.71	——	5,452	"
			White wood; partially seasoned.						
24	Round.....	12.01	8.46	9.61	9.65	54.02	70.95	3,333	"
25	"	11.97	6.38	7.40	7.72	29.78	44.62	2,687	"
			White oak; partially seasoned.						
26	Round.....	12.01	9.13	10.15	11.01	63.28	39.01	3,003	"
27	"	12.01	8.37	0.40	10.23	52.83	80.00	3,786	"
28	"	12.01	7.55	8.75	9.05	42.58	62.14	3,758	"
29	"	12.01	6.60	7.79	8.06	32.02	48.83	3,435	"
30	Cylindrical.	12.00	10.00	——	——	76.70	——	2,478	"
31	" .	6.33	10.00	——	——	76.70	——	2,738	"
32	" .	2.00	9.98	——	——	78.23	——	3,132	"
33	" .	2.00	8.18	——	——	52.55	——	3,140	"
34	" .	1.98	7.73	——	——	46.93	——	3,303	"
35	" .	11.93	8.20	——	——	50.92	——	1,964	"

TABLE I.—*Continued.*

No.	Form.	Length, Feet.	Diameter in Inches at			Section Area in Sq. Ins. at		Ultimate Resistance, Lbs. per Sq. Inch.	Ends.
			Top.	Mid.	Base.	Top.	Base.		
		White oak; used in mill 6½ years.							
36	Round.....	12.08	5.85	—	6.84	23.89	33.76	4,604	Flat.
37	"	12.08	5.85	—	6.85	24.05	34.02	6,029	"
38	"	12.11	5.87	—	6.70	23.92	32.12	4,680	"
39	"	12.05	6.02	—	6.75	25.47	32.79	2,945	"
40	"	11.72	6.10	—	6.83	26.08	33.50	3,451	"
41	"	12.01	5.97	—	6.74	25.09	32.78	4,225	"
42	"	12.07	5.75	—	6.88	22.98	34.19	3,264	"
		White oak; used in mill 25 years.							
43	Cylindrical.	13.87	10.56	—	—	84.74	—	4,602	"
44	" .	14.00	10.54	—	—	84.83	—	4,951	"
45	" .	13.89	10.54	—	—	84.40	—	4,266	"
46	" .	13.80	10.50	—	—	83.75	—	3,881	"
47	" .	13.92	10.20	—	—	79.17	—	4,674	Flat, pintle.
48	" .	13.89	10.80	—	—	82.68	—	4,838	" "
49	Round.....	13.67	9.25	—	9.50	64.36	68.04	3,434	Flat.
50	Cylindrical.	13.85	9.55	—	—	68.65	—	4,618	"
51	" .	13.65	9.40	—	—	66.56	—	3,981	"
52	" .	13.90	9.35	—	—	65.82	—	3,266	Pintle ends.
53	Round.....	11.51	5.98	—	7.20	26.03	38.66	6,147	Flat
		White oak; thoroughly seasoned; 1 year old.							
54	Cylindrical.	12.00	7.74	—	—	45.04	—	3,219	Flat.
55	" .	11.12	10.95	—	—	91.16	—	1,865	One flat; one round.
56	" .	2.00	10.91	—	—	95.48	—	4,450	Flat.

the fibres at the small end. In all such columns the ultimate resistance is *per square inch of the small end.*

Some of the square columns had their corners slightly beveled.

In his report to the Boston Manufacturers' Mutual Fire Ins. Co., Prof. Lanza says: "The immediate location of the fracture was generally determined by knots; . . ." but states that, whether knotty or straight grained, failure took place in the tapered columns at the small ends. Tapering a column, therefore, to the extent shown in these cases, is a source of weakness.

Tables II and III contain the results of Col. Laidley's tests, some of which belong to short blocks. These tests

TABLE II.

YELLOW PINE.

No.	Length, Inches.	Form of Section.	Section Dimensions, Inches.	Ultimate Resistance per Sq. In.	
				Lbs.	
1	20.4	Circular.	10.2 diam.	6,676	Straight-grained and seasoned 20 years.
2	119.95	Square.	11 × 11	6,230	
3	119.90	"	11 × 11	6,552	
4	20.0	"	10.4 × 10.4	7,936	
5	16.0	"	8 × 8	8,165	
6	8.0	"	4 × 4	7,394	
7	3.0	"	1.5 × 1.5	5,533	
8	6.0	"	3 × 3	8,644	
9	6.0	"	3 × 3	8,133	
10	3.0	"	1.5 × 1.5	8,389	
11	3.0	"	1.5 × 1.5	8,302	
12	3.0	"	1.5 × 1.5	6,355	
13	14.0	"	4.6 × 4.6	9,947	
14	17.2	"	4.6 × 4.6	10,250	
15	19.1	"	5.3 × 5.3	7,820	
16	180.0	Rectangular.	16 × 13.65	3,070	
17	180.0	"	16.2 × 7.0	2,795	
18	180.0	"	17 × 8.75	3,180	

Nos. 13, 14, and 15 were pine of very slow growth.
Nos. 16, 17, and 18 were very green and wet.

TABLE III.

SPRUCE THOROUGHLY SEASONED.

No.	Length, Inches.	Form of Section.	Section Dimensions, Inches.	Ultimate Resistance per Sq. In.
				Lbs.
1	24	Rectangular.	5.4 × 5.4	4,946
2	24	"	5.4 × 5.4	4,811
3	36	"	5.4 × 5.4	4,874
4	36	"	5.4 × 5.4	4,500
5	60	"	5.4 × 6.4	4,451
6	60	"	5.4 × 6.4	4,943
7	120	"	5.4 × 5.4	3,967
8	120	"	5.4 × 5.4	4,908
9	60	"	5.4 × 5.4	5,275
10	30	"	5.4 × 5.4	5,372
11	15	"	5.4 × 5.4	5,754
12	121.2	Circular.	12.4 diam.	4,681

were made during 1881, and a detailed account of them is given in "Ex. Doc. No. 12, 47th Congress, 1st Session."

These experiments give some important deductions.

In the first place, within the limits of the ratio of length to diameter, or shortest side of rectangular section, appearing in these tests, the ultimate resistance is essentially independent of the length. This is the result of the action of causes noticed in the consideration of wrought-iron columns composed of ⊏s. The ultimate resistance of any such column, therefore, is to be obtained by multiplying the area of its cross-section by the ultimate resistance, per square inch, of short blocks.

In Prof. Lanza's experiments the greatest ratio of length to radius of gyration was about 86. Below this value the general conclusion just given may be expected to hold, but probably not much above it.

In Col. Laidley's tests the greatest value of the same ratio was about 90 (No. 17 of Table II), at which there seemed to be a little decrease in ultimate resistance.

Again, it is to be observed that Prof. Lanza's results are much less than those of Col. Laidley for the same timber. The columns of the former were of ordinary merchant material, with the usual accompaniment of knots, weak spots, crooked grain, etc., while the latter experimented with fine, straight-grained timber.

The slow-growth specimens (13, 14, and 15 of Table II) gave much the highest results, while the wet and unseasoned ones (16, 17, and 18) gave the lowest of all.

Hence the ultimate resistance of timber columns will depend upon quality and condition of material, mode of growth, degree of seasoning, etc., etc.

Table II also shows, what has been observed elsewhere, that smaller specimens give higher results than larger ones.

Formula of C. Shaler Smith, C.E.

During the winter of 1861–62, Mr. C. Shaler Smith conducted a series of over 1200 tests of full-size yellow-pine square and rectangular columns for the Ordnance Dept. of the Confederate Government. The results of these tests have never been published, but Mr. Smith has kindly furnished the writer with the following summary.

The tests were grouped as follows:

"1st. Green, half-seasoned sticks answering to the specification 'good, merchantable lumber.'

"2d. Selected sticks reasonably straight and air-seasoned under cover for two years and over.

"3d. Average sticks cut from lumber which had been in open-air service for four years and over."

If l = length of column in inches,
d = least side of column section in inches,
and p = Ult. Comp. resistance in lbs. per sq. in.;

then the formulæ found for these three groups were:

$$\text{For No. 1: } p = \frac{5400}{1 + \frac{1}{250}\frac{l^2}{d^2}}.$$

$$\text{For No. 2: } p = \frac{8200}{1 + \frac{1}{300}\frac{l^2}{d^2}}.$$

$$\text{For No. 3: } p = \frac{5000}{1 + \frac{1}{250}\frac{l^2}{d^2}}.$$

But in order to provide against ordinary deterioration while in use, as well as the devices of unscrupulous builders, Mr. Smith recommends the formula for group No. 3 as the proper one for general application. He also recommends that the factor of safety shall be $\sqrt{\frac{l}{d}}$ until 25 diameters are reached, and *five* thenceforward up to 60 diameters. This last limit he regards as the extreme for good practice.

Mr. Trautwine computed his tables from tests of group No. 3.

Addendum to Article 63.

Tables IV and V have been formed by digesting the results of tests of timber columns made at Watertown, Mass., and found in "Ex. Doc. No. 1, 47th Congress, 2d Session."

Each result in both Tables is usually a mean of from two to four tests, although a few belong to one test only. All timber, both of yellow and white pine, was ordinary merchantable material, with about the usual defects, knots, etc., and failure frequently took place at the latter; it was all well seasoned, and all columns were tested with flat ends.

TABLE IV.

YELLOW-PINE COLUMNS WITH FLAT ENDS.

Length.		Size of Stick, Inches.	$\frac{l}{d}$.	Ultimate Compressive Resistance, Lbs. per Sq. In.	Length.		Size of Stick, Inches.	$\frac{l}{d}$.	Ultimate Compressive Resistance, Lbs. per Sq. In.
Ft.	Ins.				Ft.	Ins.			
15	0	8.25 × 16.25	21.7	3,445	15	0	5.0 × 12	35.6	3,764
10	0	5.5 × 5.5	22	4,738	23	4	7.7 × 9.7	36.4	3,304
16	8	7.7 × 9.7	26.7	4,384	17	6	5.5 × 5.5	38.2	3,242
15	0	6.6 × 15.6	27.0	3,593	15	0	4.5 × 11.6	41	2,462
12	6	5.5 × 5.5	27.3	5,077	26	8	7.4 × 9.4	43	2,893
15	0	5.9 × 12.0	30.8	3,546	15	0	4.0 × 11.4	44	3,065
20	0	7.6 × 9.6	31.2	3,496	20	0	5.4 × 5.4	44.3	2,867
15	0	5.7 × 11.7	31.9	3,106	22	6	5.5 × 5.5	50	2,065
15	0	5.6 × 15.6	32.1	3,656	25	0	5.5 × 5.5	55	1,856
15	0	5.5 × 5.5	32.8	3,962	27	6	5.3 × 5.3	62.3	1,709

Flat-end yellow-pine columns were observed to begin to fail with deflection at a length of about 22*d*, *d* being the width or least dimension of the normal cross-section. All columns were of rectangular section, and *l* in the following table is the length. Table IV, therefore, includes no short column, i.e., one which failed by compression alone with no deflection.

About sixteen of the latter were tested with the following results:

Short yellow-pine columns; $l \div d$ below 22.........	maximum = 5,677	lbs. per sq. in.
	mean = 4,442	" " "
	minimum = 3,430	" " "

Each of the preceding tests was made on a single rectangular stick. A number of tests, however, were made on compound columns formed by bolting together from two to three rectangular sticks, with bolts and packing or separating blocks at the two ends and at the centre. The bolts were parallel to the smaller sectional dimensions of the component sticks. As was to be expected, those compound columns possessed essentially the same ultimate resistance per square inch as each component stick considered as a column by itself, as the following results show. *l* is the length of the column and *d* the smallest dimension or width of one member of the composite column. All had flat ends.

$l \div d$.	Number of Tests.		
32.1.........	18.......	maximum = 4,559	lbs. per sq. in.
		mean = 3,841	" " "
		minimum = 2,756	" " "
36..........	18.......	maximum = 3,357	" " "
		mean = 3,122	" " "
		minimum = 2,942	" " "

Table V gives the results for white-pine columns, and corresponds with Table IV in that it shows only the failures with deflection, which was observed to begin with those columns at a length of 32*d*. *l* and *d* possess the same

TABLE V.

WHITE-PINE COLUMNS WITH FLAT ENDS.

Length.	Size of Stick, Inches.	$\frac{l}{d}$.	Ultimate Compressive Resistance, Lbs. per Sq. In.	Length.	Size of Stick, Inches.	$\frac{l}{d}$.	Ultimate Compressive Resistance, Lbs. per Sq. In.
Ft. Ins.				Ft. Ins.			
15 0	5.6×15.6	32	1,874	17 6	5.4×5.4	40	1,841
20 3	7.4× 9.3	32.4	2,448	26 8	7.5×9.3	42.7	2,113
15 0	5.6×11.5	32.7	2,432	20 0	5.3×5.3	45	1,455
15 3	5.4× 5.4	33	2,744	22 6	5.2×5.2	52	1,501
23 4	7.7× 9.6	36.4	2,072	25 0	5.3×5.3	57	952
15 0	4.5×11.6	40	1,672	27 6	5.4×5.4	62	1,081

signification as in Table IV, the column $l \div d$ showing the ratios between the lengths and least widths.

Thirty columns with lengths less than $32d$ were tested to destruction. These sticks failed generally at knots by direct compression and without deflection. The results of these thirty tests were as follows:

Short white-pine columns; $l \div d$ below 32......... { maximum = 3,700 lbs. per sq. in.
mean = 2,414 " " "
minimum = 1,687 " " "

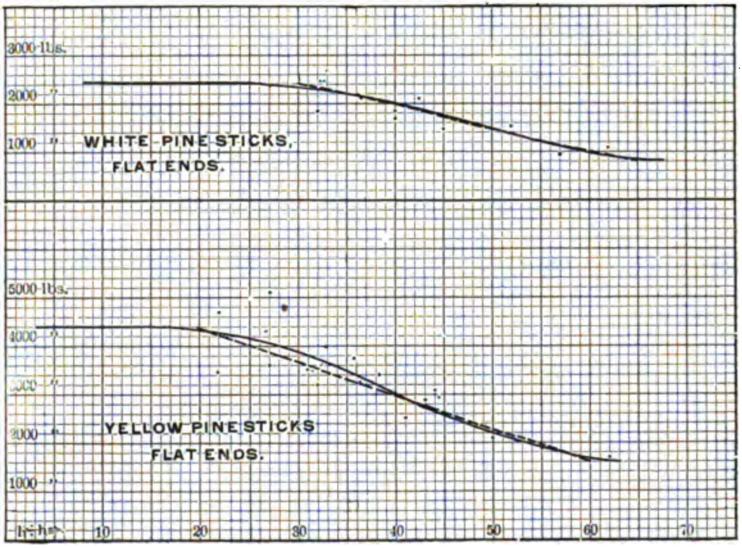
All the preceding white-pine columns were single sticks, but a large number of built posts composed of two to four white-pine sticks bolted together, with spacing blocks at the two ends and at the centre, were also tested with the results shown below. $l \div d$ is the ratio of length over least width of a single stick of the set forming the composite column.

$l+d$.	Number of Tests.	
32.1........	15.......	maximum = 2,273 lbs. per sq. in. mean = 1,980 " " " minimum = 1,661 " " "
36...........	9.......	maximum = 2,255 " " " mean = 1,999 " " " minimum = 1,804 " " "
40...........	6.......	maximum = 2,021 " " " mean = 1,830 " " " minimum = 1,419 " " "

A comparison of these results with those given in Table V shows that these composite or built columns were the same in strength per square inch with the single sticks of which they were composed, the latter being considered single columns.

All the white-pine composite columns were tested with flat ends and were built up with the greatest widths of individual sticks adjacent to each other.

Plate F.



The results in Tables IV and V are shown graphically in Plate F. One ordinate gives the values of $l \div d$, and the other the ultimate resistance in pounds per sq. in.

The full curved lines running into horizontal tangents at the left represent about mean lines through the points indicating the actual column tests.

The broken lines represent the following empirical formulæ; in which p is either the ultimate resistance or working stress in pounds per sq. in.

For yellow pine . . . $p=5800-70(l\div d)$
" white " . . . $p=3800-47(l\div d)$

For wooden railway structures there may be used:

For yellow pine . . . $p=750-9(l\div d)$
" white " . . . $p=500-6(l\div d)$

For temporary structures, such as bridge false works *carrying no traffic:*

For yellow pine . . . $p=1500-18(l\div d)$
" white " . . . $p=1000-12(l\div d)$

The preceding formulæ are to be used only between the limits of $20\frac{l}{d}$ *and* $60\frac{l}{d}$ *for yellow pine and* $30\frac{l}{d}$ *and* $60\frac{l}{d}$ *for white pine.*

For short columns below $20\frac{l}{d}$ and $30\frac{l}{d}$ there are to be used for yellow and white pine respectively:

	Ultimate.	Railway Bridges.	Temporary Structures.
Yellow pine. . . .	$p=4400$.	550.	1100 lbs. per sq. in.
White "	$p=2400$.	300.	600 " " "

All the preceding values are applicable to good average lumber for the engineering purposes indicated.

Table VI exhibits a number of results of the tests of short timber columns taken from the "U. S. Reports of Tests of Metals and Other Materials" for 1894, 1896, 1897, and 1900. It will be observed that the ratios of length over thickness, i.e., minimum dimension of cross-section, are less than 22, and with two exceptions much less. These columns do not, therefore, come within the range of application of such formulæ as those given on the preceding page for yellow pine and white pine.

TABLE VI.

SHORT TIMBER COLUMNS.

Timber.	Dimensions, Inches.			$\frac{l}{d}$.	Ultimate Compressive Resistance, Lbs. per Sq. In.			No. of Tests.	
	Leng'h	Bre'dth	Thickness.		Max.	Mean.	Min.		
Long-leaf pine....	120	9.8	9.8	12	4,976	4,574	4,200	5	Butt sticks.
" "	120	9.8	9.8	12	3,800	3,558	3,369	5	Top "
" "	120	9.8	9.8	12	4,200	3,957	3,714	2	Middle "
Short-leaf pine...	120	9.8	9.8	12	3,925	3,481	3,037	2	Butt "
" " ...	120	9.8	9.8	12	3,400	3,000	2,600	2	Top "
" " ...	120	9.8	9.8	12	4,000	3,568	3,135	2	Middle "
Spruce..........	120	9.6	9.6	12	3,174	2,589	1,900	10	
Long-leaf pine*...	131	9.5	9.5	14	7,354	6,093	4,960	12	Old posts.
Cypress.........	120	8	8	15	3,457	3,308	3,113	3	
White pine......	58	9.5	7.9	8	—	3,652	—	1	Probably 170 years old.
" "	71	9.5	7.9	9	—	2,917	—	1	
Red oak.........	48	4 to 6	3.5	14	6,247	5,160	2,917	5	
Douglas fir.......	60	14	2.8	22	—	6,211	—	1	
" "	60	8 & 14	3	20	7,882	6,725	5,568	2	
" "	60	12	4.1	15	—	6,220	—	1	
White oak.......	121	10	8	15	—	3,697	—	1	
" "	106	10 & 12	10	11	4,230	4,214	4,197	2	
" "	74	7.5	7.9	10	—	4,372	—	1	
" "	69	10	8	9	—	4,042	—	1	

* Well seasoned and dry; 12 years old. Had been in a fire and corners were partially charred.

All posts represented in this table contained probably 15 to 18 per cent. of moisture, or perhaps more.

The long-leaf and short-leaf pine tests show that columns taken from the butts of trees are stronger than those taken either from the middle or the tops, the top sticks, as a rule, having the least ultimate resistance per square inch of all. The white-pine and red-oak sticks yield interesting results on account of their age, as they were taken from some wooden trusses of the Old South Church, Boston, Mass., a building constructed in 1729. The timber was so housed as to be completely protected and kept very dry. The results show no loss of resistance as compared with tests of the same kind of timber at the present time.

The effect of immersion in water on the resistance of timber is illustrated by tests made at the Watertown Arsenal. A post similar to one of the old long-leaf pine columns, 12 of which were tested in a seasoned condition giving the average shown in the Table of 6093 pounds per square inch, was submerged in water for a period of 130 days and then tested with the result of failing at 3800 pounds per square inch.

The values shown in Table VI correspond closely to the results shown for yellow pine and white pine on pages 539 and 540, so far as they may properly be compared.

CHAPTER IX.

SHEARING AND TORSION.

Art. 64.—Coefficient of Elasticity.

It has already been shown in some of the articles of the first portion of this book on shearing and torsion that the coefficients of elasticity for those two stresses are the same; and, indeed, that those two stresses are identical in character. The coefficients of elasticity given in this article are then derived chiefly from experiments in torsion.

A considerable number of coefficients of elasticity for torsion may be found in Table III of Art. 66. These values belong chiefly to a wide range of steel, wrought iron, and cast iron. It is important to observe that the value of the coefficient does not change perceptibly with the grade of steel; it is essentially the same for rivet and crucible steel. Indeed it is practically the same for steel and wrought iron. This is in accordance with the general theory of elasticity in amorphous solid bodies. The values of G, both in this article and in Art. 66, Table III, indicate as a close approximation

$$G = .45E.$$

In "Der Civilingenieur," Heft 2. 1881, the results of some interesting and important experiments on cast-iron rods or prisms of various cross-sections, by Prof. Bauschinger, are given in full detail. The rods or prisms were about 40 inches long, and were subjected to torsion, while the twisting of two sections about 20 inches apart, in reference to each other, was carefully observed. The results for four different cross-sections will be given—i.e., circular, square, elliptical (the greater axis was twice the less), and rectangular (the greater side was twice the less). In each case the area of cross-section was about 7.75 square inches. The angle α is the angle of torsion—i.e., the angle twisted or turned through by a longitudinal fibre whose length is unity and which is at unit's distance from the axis of the bar.

Section.	α.	G.
Circular	0.007 degree	7,466,000 lbs. per sq. in.
	0.07 "	6,157,000 " " "
Elliptical	0.009 "	7,437,000 " " "
	0.076 "	6,228,000 " " "
Square	0.008 "	7,039,000 " " "
	0.073 "	5,987,000 " " "
Rectangular	0.01 "	6,996,000 " " "
	0.08 "	5,716,000 " " "

The formula by which G is computed, when the torsional moment and angle α are given, is the following:

$$G = \frac{M}{\alpha} \cdot c\frac{I_p}{A^4}, \qquad (1)$$

in which M is the twisting moment, A the area of the cross-section, I_p the polar moment of inertia of that cross-section, and c a coefficient which has the following values:

$4\pi^2 = 39.48$ for circle and ellipse,
42.70 " square,
42.00 " rectangle,

as shown in Appendix I.

Bauschinger's experiments show that the coefficient of shearing elasticity for cast iron may be taken from 6,000,000 to 7,000,000 pounds per square inch; also that it varies for different ratios between stress and strain.

It has been shown in Art. 5, that if E is the coefficient of elasticity for direct stress, and r the ratio between direct and lateral strains, for tension and compression, that G may have the following value:

$$G=\frac{E}{2(1+r)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Prof. Bauschinger, in the experiments just mentioned, measured the direct strain for a length of about 4 inches, and the accompanying lateral strain along the greater axis of the elliptical and rectangular cross-sections, and thus determined the ratio r between the direct and lateral strains per unit in each direction. The following were the results:

COMPRESSION.

Section.	r.	G.
Circular	0.22	6,541,000 lbs. per sq. in
Elliptical	0.23	6,541,000 " " "
Square	0.24	6,442,000 " " "
Rectangular	0.24	6,499,000 " " "

TENSION.

Section.	r.	G.
Circular	0.23	6,570,000 lbs. per sq. in.
Elliptical	0.21	6,811,000 " " "
Square	0.26	6,399,000 " " "
Rectangular	0.22	6,527,000 " " "

The values of E are not reproduced, but they can be calculated indirectly from eq. (2) if desired.

It is seen that the values of G, as determined by the different methods, agree in a very satisfactory manner, and thus furnish experimental confirmation of the fundamental equations of the mathematical theory of elasticity in solid bodies.

The fact that G is essentially the same for all sections is also strongly confirmatory of the theory of torsion in particular.

These experiments show that, for cast iron, the lateral strains are a little less than one quarter of the direct strains. If r were one quarter, then $G = \frac{2}{5}E$, or $E = \frac{5}{2}G$.

In the "Journal of the Franklin Institute," for 1873, Prof. Thurston gives the following values of G, as determined from experiments with his torsion machine:

White pine............	$G =$ 220,000	lbs.	per	sq. in.
Yellow pine, sap.......	$G =$ 495,000	"	"	"
Yellow pine, heart......	$G =$ 495,000	"	"	"
Spruce................	$G =$ 211,000	"	"	"
Ash..................	$G =$ 410,000	"	"	"
Black walnut..........	$G =$ 582,000	"	"	"
Red cedar............	$G =$ 890,000	"	"	"
Spanish mahogany.....	$G =$ 660,000	"	"	"
Oak..................	$G =$ 570,000	"	"	"
Hickory..............	$G =$ 910,000	"	"	"
Locust...............	$G =$ 1,225,000	"	"	"
Chestnut.............	$G =$ 355,000	"	"	"

The specimens were small ones, and the timber was seasoned.

Art. 65.—Ultimate Resistance.

Before considering the ultimate shearing resistance of special materials it will be well to notice the two different methods in which a piece may be ruptured by shearing.

If the dimensions of the piece in which the shearing force or stress acts are very small, i.e., if the piece is very thin, the case is said to be that of "simultaneous" shearing. If the piece is thick, so that those portions near the jaws of the shear begin to be separated before those at some distance from it, the case is said to be that of "shearing in detail." In the latter case failure extends gradually, and in the former takes place simultaneously over the surface of separation. Other things being the same, the latter case (shearing in detail), will give the least ultimate shearing resistance per unit of the whole surface.

In reality no plate used by the engineer is so thin that the shearing is absolutely simultaneous, though in many cases it may be essentially so.

Wrought Iron.

The following averages (each result being an average of six tests) are from Chief Engineer Shock's experiments, in 1868, on ordinary commercial "rounds" ("Steam Boilers," by Wiliam H. Shock, Chief Engineer, U. S. N.), in which S is the ultimate shearing resistance in pounds per square inch:

	S.	
Diam. of Round.	Single Shear.	Double Shear.
0.5 inch	44,150 lbs.	41,090 lbs.
0.625 "	39.250 "	38,670 "
0.75 "	39,550 "	39,770 "
0.875 "	41,500 "	37,890 "
1.00 "	40,700 "	37,650 "

Although these figures show some irregularities, the general result is unmistakable, and shows a decrease of S with an increase of diameter.

The results of experiments at Bristol, England, by Mr. Jones ("Proc. Inst. Mech. Engrs.," 1858), on punching plate iron, are as follows:

Thickness of Plate.	Diam. of Hole.	S.
0.437 inch	0.250	54,700 lbs. per sq. in.
0.625 "	0.500	60,900 " " "
0.625 "	0.750	52,900 " " "
0.875 "	0.875	51,700 " " "
1.000 "	1.000	55,100 " " "

Mr. C. Little found the following for English "hammered scrap bars and rolled iron," *with parallel cutters or shears*:

Area Cut.	Direction.	S.
0.50×3.00 ins.	Flat	49,950 lbs. per sq. in.
0.50×3.00 "	Edge	51,750 " " "
1.00×3.00 "	Flat	51,750 " " "
1.00×3.00 "	Edge	50,850 " " "
1.00×3.02 "	Flat	44,350 " " "
1.00×3.02 "	Edge	46,150 " " "
1.80×5.00 "	Edge	46,150 " " "

In these experiments the edges of the shears were always parallel to each other, thus tending to produce simultaneous shearing. In ordinary workshop practice, however, the jaws of the shears make a constant angle with each other, thus shearing successive portions of the material as the jaws approach, whatever may be the dimensions of the piece, and consequently always producing shearing in detail. In the experiments (by the same authority, i.e., Mr. C. Little, "Proc. Inst. Mech. Engrs.," 1858) from which the following results were deduced, the angle between the jaws of the shears was an inclination of 1 in 8:

Bars.		Flatways.	Edgeways.
3 × 1.5	ins.	$S=40,800$	45,000 lbs. per sq. in.
4.5 × 1.375	"	$S=32,000$	40,100 " " "
3.0 × 1.00	"	$S=35,200$	47,300 " " "
5.25 × 1.75	"	$S=37,400$	50,600 " " "
6.00 × 1.50	"	$S=33,600$	41,200 " " "

As was to be expected, the "edgeways" results are much the largest, as with that position of the bar the shearing approached more nearly the simultaneous condition.

There will be found in the articles on riveted joints a large number of results of tests of wrought-iron rivets in shearing. Those tests show that the ultimate shearing resistance of wrought-iron rivets may range from about 34,000 to about 43,000 pounds per square inch when the diameter of rivet varies from ⅝ inch to 1 inch. The entire range of experimental investigation shows that the ultimate shearing resistance of wrought iron may run from .75 to .8 of the ultimate tensile resistance.

Cast Iron.

Very few experiments on the resistance of cast iron to shearing have been made, as this metal is seldom or never used to resist such a stress.

Mr. Bindon B. Stoney ("Theory of Strains in Girders and Similar Structures," p. 357 of 2d Edit.) has found, by experiment, that the ultimate shearing resistance of the cast iron with which he experimented varied from about 17,900 to 20,200 pounds per square inch. He concluded that the shearing and tensile resistances might be taken the same.

Steel.

The results of Prof. Ricketts' shearing tests on both open hearth and Bessemer steel rounds with different grades of carbon are given in Table II of Art. 43. The elastic limit is the point at which the metal first fails to sustain the scale beam. The double-shear resistance in one case exceeds the single by over six per cent. According to these tests, the ultimate shearing resistance of mild steel may be taken at three quarters of its ultimate tensile resistance. Each shear result is a mean of three tests.

The rivet steel was low, containing but .09 per cent. of carbon. While the specimens of Bessemer steel were a little higher in carbon, ranging from .11 to .17 per cent., except the last six, they were also of low or medium steel. It should be carefully noted that the results in that table show that the ultimate shearing resistances for the low or medium steels running from 44,600 pounds per square inch up to 53,260 pounds per square inch are closely three fourths the corresponding ultimate tensile resistances. On the other hand, the six specimens of high steel give ultimate shearing resistances but little over two thirds of the corresponding ultimate tensile resistances. This is a feature of the relation between the ultimate shearing and ultimate tensile resistances of different grades of steel which is commonly exhibited in tests. The high steel appears to yield an ultimate shearing resistance of sensibly less percentage of the tensile ultimate than low steel.

Mr. Kirkaldy investigated the ultimate shearing resistance of four grades of Fagersta steel, and the following results are taken from "Experimental Enquiry into the Mechanical Properties of Fagersta Steel," by David Kirkaldy, 1873. The test-piece, in each case, was turned from a 2-inch square bar, to a diameter of 1.128 inches, and each result is a mean of three experiments. S is the ultimate resistance to shearing in pounds per square inch; r is the ratio of ultimate shearing over ultimate tensile resistance of the same steel; while "d" is the detrusion or relative movement of one part of the specimen in respect to the other at the instant of separation over the entire surface.

Mark.	S.	r.	d.
1.2	61,400.00 lbs.	0.73	0.19 inch.
0.9	71,740.00 "	0.75	0.25 "
0.6	71,650.00 "	0.70	0.28 "
0.3	45,410.00 "	0.54	0.32 "

As is evident, the lower "Mark" numbers belong to the softer steels.

In each case two surfaces were sheared, as the "round" was a pin for three links, two of which pulled one way and one the other.

All of Mr. Kirkaldy's experiments seem to show that the ultimate shearing resistance of steel is about three quarters the tensile.

Table I contains the results of the experiments of Prof. A. B. W. Kennedy, as given in "Engineering" for May 6, 1881. The tensile resistance of the same steel was given in the chapter on "Tension."

The specimens were round and of mild rivet steel. The ratio of the ultimate resistance to shearing over that to tension varied from 0.80 to 0.89.

TABLE I.

RIVET STEEL.

Diameter in Inches.	Ultimate Resistance in Lbs. per Sq. In.	Mean.	Ratio of Ultimate Shear over Ultimate Tension.
1.00	54,110	54,550	0.89
1.00	54,930		
1.00	55,240		
1.00	52,830		
1.00	56,660		
1.00	53,530		
0.62	60,260	59,640	0.87
0.62	59,400		
0.62	59,600		
0.62	59,220		
0.62	59,740		
0.62	53,670	52,450	0.80
0.62	51,290		
0.62	52,670		
0.62	53,000		
0.62	51,620		

In the "Journal of the Franklin Institute" for March, 1881, Charles B. Dudley, Ph.D., gives the results of 192 tests of rail steel, the specimens, 0.625 inch round, having been taken from rails which had been subjected to service for considerable periods of time on the Penn. R. R. The tests were made by Mr. J. W. Cloud, engineer of tests for the Penn. R. R. Co. The following is a summary of the results:

$$S = \begin{cases} 63,560.00 \text{ pounds per sq. in. } (\textit{greatest}). \\ 59,880.00 \text{ pounds per sq. in. } (\textit{mean}). \\ 53,380.00 \text{ pounds per sq. in. } (\textit{least}). \end{cases}$$

The percentages of carbon varied from .282 to .438, showing that the steel was rather high. The ultimate tensile resistances were such that the values of the shearing resistances S were about three fourths of the former.

Table II exhibits the results of the shearing tests of the various grades of steel indicated at the left-hand side of the

TABLE II.

SHEARING OF STEEL.

Kind of Steel.	No. of Tests.	Lbs. per Sq. In.			Ratio of Shearing to Ultimate Tensile.
		Tension.		Ultimate Shearing.	
		Elastic.	Ultimate.		
Landore Siemens....	2	37,500	57,000	47,500	.829
" "	2	40,000	63,500	51,000	.800
" "	3	37,000	64,000	52,000	.811
" "	6	40,600	69,000	56,000	.807
Weardale Bessemer..	6	44,000	71,000	51,000	.715
Bessemer...........	4	51,500	78,000	64,000	.823
"	4	62,000	82,000	59,000	.721
Crucible............	2	69,500	116,000	74,000	.632
Bessemer...........	2	70,000	118,000	79,000	.670

table. The ultimate tensile resistances run from 57,000 pounds per square inch for the low steel up to 118,000 pounds per square inch for the high steel. It is noticeable that the ratio of the ultimate shearing resistance over the ultimate tensile varies from about .83 for the low steel down to about .63 for the high steel, confirming the results already indicated. The results of Table II are taken from the "Proceedings of the Institution of Mechanical Engineers for 1885," the tests having been made by Prof. A. B. W. Kennedy. On the whole, it may be stated that in the low rivet steels the ultimate shearing resistance may be taken at .8 of the ultimate tensile resistance, but in the high steels that ratio may fall below .65. As an average the ratio of the ultimate shearing intensity over the ultimate tensile intensity may be taken at .75 for the usual grades of structural steel.

Further results of the shearing of steel in rivets may be found in the articles on riveted joints; they confirm those given in this article.

Copper.

From some English experiments, Mr. Bindon B. Stoney concluded that the ultimate shearing resistance of copper was about two thirds of that of wrought iron, but the shearing resistance of copper must necessarily depend on the grade of metal, i.e., whether cast, wrought, cold drawn, or otherwise treated.

Timber.

In treating the shearing resistance of timber, it is necessary to consider whether the shearing takes place *along* the fibres or in a direction *normal* to them.

TABLE III.

ACROSS FIBRES.

Kind of Wood.	*S.*	Kind of Wood.	*S.*
Ash.	6,280	Locust.	7,176
Beech.	5,223	Maple.	6,355
Birch.	5,595	Oak, white.	4,425
Cedar, white.	1,372 to 1,519	Oak, live.	8,480
Cedar, Central American	3,410	Pine, white.	2,480
Cherry.	2,945	Pine, yellow, Northern.	4,340
Chestnut.	1,535	Pine, yellow, Southern.	5,735
Dogwood.	6,510	Pine, yel., very resinous	5,053
Ebony.	7,750	Poplar.	4,418
Gum.	5,890	Spruce.	3,255
Hemlock.	2,750	Walnut, black.	4,728
Hickory.	6,045 to 7,285	Walnut, common.	2,830

Table III contains the results of experiments by Mr. John C. Trautwine on round specimens 0.625 inch in diameter, and *across* the fibres ("Journal of the Franklin Institute," Feb., 1880). As before, *S* is the ultimate shearing resistance in pounds per square inch.

Table IV has been condensed from the results of Col. Laidley's tests at the Watertown arsenal (" Ex. Doc. No. 12, 47th Congress, 1st Session"). Usually, two such results have been selected as will give a correct idea of the resistance. In all cases except Nos. 19, 20, 23, and 33, the smaller resistance belongs to the larger shearing surface. In No. 33 the smaller resistance belongs to an unsatisfactory experiment.

TABLE IV.

No.	Kind of Material.	Shearing Area in Square Inches.	Ultimate Shear in Lbs. per Sq. Inch.	Directions to Rings of Growth.
1	Oregon pine	5.0 and 14.0	442 and 1,096	Perpendicular (2 exps.)
2	Oregon pine	10.7	820	Oblique
3	Oregon maple	14.4	436	Perpendicular
4	Oregon maple	10.9	1,028	Oblique
5	California laurel	11.0 and 14.2	549 and 1,204	Oblique (2 exps.)
6	Ava Mexicana	14.8	346	Perpendicular
7	Ava Mexicana	11.0	700	Parallel
8	Oregon ash	14.6	443	Parallel
9	Oregon ash	8.2	1,126	Perpendicular
10	Mexican white mahogany	11.0 and 15.1	438 and 1,000	Oblique (2 exps.)
11	Mexican cedar	15.0	423	Perpendicular
12	Mexican cedar	9.8	814	Parallel
13	Mexican mahogany	15.0	566	Parallel
14	Mexican mahogany	11.1	1,333	Perpendicular
15	Oregon spruce	22.9 and 34.5	261 and 356	Parallel (2 exps.)
16	Oregon spruce	5.5	315	Perpendicular
17	White pine	3.0 and 16.0	381 and 423	Perpendicular (2 exps.)
18	White pine	16.0 " 24.0	324 " 352	Parallel (2 exps.)
19	Whitewood	7.3 " 11.0	127 " 370	Oblique (2 exps.)
20	Whitewood	10.8 " 21.9	328 " 481	Parallel (2 exps.)
21	Whitewood	21.6 " 32.3	322 " 385	Perpendicular (2 exps.)
22	Yellow pine	13.0 " 13.1	317 " 399	Oblique (2 exps.)
23	Yellow pine	17.0 " 25.4	286 " 409	Perpendicular (2 exp.)
24	Ash	16.3 " 24.4	592 " 600	Parallel (2 exps.)
25	Ash	16.2 " 16.0	458 " 700	Perpendicular (2 exps.)
26	Red oak	16.0 " 23.9	743 " 745	Perpendicular (2 exps.)
27	Red oak	16.0 " 24.0	726 " 999	Parallel (2 exps.)
28	White oak	16.2 " 24.0	803 " 966	Parallel (2 exps.)
29	White oak	15.8 " 24.0	752 " 846	Perpendicular (2 exps.)
30	Yellow birch	17.0 " 17.0	563 " 815	Oblique (2 exps.)
31	Yellow birch	25.6 " 25.6	612 " 672	Perpendicular (2 exps.)
32	White maple	16.0	647	Oblique
33	White maple	15.9 and 24.0	399 and 537	Perpendicular (2 exps.)
34	Spruce	15.8 " 23.8	235 " 347	Parallel (2 exps.)
35	Spruce	16.0 " 24.0	316 " 374	Perpendicular (2 exps.)

The direction of shear is seen to be either along the fibres or tangential to the rings of growth, in the latter case being either perpendicular or oblique to the fibres. When the shear is along the fibres the results are generally reasonably in harmony, for the same timber, with those in Table V.

The values shown in Table V were found by Prof. J. B. Johnson in his experimental work for the Department of Agriculture, and they have been taken from "U. S. Forestry Circular, No. 15." As indicated in the table the results belong to unseasoned timber, although the specimens were selected and were not large. Each value is a mean of a number of tests.

TABLE V.

SHEARING RESISTANCE OF TIMBER ALONG FIBRES.

No.	Species.	Shearing with Grain not Reduced for Moisture.	No.	Species.	Shearing with Grain not Reduced for Moisture.
1	Long-leaf pine.......	700	17	Black oak...........	1,100
2	Cuban pine..........	700	18	Water oak...........	1,100
3	Short-leaf pine.......	700	19	Willow oak...........	900
4	Loblolly pine........	700	20	Spanish oak.........	900
5	White pine..........	400	21	Shagbark hickory.....	1,100
6	Red pine............	500	22	White hickory........	1,100
7	Spruce pine.........	800	23	Water hickory........	1,000
8	Bald cypress........	500	24	Bitternut hickory.....	1,000
9	White cedar.........	400	25	Nutmeg hickory......	1,100
10	Douglas spruce......	500	26	Pecan hickory........	1,200
11	White oak..........	1,000	27	Pignut hickory.......	1,200
12	Overcup oak........	1,000	28	White elm...........	800
13	Post oak............	1,000	29	Cedar elm............	1,300
14	Cow oak............	900	30	White ash.............	1,100
15	Red oak............	1,100	31	Green ash............	1,000
16	Southern red oak.....	900	32	Sweet gum...........	800

All the preceding values belong to small specimens or selected sticks, as has been observed; consequently they are generally somewhat higher than can be justified for use in engineering practice. The amount of reduction necessary to meet the requirements of that usage is a matter of judgment based on experience. Probably as reasonable values as can be assigned both for ultimate resistances and for working stresses under ordinary circumstances are those given in Table VI. They are taken from the "Report of the Committee on Strength of Bridge and Trestle Timbers," for 1895, of the Association of Railway Superintendents of Bridges and Buildings to which allusion has been made in preceding articles. The working stresses (i.e., greatest permissible in design) are taken at one fourth of the ultimate. This fraction needs to be varied for different classes of work.

TABLE VI.

ULTIMATE RESISTANCES AND WORKING STRESSES FOR TIMBER IN SHEARING.

Timber.	Pounds per Square Inch.			
	Ultimate Resistance.		Working Stresses.	
	Across Grain.	With Grain.	Across Grain.	With Grain.
White oak	4,000	800	1,000	200
White pine	2,000	400	500	100
Southern long-leaf or Georgia yellow pine	5,000	600	1,250	150
Douglas and yellow fir	——	600	——	150
Northern or short-leaf yellow pine	4,000	400	1,000	100
Canadian (Ottawa) white pine	——	350	——	100
Canadian (Ontario) red pine	——	400	——	100
Spruce and Eastern fir	3,000	400	750	100
Hemlock	2,500	350	600	100
Cedar	1,500	——	400	——
Chestnut	1,500	600	400	150
California redwood	——	400	——	100

Natural Stones.

The ultimate shearing resistance of stones has not as great practical value as the ultimate compressive or the ultimate bending resistance, yet there are occasional structural conditions under which it is necessary to ascertain what shearing capacity may be relied upon. Valuable data for this purpose are shown in Table VII taken from the "U. S. Report of Tests of Metals and Other Materials" for 1894 and 1899. The sheared surfaces were about 6 inches by 4 inches in area. Generally one such surface was sheared, but occasionally two.

TABLE VII.

SHEARING RESISTANCE OF NATURAL STONES.

Stone.	Ultimate Shearing Resistance, Lbs. per Square Inch.		
	Maximum.	Mean.	Minimum.
Brandford granite, Conn.	1,925	1,834	1,742
Milford Granite, Mass.	2,872	2,554	2,236
Troy granite, N. H.	2,231	2,219	2,197
Milford pink granite, Mass.	——	1,825	——
Pigeon Hill granite, Mass.	2,047	1,549	1,052
Creole marble, Ga.	——	1,369	——
Cherokee marble, Ga.	——	1,237	——
Etowah marble, Ga.	——	1,411	——
Kennesaw marble, Ga.	——	1,242	——
Marble Hill marble, Ga.	1,501	1,332	1,163
Tuckahoe marble, N. Y.	1,554	1,490	1,426
Mount Vernon limestone, Ky.	2,016	1,705	1,389
Cooper sandstone, Oregon.	——	1,831	——
Maynard sandstone, Mass.	1,287	1,204	1,120
Kibbe sandstone, Mass.	1,308	1,150	992
Worcester sandstone, Mass.	1,383	1,243	1,102
Chuckanut sandstone, Wash.	——	1,352	——
Yammerthal limestone, Buffalo.	2,518	2,127	1,735

All the results except the last are taken from the Report for 1894. Where but one value appears in the table one test only was made. In the other cases two tests were made and the mean values are means of the two shown in the columns containing the greatest and least. It will be observed that the ultimate shearing resistance is scarcely more than ten per cent. of the ultimate compressive resistance of the various stones tested.

The greatest permissible working stresses for natural stones in shear, in design work, will necessarily depend on the duty to be performed. In view of the variable character of even the best of natural stones as delivered ready for use, one eighth to one tenth of the ultimate is as much as should be taken in ordinary cases, and materially less than that under some conditions.

Bricks.

The shearing resistance of bricks, like that of natural stones, is seldom employed, but it is sometimes needed. The ultimate resistances of bricks in shearing shown in Table VIII are taken from the "U. S. Report of Tests of Metals and Other Materials" for 1894.

TABLE VIII.

BRICKS IN SHEARING.

Kind of Brick.	Ultimate Shearing Resistance, Lbs. per Square Inch.	Number of Sheared Surfaces.
Hydraulic Press Brick Co., St. Louis, No. 6	1,011	1
" " " " " " 511	642	2
" " " " " brown	1,047	1
" " " " Chicago, red	784	1
Northern Hydraulic Press Brick Co., Minneapolis, dark red	714	1
Eastern " " " " Philadelphia, 210	1,167	1
" " " " " " 220	1,097	1
" " " " " " 390	988	2
Philadelphia and Boston Face Brick Co., Boston, gray	433	2
" " " " " " " "	639	1

In these shearing tests the sheared surfaces were each about 2.25 by 4 inches in dimensions.

The ultimate shearing resistances in Table VIII range scarcely 10 per cent. of the ultimate compressive resistances of the same materials shown in Table II of Art. 55.

Working shearing stresses for design operations should not be taken more than one eighth to one tenth of the ultimate values found in Table VIII.

Art. 66.—Torsion.

Coefficients of Elasticity.

The coefficients of elasticity for torsion or shearing can be found either in Art. 64 or in the Addendum to this article.

Ultimate Resistance and Elastic Limit.

WROUGHT IRON.

In 1866 Mr. Kirkaldy tested four hammered Swedish iron bars turned to a diameter of 1.5 inches for a length of seven diameters. The average ultimate moment of torsion was produced by a weight of 2,677 pounds with a leverage of 12 inches; hence, in eq. (20) of Art. 8, $M = 2,677 \times 12 = 32,124$. Putting $2r_0 = d = 1.5$ inches in that equation, there will result

$$T_m = 5.1\frac{M}{d^3} = 48,540 \text{ pounds per square inch.}$$

This is the greatest intensity of torsional shear in the section.

If T_m be taken at 48,000 the diameter of a wrought-iron shaft required to resist an ultimate moment M will be

$$d = 0.047\sqrt[3]{M}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If the working moment be taken at one eighth the ultimate, the diameter required will be

$$d = 0.047\sqrt[3]{8M_1} = 0.094\sqrt[3]{M_1}, \quad . \quad . \quad . \quad . \quad (2)$$

in which M_1 is the working moment.

If H is the number of horse-powers per minute to be transmitted by the shafting, and n the number of revolutions which it is to make,

$$M_1 = \frac{12 \times 33000}{2\pi} \cdot \frac{H}{n}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Putting this value in eq. (2),

$$d = 3.742\sqrt[3]{\frac{H}{n}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

This value of d will be much too small in the case of long shafting required in the distribution of power, in consequence of the bending caused by the belting.

The mean torsional moment at the elastic limit, in Mr. Kirkaldy's four experiments, was about 0.4 the ultimate.

In 1846 Major Wade ("Experiments on Metals for Cannon") tested three wrought-iron circular cylinders about 1.9 inches in diameter and 15 inches long, with the following results:

$$T_m = \frac{5.1M}{d^3} = 28{,}325 \text{ lbs. per sq. in.}$$
$$= 27{,}525 \text{ " " "}$$
$$= 27{,}800 \text{ " " "}$$
$$\overline{83{,}650}$$
$$\text{Mean} = 27{,}900 \text{ (nearly).}$$

If the mean be taken at 28,000,

$$d = 0.056\sqrt[3]{M}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

It is seen that Major Wade found T_m much less than Kirkaldy's value for Swedish iron, and d in eq. (5) is correspondingly greater. If H and n carry the same signification as before, and if 8 is the safety factor,

$$d = 4.49\sqrt[3]{\frac{H}{n}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

In all these results, the moments are supposed to be given in *inch-pounds*, and the resulting values of d are consequently in inches.

CAST IRON.

Major Wade also made tests on circular cylinders of cast iron about 1.9 inches in diameter and 15 inches long.

If d is the diameter $= 2r_0$ in eq. (20) of Art. 8, he found the following results with the grades of iron shown:

2d fusion............ $T_m = 31{,}500$ lbs. per sq. in.
3d fusion............ $T_m = 44{,}775$ " " "
2d and 3d fusion..... $T_m = 49{,}735$ " " "
2d fusion............ $T_m = 40{,}020$ " " "
3d fusion............ $T_m = 53{,}380$ " " "
2d fusion............ $T_m = 49{,}526$ " " "
3d fusion............ $T_m = 46{,}230$ " " "
Mean $= 45{,}000$ (nearly).

Hence the diameter in inches for the ultimate moment M in *inch-pounds* is:

$$d = \sqrt[3]{\frac{5.1}{45000} M} = 0.048\sqrt[3]{M}. \quad . \quad . \quad . \quad . \qquad (7)$$

These values of T_m are very high, because the iron with which Major Wade experimented was evidently of a special character and extraordinarily strong.

The same experimenter tested some square sections, for which, by the exact theory of Appendix I:

$$T = 5\frac{M}{b^3}; \ (b = \textit{side of square}). \quad . \quad . \quad . \qquad (8)$$

The following are from Major Wade's results:

$b = 1.00$ inches; $M = 8,750$ inch-pounds; $T_m = 43,750$ pounds.
$b = 1.42$ inches; $M = 23,000$ inch-pounds; $T_m = 40,210$ pounds.
$b = 1.75$ inches; $M = 54,000$ inch-pounds; $T_m = 50,370$ pounds.

The mean of these results is: $T = 44,800$ (nearly). Hence for the ultimate moment in *inch-pounds*

$$b = \sqrt[3]{\frac{5M}{44800}} = 0.0481\sqrt[3]{M}. \quad . \quad . \quad . \quad . \quad (9)$$

It is to be observed that, according to these experiments, T_m is the same for circular and square sections; a result very different from that of Prof. Bauschinger's experiments, as will presently be seen.

Four of Major Wade's experiments on hollow circular cylinders are next to be given.

By referring to eq. (20) of Art. 8, the resisting moment of such a cylinder, if d is the external and d_1 the internal diameter, will be

$$M = \frac{T_m d^3 - T'_m d_1^3}{5.1} = \frac{T_m}{5.1d}(d^4 - d_1^4). \quad . \quad . \quad (10)$$

$$\therefore \quad T_m = \frac{5.1dM}{d^4 - d_1^4}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

For the first case:

$d = 3.25$ ins.; $d_1 = 2.61$ ins.; $M = 95,000$ in.-lbs.

Substituting in eq. (11):

$T_m = 24,170$ lbs. per sq. in. (nearly).

For the second case:

$d = 2.21$ ins.; $d_1 = 1.54$ ins.; $M = 49,500$ in.-lbs.

Substituting in eq. (11):

$T_m = 30,610$ lbs. per sq. in. (nearly).

For the third case:

$d = 1.81$ ins.; $d_1 = 0.91$ in.; $M = 37,250$ in.-lbs.

Substituting in eq. (11):

$$T_m = 34,220 \text{ lbs. per sq. in. (nearly).}$$

For the fourth case:

$d = 1.30$ ins.; $d_1 = 0.65$ in.; $M = 13,000$ in.-lbs.

Substituting in eq. (11):

$$T_m = 32,180 \text{ lbs. per sq. in. (nearly).}$$

These results indicate that T_m decreases as the thickness of the wall of the hollow cylinder decreases and as the exterior diameter increases.

Professor Bauschinger ("Der Civilingenieur," 1881, Heft 2) tested cylinders about 40 inches long, and with the following cross-sections and approximate dimensions:

Circle...........	Diameter =	3.25	inches.
Ellipse..........	Diameters =	2.30 4.40	" "
Square..........	Sides =	3.00 3.00	" "
Rectangle.......	Sides =	2.04 4.10	" "
Rectangle.......	Sides =	1.02 4.10	" "

The ultimate twisting moments substituted in eqs. (83), (41), (73), (75), and (77) of Art. 6, Appendix 1, give

For circle.................. $T_m = 27,730$ lbs. per sq. in.
For ellipse.................. $T_m = 36,120$ " " "
For square.................. $T_m = 37,160$ " " "
For rectangles (sides 2 to 1)... $T_m = 36,370$ " " "
For rectangles (sides 4 to 1)... $T_m = 37,090$ " " "

These experiments give T_m considerably less value for the circular cross-section than for the others.

The U. S. Board, however, found the following values for four cast-iron cylinders one inch long and 0.565 inch in diameter:

T_m = 35,980, 34,110, 34,280, and 33,770 lbs. per sq. in.

Elas. Lim. = 60, 55, 64, and 62 per cent. of T_m, respectively.

STEEL.

In connection with the torsional resistance of steel, tests of circular cylinders only are to be considered. Those first to receive attention were made by Mr. Kirkaldy on English steel, in 1866–1870, and the results have been deduced from his data.

As the sections are all circular, eq. (20) of Art. 8 is the only one needed:

$$T_m = \frac{5.1M}{d^3}. \qquad (12)$$

In this equation T_m is the greatest intensity of torsional shear, in any section, in pounds per square inch, "d" the diameter of the shaft or cylinder in inches, and M the twisting moment in *inch-pounds.*

In all the following experiments the lever arm of the twisting couple was 12 inches; hence if P is the twisting force, $M = 12P$, and eq. (12) becomes

$$T_m = \frac{61.2P}{d^3}. \qquad (13)$$

The mean of four experiments with Bessemer steel gave for the ultimate resistance

$$P = 2{,}307 \text{ lbs., with } d = 1.25 \text{ inches;}$$
$$\therefore T_m = 72{,}298 \text{ lbs. per sq. in.} \qquad (14)$$

The length was 10 inches.

The mean of some results with Krupp's cast steel in specimens 1.25 inches in diameter, and 2.5 inches for torsion length, gave

$$P=2{,}867 \text{ lbs}; \quad \therefore \quad T_m=89{,}847 \text{ lbs.} \quad . \quad . \quad (15)$$

The following set of results were obtained from 2-inch square bars turned down to 1.382 inches in diameter for a length of 11 inches, and gives the means of the number of tests indicated.

Specimens.		P (Ultimate).	T_m (Ultimate).	Elastic Strain.
5 Hammered tires,	Bessemer Steel.	....3,450 lbs.	....80,006 lbs.	....0.014
5 " axles,		....3,293 "	....76,365 "	....0.011
4 " rails,		....3,248 "	....75,321 "	....0.012
4 Rolled tires, axles, and rails.		....3,226 "	....74,811 "	....0.008
5 Hammered tires,	Crucible Steel.	....3,562 "	....82,603 "	....0.014
4 " axles,		....3,786 "	....87,797 "	....0.013
1 " rail,		....4,054 "	....94,012 "	....0.016
1 Rolled rail.		....3,002 "	....69,616 "	....0.012

(16)

The elastic strain is the fraction of a complete turn made by the specimen at the elastic limit.

The mean of the Bessemer steels in (14) and (16) give

$$\text{Mean } T_m=75{,}760 \text{ lbs. per sq. in.}$$

Hence if M is the breaking moment of the twisting couple in *inch-pounds*, the following will be the diameter of the shaft in inches:

$$d=\sqrt[3]{\frac{5.1M}{75760}}=0.0407\sqrt[3]{M}; \quad . \quad . \quad . \quad . \quad (17)$$

or if n is the safety factor, and M_1 the greatest working moment,

$$d=0.0407\sqrt[3]{nM_1}. \quad . \quad . \quad . \quad . \quad . \quad (18)$$

The mean of the crucible steel results in (16), with the exception of the last, is

$$\text{Mean } T_m=88{,}140 \text{ lbs.}$$

Hence the diameter (in inches) of the shaft which will just sustain the breaking moment M, in *inch-pounds*, is

$$d=\sqrt[3]{\frac{5.1M}{88140}}=0.0387\sqrt[3]{M}; \quad . \quad . \quad . \quad . \quad (19$$

or, if n is the safety factor, and M_1 the greatest working moment,

$$d=0.0387\sqrt[3]{nM_1}. \quad . \quad . \quad . \quad . \quad . \quad (20)$$

In all the preceding experiments the elastic limit varied from 40 to 47 per cent. of T_m (ultimate) as given in (14), (15), and (16).

In 1873 Kirkaldy made some experiments on specimens of Fagersta steel which possessed a length of about 9 inches and a diameter of 1.128 inches, the length of the twisting lever being still 12 inches. Eq. (13) then gives the following results, each being a mean of three tests:

Mark.	P (Ultimate).	T_m (Ultimate).	Strain.
1.2	2,120 lbs.	90,397 lbs.	0.[illegible]9
0.9	2,336 "	99,607 "	0.79
0.6	2,261 "	96,409 "	1.02
0.3	1,520 "	64,813 "	3.22

The "*strain*" is the number of complete turns made by the specimen at the place and instant of rupture.

The specimens with the higher "mark" numbers were the higher steels.

The elastic limit varied from 46 to 58 per cent. of the ultimate T_m.

The diameter of a shaft for any of these grades may readily be computed by the use of these values of T_m in equations similar to eqs. (17) to (20).

The following values were determined by the Committee on Chemical Research of the U. S. Board, "Ex. Doc. 23, House of Rep., 46th Congress, 2d Session," with specimens 1 inch long turned to diameters of 0.625 and 0.565 inch, and tested in a Thurston machine:

T_m.	Elastic Limit in Per Cent. of T_m.	Ultimate Angle of Torsion.	
100,990 lbs. per sq. in.	34	149°.0	
95,230 " " "	34	142°.3	
110,260 " " "	33	68°.4	
115,780 " " "	42	56°.1	
52,375 " " "	34	278°.2	
71,420 " " "	45	220°.8	
88,210 " " "	39	99°.5	
55,885 " " "	35	165°.0	
119,040 " " "	40	84°.9	
75,430 " " "	44	180°.7	
91,690 " " "	39	53° to 113°	Tool steels; annealed.
96,450 " " "	36	48° to 84°	
109,010 " " "	29	61° to 143°	
107,315 " " "	32	42° to 123°	
109,590 " " "	32	73° to 141°	

Each of the last five results is a mean of eight tests.

The first portion of these results would possess more value if the test specimens had been larger.

With these values of T_m, the diameter of a shaft, with the torsion moment M in inch-pounds, becomes

$$d = \sqrt[3]{\frac{5.1M}{T_m}} = 1.721\sqrt[3]{\frac{M}{T_m}}.$$

Copper, Tin, Zinc, and their Alloys.

The following values of T_m have been computed by the aid of eq. (12) from data determined by Prof. R. H. Thurston, and given by him in the works already cited in connection with tension and compression. The test specimens were 0.625 inch in diameter, with a torsion length of 1.00 inch, and were tested in his torsion machine. The ultimate shearing resistances of these alloys in torsion are thus seen to vary as widely as their tensile resistances.

TABLE I.

Composition.		Ultimate Torsive Shear, T_m.	Elastic Limit, Per Cent of T_m.	Ultimate Torsion Angle.
Cu.	Sn.			
		Pounds.		Degrees.
100	00	35,910	35	153.0
100	00	28,430	40	52 to 154
00	100	3,196	45	557.0
00	100	3,297	33	691.0
90	10	43,943	41	114.5
80	20	47,671	62	16.3
70	30	4,407	100	1.5
62	38	1,770	100	1.0
52	48	686	100	1.0
39	61	5,881	100	1.7
29	71	5,257	100	2.34
10	90	5,761	63	131.8
90	10	25,027	49	57.2
90	10	31,851	57	72.6

T_m is in pounds per square inch.

Table I relates to alloys of copper and tin, and Table II to alloys of copper and zinc.

None but specimens with circular sections were tested.

TABLE II.

Percentage of		Ultimate Torsive Shear, T_m.	Elastic Limit, Per Cent of T_m.	Ultimate Torsion Angle.
Copper.	Zinc.			
		Pounds.		Degrees.
90.56	9.42	35,100	17.2	458.0
81.90	17.99	41,575	27.5	345.0
71.20	28.54	41,000	24.0	269.0
60.94	38.65	48,520	29.4	202.0
55.15	44.44	52,320	32.7	109.0
49.66	50.14	43,154	36.0	38.0
41.30	58.12	4,588	100.0	1.8
32.94	66.23	7,241	100.0	1.2
20.81	77.63	16,374	100.0	0.8
10.30	88.88	22,500	85.6	7.1
0.00	100.00	9,186	38.1	141.5

With the preceding values of T_m, the following expression for the diameter in inches may be written, if M is given in *inch-pounds*:

$$d=\sqrt[3]{\frac{5.1M}{T_m}}=1.721\sqrt[3]{\frac{M}{T_m}}.$$

TIMBER.

In the July, 1873, number of Van Nostrand's Magazine, Prof. Thurston gave the results of some experiments on timber test specimens of circular section, 0.875 inch in diameter. Eq. (12) may be written as follows:

$$M=\frac{T_m}{5.1}d^3=Cd^3. \qquad (21)$$

Prof. Thurston determined the values of C, and the values of $T_m = 5.1\ C$ have been computed from them:

	T_m (Per Eq. In.).
White pine	1,530 lbs.
Yellow pine, sap	2,142 "
Yellow pine, heart	2,448 "
Spruce	1,836 "
Ash	2,632 "
Black walnut	3,366 "
Red cedar	1,958 "
Spanish mahogany	3,978 "
Oak	3,244 "
Hickory	5,202 "
Locust	4,896 "
Chestnut	2,142 "

It is presumed that the axis of torsion was parallel to the fibres, which would cause the shear to take place across the latter.

It is interesting to observe that T_m is generally considerably less than the ultimate resistance to simple shear as given in Table III of Art. 65.

If d is in inches and M in *inch-pounds*, there may again be written

$$d=\sqrt[3]{\frac{5.1M}{T_m}}=1.721\sqrt[3]{\frac{M}{T_m}}.$$

If M is given in *foot-pounds*, $12M$ is to be written for M. If M_1 is the greatest working moment, and n the safety factor, nM_1 is to be written for M.

Relation between Ultimate Resistances to Tension and Torsion.

In the "Trans. Am. Soc. of Civ. Engrs.," Vol. VII, 1878, Prof. Thurston gave the results of some of his experiments which were made with a view to the determination of this relation. If M is the ultimate torsional moment in *foot-pounds* of specimens one inch long and 0.625 inch in diameter; θ the angle of torsion corresponding to this greatest moment M; and T the ultimate tensile resistance in pounds per square inch, he deduced from a large number of steel specimens of wide range in grades the following formula:

$$T=M\left(\frac{900-\theta}{3}\right).$$

No experiments were made in which θ was greater than 300°.

T is thus seen to increase as M increases and as θ decreases.

Addendum to Article 66.

There may be found in Vol. XC of the "Proceedings of the Institution of Civil Engineers" for 1886–87 a paper by Messrs. J. Platt and R. F. Hayward, containing comparative experimental values of the elastic limits and ultimate

TABLE III.

TORSION AND SHEARING.

Metal.	All Values are in Pounds per Square Inch.				
	$E.L.$	T_m.	G.	Ultimate Tensile Resistance.	S.
Wrought iron: Crown best	20,530	56,400	12,800,000	48,400	42,050
Bessemer steel..........	45,400	100,000	12,880,000	116,930	78,880
Crucible steel...........	43,300	94,900	13,660,000	116,850	74,500
Landore rivet steel: .18 C., .6 Mg...............	22,900	66,970	13,070,000	64,060	51,570
Crown rivet iron.........	23,250	64,630	13,700,000	56,020	47,870
Steel casting............	23,400	77,820	13,040,000	85,720	60,160
Siemens-Martin steel.....	22,600	62,960	13,400,000	57,700	46,900
Wrought iron: S. C. Crown	22,920	66,160	13,920,000	55,010	46,510
Muntz metal............	19,500	58,500	6,790,000	57,020	42,000
Gun metal: Cu, 64; zinc, 2; tin, 8..............	12,100	35,500	5,610,000	30,720	27,960
Cast iron, turned........	——	33,040	7,163,000	25,610	11,860
" " "	——	38,200	6,602,000	——	11,420
" " skin on........	——	40,600	7,620,000	——	8,810
" " " "........	——	34,330	7,050,000	——	——

$E.L.$ indicates elastic limit in torsion;
T_m " ultimate torsion shearing resistance;
G " coefficient of elasticity in torsion;
S " ultimate resistance in simple shear.

resistances in tension, torsion, and shearing, as well as the coefficients of elasticity in torsion of steel, wrought iron, and other metals, and Table III contains the principal results. Each value in the table is the mean of two to six or more tests. All the torsion specimens except those in cast iron were $\frac{5}{8}$ inch in diameter for a length of 13 inches. Some of the cast-iron specimens were $1\frac{1}{8}$ inch in diameter and were tested as cast, i.e., they were not turned down. The remaining cast-iron specimens were turned down to $\frac{5}{8}$ inch in diameter.

It will be observed that the ultimate shearing resistance, S, is about three fourths only of the ultimate torsion shear T_m, although it might reasonably be anticipated that they would be the same. Messrs. Platt and Hayward made their simple shearing tests by turning grooves one inch apart in inch rounds with a round-nosed tool, thus securing double shear at two adjacent sections located by the grooves. This arrangement would produce some bending at each section of failure and almost certainly prevent equal division of the total shearing loads between the two sections, thus, in a measure, producing failure in detail. Both these influences would reduce the apparent ultimate shearing intensities S. The latter intensities (given in Table III) cannot, therefore, be accepted as correct.

The high values of the ultimate tensile resistance of the cast iron and of the ultimate T_m, in the same table, show conclusively that the cast iron employed in these tests was of far greater resistance than that ordinarily used in engineering structures which has an ultimate tensile resistance of 16,000 to 18,000 pounds per square inch. It is probably essentially correct to take the ultimate torsion shear T_m proportionate to the ultimate tensile resistance.

The coefficient of torsion elasticity, G, was computed by eq. (17) of Art. 8, P_e being the twisting moment M_t and the strain ϕ_0 being measured on the specimen when under test.

These and other similar tests show that G may be taken as $.45E$ (E being the coefficient of elasticity for tension) with essential accuracy.

Table IV contains experimental values of the elastic limit and ultimate torsion shearing resistance of the alloys of aluminum, tin, and copper shown in the table. They were determined by Messrs. Gebhardt and Ward in 1896 in the mechanical laboratory of Sibley College at Cornell

University and reported to the Am. Soc. Mech. Engrs. in 1898.

TABLE IV.

ALUMINUM ALLOYS—TORSIONAL RESISTANCE.

Composition Per Cent. by Weight.			Angle of Torsion Deg.		Ultimate Shearing Stress per Sq. In.		General Character.
Al.	Tn.	Cu.	Elastic Limit.	Maxi-mum.	Elastic Limit.	Maxi-mum.	
——	——	100	2	130	4,300	25,000	
2.5	2.5	95	4	200	10,710	33,075	Very soft; ductile.
3.75	3.75	92.5	6	198	11,827	35,802	Soft; ductile.
5	5	90	7	175	15,525	45,155	Slightly tough; ductile.
6.25	6.25	87.5	4	37	30,282	63,440	Tough: medium ductility.
7.5	7.5	85	3.5	22	25,447	37,062	Very tough; rather hard.
8.75	8.75	82.5	7	10	18,413	18,413	Hard; somewhat brittle.
10	10	80	6	8	15,230	15,230	Very hard; brittle.
*11	11	78	5.8	5.8	13,717	13,717	" " exceed'gly brittle.
*20	20	60	1	1	2,321	2,321	" " " "
				Scattering.			
2	10	88	3	147.5	14,000	43,987	Somewhat soft; ductile.
10	1	89	5	52	21,740	50,000	Tough; medium ductility.
12	2	86	9	9	32,984	32,984	Very tough; hard.
13	2	85	8	12	32,723	37,003	" " "
85	7.5	7.5	3	37	8,703	17,630	Very soft, somewhat ductile.
27.1	119	69.6	2.5	20	2,800	2,800	" " spongy.
100	——	——	2	160	4,005	12,911	

* Could not be machined.

The results of the table show that the alloys yielding other resistances of considerable value will also exhibit proportionate torsion resistances, as might be anticipated.

CHAPTER X.

BENDING OR FLEXURE.

Art. 67.—Coefficient of Elasticity.

THE coefficient of elasticity, as determined by experiments in flexure, can scarcely be considered other than a conventional quantity. If the coefficients of elasticity for pure tension and compression were exactly equal to each other, and if all the hypotheses involved in the common theory of flexure were true, then, indeed, the coefficient of elasticity for flexure would possess actual existence and be the same as that for either tension or compression.

These conditions, however, never exist, and the quantities found in this chapter under the name "coefficient of elasticity" possess value chiefly as empirical factors which enable the deflections in the different cases to be estimated with sufficient accuracy for all ordinary purposes.

The formulæ to be used in the determination of the coefficients of elasticity for flexure have already been established, and their use will be shown in succeeding articles.

Art. 68.—Formulæ for Rupture.

As with the formulæ for the coefficient of elasticity, so with the formulæ for rupture in bending; they are all deductions from the common theory of flexure, and, strictly speaking, are subject to all the limitations involved in it.

If K and K' are the greatest intensities of stress in the section of rupture and at the instant of rupture; y the variable normal distance of any fibre from the neutral surface; y_1 and y' the greatest values of y; b the variable width of the section (normal to y); and M the resisting moment at the instant of rupture; then the general formula for rupture by bending, as given by eq. (1) of Art. 26, is

$$M = \frac{K}{y_1}\int_0^{y_1} y^2 b dy + \frac{K'}{y'}\int_{-y'}^{0} y^2 b dy. \quad . \quad . \quad . \quad (1)$$

This equation is based on the supposition that the coefficients of elasticity for tension and compression are not equal. Although this supposition is strictly true, yet equality is almost invariably assumed, particularly in the treatment of solid beams. Fortunately this assumption is not far wrong in those materials which are most valuable to the engineer.

Eq. (1), however, will hereafter be applied to some cast-iron flanged beams.

If the tensile and compressive coefficients of elasticity are equal, $\frac{K}{y_1} = \frac{K'}{y'}$; or if K is the greatest intensity of stress in the section which exists in the fibre at the greatest normal distance, d_1, from the neutral surface, then $\frac{K}{y_1} = \frac{K}{d_1}$, and eq. (1) becomes

$$M = \frac{KI}{d_1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

This is eq. (10) of Art. 9, and is the one almost invariably used in engineering practice.

In eq. (2) I is the moment of inertia of the cross-section of the beam about its neutral axis. By introducing the value of I for each particular shape of section, simple working forms of eq. (2) may easily be obtained. This will be done for two sections in the following article.

Art. 69.—Solid Rectangular and Circular Beams.

While the rectangular form of cross-section almost invariably characterizes timber beams, similar ones of iron, steel, and other metals are only occasionally seen. Beams of iron and steel with circular cross-sections, however, are quite common as pins in pin connection bridges.

If ΣPx represents the moment of the external forces about the neutral axis of any section, eq. (2) of the preceding article becomes

$$\Sigma Px = \frac{KI}{d_1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The following are the values of I and d_1 for rectangular and circular sections, h being the side of the rectangle normal and b that parallel to the neutral axis, while r is the radius of the circular section and A the area in each case:

$$\text{Rectangular:} \quad \begin{cases} I = \dfrac{bh^3}{12} = \dfrac{Ah^2}{12}, \\ d_1 = \dfrac{h}{2}. \end{cases}$$

$$\text{Circular:} \quad \begin{cases} I = \dfrac{\pi r^4}{4} = \dfrac{Ar^2}{4}, \\ d_1 = r. \end{cases}$$

If the beams are supported at each end and loaded by a weight W at the centre of the span (or distance between supports), which may be represented by l, then the moment at the centre of the beam becomes

$$\Sigma Px = M = \frac{Wl}{4}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

There will then result from eq. (1):
For rectangular sections:

$$M = \frac{Wl}{4} = \frac{Kbh^2}{6} = \frac{KAh}{6}. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

For circular sections:

$$M = \frac{Wl}{4} = \frac{\pi K r^3}{4} = \frac{KAr}{4}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The quantity K is called the *modulus of rupture for bending*, and if experiments have been made, so that W is known, eq. (3) gives

$$K = \frac{3}{2}\frac{Wl}{Ah} = \frac{3}{2}\frac{Wl}{bh^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and eq. (4)

$$K = \frac{Wl}{Ar} = \frac{Wl}{\pi r^3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

If the rectangular section is square, $bh^2 = b^3 = h^3$.

Wrought Iron.

If the beam is simply supported at each end and carries a load W at the centre, while E is the coefficient of elasticity and w the deflection at the centre, eq. (28) of Art. 22 gives

$$w = \frac{Wl^3}{48EI}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

If, in any given experiment, w is measured, E may then be found by the following form of eq. (7):

$$E = \frac{Wl^3}{48wI}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

If the section is rectangular

$$E = \frac{Wl^3}{4wbh^3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

These equations enable the coefficient of elasticity E to be computed readily from experimental observations. It is only necessary to measure accurately the deflection w produced by the load or weight W and then introduce all the known quantities in eq. (8) or eq. (9).

A bar of wrought iron 3 inches deep and 1 inch wide was placed on supports 48 inches apart and loaded with a weight of 400 pounds at mid-span. The measured deflection was .0155 inch. Hence

$$E = \frac{400 \times 48 \times 48 \times 48}{4 \times 1 \times 3 \times 3 \times 3 \times .0155} = 26,430,000.$$

Other applications may be made in precisely the same way.

In testing short solid wrought-iron or other beams to failure, it has been observed that when the beams are very short the ultimate extreme fibre stress K is far greater than the ultimate resistance to tension, and that the difference increases rapidly where the length of span decreases to a few times the depth of beam. This feature of the flexure of solid beams is fully illustrated by what follows.

Kirkaldy tested four square Swedish iron bars to rupture. By the aid of eq. (5) and the data given above, the greatest, mean, and least results were as follows:

	W.	K.	Final Deflection.
Greatest	15,885 lbs.	74,475 lbs. per sq. in.	5.85 ins.
Mean	14,516 "	68,044 " " "	5.35 "
Least	13,338 "	62,522 " " "	4.98 "

The ultimate tensile resistance of the same iron was about 45,000 pounds per square inch. These experiments would seem to show that K, for square bars under similar circumstances of span and depth, may be taken about 1.5 times the ultimate resistance to tension.

The results in the following table were computed by the aid of eq. (6) for some circular beams of "Burden's Best" iron which were tested at the Rensselaer Polytechnic Institute in November, 1882. As beams cannot be actually broken under such circumstances, the "ultimate" value of K was taken with a final deflection of one to one and one quarter the diameter.

CIRCULAR BEAMS OF "BURDEN'S BEST" WROUGHT IRON.

Kind.	Diameter.	Span.	W.		K.	
			Elastic.	Ultimate.	Elastic.	Ultimate.
	Ins.	Ins.	Lbs.	Lbs.	Lbs.	Lbs.
Turned	1.25	12	3,000	6,000	46,950	93,900
Turned	1.25	8	4,400	10,500	45,900	109,500
Turned	1.25	12	——	——	54,760	93,870
Turned	1.25	8	——	——	52,150	114,700
Rough	1.00	12	——	——	55,000	91,700
Rough	1.00	8	——	——	57,000	101,900
Turned	1.00	12	——	——	55,000	91,600
Turned	1.00	8	——	——	——	107,000
Rough	1.00	12	1,700	3,000	51,950	91,680
Rough	1.00	8	2,800	4,800	57,000	97,800
Turned	0.75	12	700	1,100	47,100	74,050
Turned	0.75	8	1,200	1,900	53,880	85,310
Turned	0.75	12	700	1,100	47,100	74,050
Turned	0.75	8	1,300	1,900	58,370	85,310

The "elastic limit" is taken at that point beyond which the metal "flows," and is indicated by the incapability of the specimen to hold up the scale beam beyond it, under a small increase of stress; in other words, it is that point at which the specimen "breaks down."

These experiments show conclusively that "ultimate" K decreases as the ratio of span over diameter increases, but they are not sufficiently extended to establish the limits of application of the observation.

Cast Iron.

All the following results for cast-iron beams are found from Major Wade's experiments ("Strength and Other Properties of Metals for Cannon," 1856). His test bars were nearly two inches square in section or two inches in diameter, and were twenty-four inches long. They were loaded at the centre, and the distance between supports was twenty inches. The following table gives results for square bars. K is given in pounds per square inch, and is found by the aid of eqs. (5) and (6). "*Def.*" is the final deflection.

Although Major Wade made many other experiments of the same kind, these may be considered representative ones.

It is highly interesting to observe that K and the final deflection are materially larger for circular beams than for square ones.

By comparing these values of K with the ultimate tensile resistances found by Major Wade, and which have been given under the head of "Tension," it will be seen that no great error will be involved if K is taken at *twice the ultimate tensile resistance for square bars,* and *two and a quarter times the same quantity for bars with circular section.*

BARS WITH SQUARE SECTION, EQ. (5).

Kind of Iron.	Hours in Fusion.	*W.*	*K.*	Def.
		Lbs.	Lbs.	In.
Richmond iron; 2d fusion.	0	11,587	42,130	0.156
	1	12,487	45,110	0.152
	2	15,019	52,870	0.152
	2	15,525	55,930	0.147
Stockbridge iron; 2d fusion.	½	11,812	42,760	0.162
	1	14,512	52,670	0.195
	1½	16,481	60,500	0.202
	2	19,462	69,680	0.230
Franklin iron; 3d fusion.	0	12,987	49,070	0.250
	0	13,365	50,120	0.217
	1	15,363	57,330	0.220
	1	14,616	54,550	0.195
	2	13,788	48,730	0.152
	2	14,850	50,720	0.170
	3	16,056	56,050	0.175
	3	16,722	60,410	0.170

BARS WITH CIRCULAR SECTION, EQ. (6).

Franklin iron; 3d fusion.	½	10,437	70,600	0.237
	1½	8,665	57,720	0.166
	3	11,112	70,740	0.254
	3¾	10,606	71,740	0.240
Franklin iron; 2d fusion.	1	7,920	52,360	——
	2	9,270	63,670	.240
	3	9,481	64,820	.262
	4	7,920	52,360	——

Whether these ratios will hold for iron of inferior quality to that used by Major Wade can only be determined by further experimenting.

Steel.

Some circular Bessemer steel beams with 12- and 8-inch spans were tested at the Rensselaer Polytechnic Institute in November, 1882, with the results which are given in the next table. The "elastic limit" is that point at which the specimen "breaks down." The "ultimate" value was that for which the deflection was equal to one or one and a quarter the diameter.

CIRCULAR BESSEMER STEEL BEAMS, EQ. (6).

Kind.	Diameter.	Span	*W.*		*K.*	
			Elastic.	Ultimate.	Elastic.	Ultimate.
	In.	Ins.	Lbs.	Lbs.	Lbs.	Lbs.
Turned.	1.00	12	——	——	86,000	146,750
"	1.00	8	——	——	85,300	152,800
"	1.00	12	2,500	4,500	76,400	137,520
"	1.00	8	3,750	7,500	76,400	152,800
"	0.75	12	1,150	1,800	77,400	122,200
"	0.75	8	1,800	3,300	80,800	148,200
"	0.75	12	1,150	1,700	77,400	114,400
"	0.75	8	1,800	3,300	80,800	148,200

The "ultimate" *K* is seen to decrease as the ratio of length over diameter increases.

The following table contains results computed from the experiments of the "Steel Committee" of the British Institution of Civil Engineers; the experiments were made in 1868. The bars were 1.9 inches square in section, and the distance between supports was twenty inches.

BESSEMER STEEL, EQ. (5).

Kind and Number of Tests.	*K.*	Elastic over Ultimate.	Final Deflection in Inches.
	Lbs. per Sq. In.		
5, tires, hammered.	129,030	0.573	3.82
5, axles, "	129,325	0.615	4.08
4, rails, "	125,900	0.612	3.94
4, tires, axles, rails; rolled.	115,120	0.563	4.03

CRUCIBLE STEEL, EQ. (5).

5, tires, hammered.	143,530	0.574	3.32
4, axles, "	152,055	0.539	3.35
1, rail, "	175,470	0.436	3.65
1, axle, rolled.	118,160	0.538	3.84

Each result is an average of the number of tests shown in the left column.

The ratio "elastic over ultimate" is the value of K at the "elastic limit" divided by its ultimate value as given in the table.

A comparison of the results found in the preceding table with the ultimate tensile resistances of the same materials (not given) shows that K *is about* 1.66 *times the ultimate tensile resistance for square Bessemer-steel bars, and about* 1.85 *times the same quantity for square crucible-steel bars.*

Mr. J. W. Cloud, of the Penn. R. R. Co., made bending tests of the Bessemer rail-steel, whose ultimate shearing resistances are given in Art. 65. His test pieces were 12 inches long, 1.5 inches wide, and 0.5 inch thick. The load was applied in the direction of the thickness, and midway between supports 10 inches apart. The greatest, mean, and least results of all his tests, with carbon ranging from .282 to .438 per cent., are the following:

	W.	K.
Greatest	3,349 lbs.	133,960 lbs. per sq. in.
Mean	3,026 "	121,040 " " "
Least	2,765 "	110,600 " " "

With these rectangular specimens of Bessemer rail-steel, supported flatwise, therefore, K may be taken about 1.6 the ultimate tensile resistance.

The following table contains the results of Kirkaldy's experiments on square bars of Fagersta steel. These bars were 1.9 inches square in section, and rested on supports 20 inches apart. W is the breaking weight at centre, and K is computed by the aid of eq. (5). The column "*Elastic over ultimate*" contains the ratios of the values of K at the "elastic limit" divided by the ultimate values given in the table.

FAGERSTA STEEL SQUARE BARS.

Mark.	W, Pounds.		K, Pounds per Square Inch.		Elastic over Ultimate.		Final Deflection, Inches.
1.2	30,496	Mean =32,589	133,380	Mean =144,000	0.669	Mean =0.66	0.75
1.2	32,896		143,880		0.695		0.72
1.2	35,376		154,710		0.616		0.87
0.9	43,820	Mean =43,833	191,640	Mean =191,700	0.500	Mean =0.496	1.46
0.9	44,552		194,850		0.476		1.62
0.9	43,128		188,640		0.512		1.38
0.6	40,260	Mean =38,145	176,100	Mean =167,040	0.467	Mean =0.48	3.15
0.6	36,200		158,310		0.491		3.56
0.6	38,120		166,740		0.482		3.22
0.3	24,420	Mean =25,283	106,800	Mean =110,580	0.561	Mean =0.623	5.22
0.3	23,280		101,820		0.653		5.05
0.3	28,150		123,120		0.654		5.05

The "*Mark*" shows the character of the steel; 1.2 is the hardest, and 0.3 the softest.

K is about 1.6 times the ultimate tensile resistance for the grades 1.2 and 0.6, and 1.8 times the same quantity for the grades 0.9 and 0.3.

Comparison of Modulus of Rupture for Bending with Ultimate Resistances.

The experiments on solid steel and iron beams which have been cited show the somewhat remarkable result that in general the ultimate intensity of stress K in the extreme fibres has the value neither of the ultimate resistance to tension nor of that to compression; nor, indeed, does there seem to be anything like a well-defined relation between those quantities. If those resistances have widely different values, K may be found between them, but in other cases it may considerably exceed the greater of the two. In no case, however, is it found less than the least of the two ultimate resistances. These investigations also show that

K varies to some extent with the shape of cross-section, although it varies chiefly with the length of span. In the absence of experiments conducted in a manner proper to the solution of this problem, it is difficult to assign definitely the reason for the facts as they appear. The explanation is probably due to a number of causes.

It has been shown conclusively, as will appear by referring to the exact theory of flexure, that the common theory of flexure does not apply even approximately to very short beams. In the preceding cases, where the results of tests of solid steel, cast- and wrought-iron beams are shown, the spans have ranged from five or six to ten or twelve times the depth of beam, and within those limits the value of *K* has ranged from possibly one and one-half to nearly two and one-half times the ultimate resistance to either direct tension or compression. In engineering practice the spans of solid circular steel beams, as in the case of pins, may be found considerably less than the depth of the beam. In all these exceedingly short beams the distribution of stress according to the common theory of flexure no longer holds; at the same time the supporting effect of the direct transverse shear acts in a high degree. The greatest intensity of direct stress of tension or compression in the normal cross-section of greatest bending, as shown by the ordinary formulæ, must be much different from (i.e., greater than) the actual. Again, the varying intensity in adjacent fibres prevents perfect freedom in lateral strains and causes a corresponding increase of resistance. In the cases considered the volume of greatest intensity of stress is small, thus placing the part first ruptured somewhat in a condition of a very short specimen. After the elastic limit is passed it is highly probable in consequence of the flow of the material that the law of the variation of stress intensity changes, becoming such that with the same greatest intensity at the surface of the solid beam the resisting moment is considerably increased.

Finally, it has been shown that the experimentally determined ultimate resistances to tension and compression are, in reality, mean intensities, and not the greatest which the material is capable of exerting at any one point, or along any one line, as in the extreme fibres of a bent beam. On this ground alone, K ought to be considerably greater than either T or C, as determined from the usual cross-sections.

All these influences act to produce the results observed in the preceding tests of solid steel and iron beams, but experimental investigations have not yet been carried far enough to give quantitative values to them.

Although the preceding observations have been applied to steel and iron beams only, the same general features may be expected to characterize short solid beams of any material. In the case of timber and stone the varying character of the material, especially the fibrous nature of timber, will either partially or wholly obscure the characteristic features of the tests of very short beams.

Alloys of Aluminum.

Table IX of Art. 44, in the fifth column from the left side, exhibits values of the ultimate stress in the extreme fibres of small beams of varying proportions of aluminum-zinc alloys. As might be anticipated, beams of either of those metals showed comparatively low resistance, but with aluminum varying from 80 down to 50 per cent. and zinc from 20 up to 50 per cent. the resistance was excellent, the maximum being found with Al 75 and zinc 25.

Table XII of Art. 44 exhibits the ultimate fibre stresses in small beams of the alloys of aluminum with copper, zinc, manganese and chromium. The rolled bars of *Al* 96 and *Cu* 4 give excellent results; as does the cast bar of *Al* 75.7, *Cu* 3, zinc 20 and *Man* 1.3. The remaining values of the transverse resistances in the table are self-explanatory.

Copper, Tin, Zinc, and their Alloys.

In the following table are given the data and the results of the experiments of Prof. R. H. Thurston, as contained in his various papers, to which reference has already been made. The distance between the points of support was twenty-two inches, while the bars were about one inch square in section, and of cast metal.

The modulus of rupture, K, is found by eq. (5), in which, however, in many of these cases, W is the weight applied at the centre, added to half the weight of the bar. When K is large and the specimens small, this correction for the weight of the bar is unnecessary; otherwise it is advisable to introduce it.

The coefficient of elasticity, E, is found by eq. (9), in which W is the centre load added to five eighths of the weight of the bar.

The manner in which both these corrections arise is completely shown in *Case* 2 of Art. 22.

E, for any particular bar, has a varying value for different degrees of stress and strain. Those given in the table may be considered average values within the elastic limit.

As usual, "*elastic over ultimate*" is the ratio of K at the elastic limit over its ultimate value.

An examination of the ultimate tensile and compressive resistances of these same alloys, as given in preceding pages, shows that the ratio of K over either of those resistances is very variable. It is usually found between them, but occasionally it exceeds both.

SQUARE BARS.

Percentage of			K. Lbs. per Sq. In.	Elastic Limit Deflection.	Final Deflection.	E. Lbs. per Sq. In.
Cu.	Sn.	Zn.				
					Ins.	
100	0.00	0.00	29,850	——	8.00	9,000,000
100	0.00	0.00	25,920	0.14 to 0.41	1.38 to 8.00	10,530,600
100	0.00	0.00	21,251	0.346	2.31	13,986,600
100	0.00	0.00	29,848	0.140	Bent.	10,203,200
90	10.00	0.00	49,400	0.400	Bent.	14,012,135
90	10.00	0.00	56,375	0.41	3.36	——
80	20.00	0.00	56,715	0.657	0.492	13,304,200
70	30.00	0.00	12,076	1.00	0.062	15,321,740
61.7	38.3	0.00	2,761	1.00	0.032	9,663,990
48.0	52.0	0.00	3,600	1.00	0.019	17,039,130
39.2	60.8	0.00	8,400	1.00	0.060	12,302,350
28.7	71.3	0.00	8,067	0.583	0.121	9,982,832
9.7	90.3	0.00	5,305	0.25	Bent.	7,665,988
0.00	100	0.00	3,740	0.273	Bent.	6,734,840
0.00	100	0.00	4,559	0.267	Bent.	5,635,590
80.00	0.00	20.00	21,193	——	3.27	11,000,000
62.50	0.00	37.50	43,216	——	3.13	14,000,000
58.22	2.30	39.48	95,620	——	1.99	11,000,000
55.00	0.50	44.50	72,308	——	——	——
92.32	0.00	7.68	21,784	0.30	Bent.	13,842,720
82.93	0.00	16.98	23,197	0.41	Bent.	14,425,150
71.20	0.00	28.54	24,468	0.51	Bent.	14,035,330
63.44	0.00	36.36	43,216	0.53	Bent.	14,101,300
58.49	0.00	41.10	63,304	0.48	Bent.	11,850,000
54.86	0.00	44.78	47,955	0.39	Bent.	10,816,050
43.36	0.00	56.22	17,691	1.00	0.0982	12,918,210
36.62	0.00	62.78	4,893	1.00	0.0245	14,121,780
29.20	0.00	70.17	16,579	1.00	0.0449	14,748,170
20.81	0.00	77.63	22,972	1.00	0.1254	14,469,650
10.30	0.00	88.88	41,347	0.73	0.5456	12,809,470
0.00	0.00	100.00	7,539	0.57	0.1244	6,984,644
70.22	8.90	20.68	50,541	——	0.4019	14,400,000
56.88	21.35	21.39	2,752	——	0.0146	14,800,000
45.00	23.75	31.25	6,512	——	0.0150	7,000,000*
66.25	23.75	10.00	8,344	——	0.0162	12,000,000*
10.00	50.00	40.00	21,525	——	Bent.	9,000,000
58.22	2.30	39.48	95,623	——	2.000	10,600,000
60.00	10.00	30.00	24,700	——	0.1267	14,500,000
65.00	20.00	15.00	11,932	——	0.0514	17,000,000
70.00	10.00	20.00	36,520	——	0.1837	15,000,000
75.00	5.00	20.00	55,355	——	Bent.	13,000,000
80.00	10.00	10.00	67,117	——	Bent.	13,500,000
55.00	5.00	44.50	72,308	——	Bent.	11,000,000
60.00	2.50	37.50	69,508	——	1.500	13,000,000
72.52	7.50	20.00	51,839	——	Bent.	12,000,000
77.50	12.50	10.00	61,705	——	0.705	13,500,000
85.00	12.50	2.5	62,405	——	Bent.	12,500,000

* These bars were about half the length of the others.

Timber Beams.

As timber beams are always rectangular in section, eq. (3) only will be needed. Retaining the notation of that equation, if the beam carries a single weight W at the centre of the span l,

$$W = \frac{2}{3}\frac{KAh}{l}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (10)$$

If the total load W' is uniformly distributed over the span,

$$W' = \frac{4}{3}\frac{KAh}{l}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (11)$$

As K is supposed to be expressed in pounds per square inch, all dimensions in eqs. (10) and (11) must be expressed in inches.

In the use of timber beams it is usually convenient to take the span l in feet, and the breadth (b) and depth (h) in inches. Placing $12l$ for l, therefore, in eqs. (10) and (11),

$$W = \frac{KAh}{18l}; \quad \text{and} \quad W' = 2\frac{KAh}{18l}. \quad . \quad . \quad . \qquad (12)$$

in which formulæ l must be taken in feet and A and h in inches.

If B be put for $\frac{K}{18}$, eq. 12 becomes

$$W = B\frac{Ah}{l}; \quad \text{and} \quad W' = 2B\frac{Ah}{l}. \quad . \quad . \quad . \qquad (13)$$

Hence when W and W' have been determined by experiment, *For single load W at centre*

$$B=\frac{Wl}{Ah} \therefore h=\frac{Wl}{AB}=\frac{18Wl}{AK}=\sqrt{\frac{Wl}{Bb}}=4.24\sqrt{\frac{Wl}{Kb}}. \quad (14)$$

For total load W′ uniformly distributed

$$B=\frac{W'l}{2Ah} \therefore h=\frac{W'l}{2AB}=\frac{9W'l}{AK}=\sqrt{\frac{W'l}{2Bb}}=3\sqrt{\frac{W'l}{Kb}}. \quad (15)$$

If the beam has a section one inch square and is one foot long, $B=W=\frac{W'}{2}$. B, therefore, may be considered *the unit of transverse rupture*; it is sometimes called *the coefficient for centre-breaking loads.*

If the depth h of the beam is given and the breadth is desired, eq. (14) gives

$$b=\frac{Wl}{Bh^2}=\frac{18Wl}{Kh^2}. \quad \ldots\ldots \quad (16)$$

Eq. 15 also gives

$$b=\frac{W'l}{2Bh^2}=\frac{9W'l}{Kh^2}. \quad \ldots\ldots \quad (17)$$

In general, whatever may be the distribution of the loading, if the bending movement is M (in inch-pounds), eq. (3) gives

$$h=\sqrt{\frac{M}{3Bb}}=2.45\sqrt{\frac{M}{Kb}}; \quad \ldots\ldots \quad (18)$$

or

$$b=\frac{M}{3Bh^2}=\frac{6M}{Kh^2}. \quad \ldots\ldots \quad (19)$$

The general observations which have already been made in connection with the ultimate resistances of timber in tension and compression are equally applicable to the flexure or bending of timber beams. The ultimate resistance of the timber as exhibited by the intensity of stress in the extreme fibre can only safely be taken when determined from tests of full-size beams as actually used in engineering structures. Such resistances or moduli when determined from small pieces selected for the purpose of test are liable to be largely in error for the reasons given in detail in Art. 57.

TABLE I.

LASLETT'S TESTS.

Sections 2 × 2 inches with span of 6 feet.

Kind of Timber.	*W*, in Lbs.	*B*, in Lbs.	Final Deflection.	Coefficient of Elas., or *E*.
			Inches.	Pounds.
Oak, English	562	422	5.10	——
Oak, English	407	305	3.95	——
Oak, English	813	610	7.71	902,600
Oak, French	877	658	6.00	1,536,800
Oak, French	831	623	7.58	1,440,000
Oak, Tuscan	758	569	7.66	605,000
Oak, Sardinian	758	569	6.50	871,400
Oak, Dantzic	474	356	6.46	——
Oak, Spanish	562	422	6.62	——
Oak, American, white	804	603	8.83	1,184,600
Oak, American, Baltimore	723	542	7.13	1,547,200
Oak, African (or teak)	1,108	831	5.14	1,010,880
Teak, Moulmein	913	685	3.38	1,378,500
Teak, Moulmein	843	632	6.49	1,172,400
Ironwood, Burmah	1,273	955	4.25	2,369,300
Chow, Borneo	975	731	2.83	2,472,300
Greenheart, Guiana	1,333	1,000	4.62	1,057,900
Sabicu, Cuba	1,293	970	3.75	2,369,300
Mahogany, Spanish	856	642	3.45	1,882,800
Mahogany, Honduras	802	602	4.06	1,187,100
Mahogany, Mexican	783	587	3.92	2,021,800
Eucalyptus, Australia:				
Tewart	1,029	772	4.75	1,791,000
mahogany	686	515	4.71	——
iron-bark	1,407	1,055	3.81	2,420,000
blue-gum	712	534	4.21	1,805,100
Ash, English	862	647	8.63	1,404,000
Ash, Canadian	638	479	7.37	——
Elm, English	393	295	5.29	——
Rock elm, Canada	920	690	8.79	1,299,700
Fir, Dantzic	877	658	5.14	1,395,400
Fir, Riga	600	450	3.63	1,763,200
Fir, spruce, Canada	670	503	5.19	1,849,200
Larch, Russia	626	470	4.33	——
Cedar, Cuba	560	420	4.37	——
Red pine, Canada	653	490	4.63	——
Yellow pine, Canada	627	470	4.66	——
Yellow pine, Canada	483	362	3.39	——
Yellow pine, Canada	304	228	3.45	——
Pitch pine, American	1,049	787	4.79	2,030,800
Pitch pine, American	930	698	4.67	1,834,300
Pitch pine, American	744	558	4.42	1,602,000
Kauri pine, New Zealand	719	539	4.00	1,636,300

E has been computed only for those cases in which *W* exceeds 700.

Tables I, II, III, and IV contain values of B and E which have been computed from data determined by the English experiments of Messrs. Laslett, Maclure, Fincham, Edwin Clark, and G. Graham Smith.

In all these tables W is the total load applied, including the weight of the beam, wherever that correction is made.

In Table I the coefficient of elasticity is computed, in all cases, for a centre load of 390 pounds. In Table II the centre load for the same computation is 1680 pounds, and in Table VII the elastic load had different values for different beams.

TABLE II.

FINCHAM'S TESTS.

3×3 inches, section; 4 feet span; very dry timber.

Kind of Timber.	W.	B.	Coefficient of Elasticity.
	Pounds.	Pounds.	Pounds.
Riga fir	4,530	670	2,293,760
Red pine	3,780	559	1,593,000
Yellow pine	2,756	408	1,550,000
Norway fir	3,292	487	1,850,000
Scotch pine	2,520	373	925,000
Kauri pine	4,110	608	1,977,400

In all cases except the four noted in Table IV the applied loads were placed at the centre of the span.

Although these experiments do not embrace a great variety of cross-section for all kinds of timber, yet Tables I, III, and IV give much larger values of B for small depths of pine and fir beams than for large ones. This is a very important consideration in connection with the ultimate resistance of beams, and probably obtains for all kinds of timber.

These experiments also showed that the coefficient of elasticity, E, varied materially in the same specimen for different deflections, and that values among the greatest may be found with large deflections; also that the "elastic limit" for flexure in timber beams is more conventional than real, since with loads about half the breaking weight, not only the deflection but the "set" varied with the time.

TABLE III.

MACLURE'S TESTS.

Specimens of Memel Fir—1849.

Breadth.	Depth.	Span.	*W*.	*B*.	Final Deflection, Inches.
Inches.	Inches.	Feet.	Pounds.	Pounds.	
1	1	1⅓	483	644	0.75
1	1	1⅓	450	600	0.75
2	2	2⅔	1,910	637	1.00
2	2	2⅔	1,311	437	1.125
3	3	9	1,104	368	3.5
3	3	9	1,482	494	4.5
6	12	12	34,720	482	2.0
9	12	12	38,080	353	2.5
12	12	12	61,600	428	3.25

TABLE IV.

TESTS BY EDWIN CLARK AND G. GRAHAM SMITH.

Timber.	Br'dth.	Depth.	Span.	*W*.	*B*.	Final Deflection.	Coefficient of Elasticity.	
	Inches.	Inches.	Feet.	Pounds.	Lbs.	Inches.	Pounds.	
American red pine. .	12	12	15	33,497	291	4.00	1,443,830	E. Clark.
American red pine. .	12	12	15	29,908	260	3.10	1,155,100	
American red pine. .	6	6	7.5	7,370	256	1.68	1,015,900	
Memel fir.	13.5	13.5	10.5	Distrib'd 68,560	293	——	2,150,500	G. Graham Smith.
Memel fir.	13.5	13.5	10.5	Distrib'd 68,560	293	——	1,561,300	
Baltic fir.	6	12	12.25	19,145	271	1.11	1,573,400	
Baltic fir.	6	12	12.25	23,625	335	1.93	1,442,300	
Pitch pine.	6	12	12.25	23,030	326	1.31	3,125,000	
Pitch pine.	6	12	12.25	23,700	336	1.31	1,431,300	
Pitch pine.	14	15	10.5	134,400	448	1.14	1,935,400	
Pitch pine.	14	15	10.5	132,610	442	——	1,693,400	
Red pine.	6	12	12.25	16,800	238	——	1,247,000	
Red pine.	6	12	12.25	19,040	270	1.94	1,247,000	
Quebec yellow pine. .	14	15	10.5	Distrib'd 68,600	229	——	1,329,750	
Quebec yellow pine. .	14	15	10.5	Distrib'd 68,600	229	——	1,329,750	
Quebec yellow pine. .	14	15	10.5	85,792	286	——	1,270,000	
Quebec yellow pine. .	14	15	10.5	76,160	254	——	——	

The quantity ordinarily termed the load at the "elastic limit" may be taken from 0.5 to 0.6 the breaking weight. In Table IV it varied from 0.50 to 0.78.

Valuable tests on timber beams have been made by Col. Laidley and Prof. Lanza; both experimented during 1881. Col. Laidley's results are given in Table V.

As was to be expected, in accordance with conclusions already drawn, the sticks of Oregon pine with the smallest depths gave values of K and B considerably larger than the others. These results emphasize the fact that for large beams K or B must be taken from tests on beams equally large, if accurate computations are to be made. With these considerations in view, Prof. Lanza's results for large spruce beams, given in Table VI, possess much value. With the exception of Nos. 11 and 12 the material was common merchantable lumber.

TABLE V.

SEASONED STICKS, LOADED AT CENTRE.

No.	Kind of Wood.	Span.	Width.	Depth.	$K = 18B$. Lbs. per Sq. Inch.	B.	Remarks.
		Ins.	Ins.	Ins.			
1	Oregon pine...........	44	3.48	3.48	11,900	661	
2	Oregon pine...........	22	1.22	1.23	13,210	734	Cross-grained
3	Oregon pine...........	22	1.21	1.20	16,570	921	
4	Oregon maple..........	44	3.63	3.63	10,560	587	
5	California laurel.......	44	3.58	3.58	8,920	496	Worm-eaten
6	Ava Mexicana.........	44	3.69	3.69	9,930	552	
7	Oregon ash............	44	3.64	3.64	8,460	470	Cross-grained
8	Mexican white mahogany	44	3.77	3.77	9,610	534	
9	Mexican cedar.........	44	3.75	3.75	7,935	441	
10	Mexican mahogany.....	44	3.75	3.75	15,830	879	

Table VII exhibits the values of the ultimate intensity of stress in the extreme fibres of timber beams as determined by Prof. J. B. Johnson for the Division of Forestry of the U. S. Department of Agriculture. He used some large beams but they were mostly small, 4 inches by 4 inches in section and 60 inches long. The values in the table are subject to the same limitations of use which have been set forth in

TABLE VI.

SEASONED SPRUCE BEAMS.

No.	Span.	Width.	Depth.	Manner of Loading.	$K = 18B$. Lbs. per Sq. Inch.	B.
	Feet.	Inches.	Inches.			
1	15.00	2.00	12.00	At centre.	5,526	307
2	6.60	2.00	9.00	" "	5,389	299
3	15.00	2.00	12.00	" "	5,237	291
4	6.67	2.75	9.00	" "	4,082	226
5	4.00	3.00	9.00	" "	3,285	183
6	10.00	3.00	9.00	" "	4,508	250
7	15.00	3.00	9.00	" "	5,651	314
8	20.00	3.90	12.00	" "	4,253	237
9	10.00	2.50	13.50	" "	3,787	210
10	16.00	3.75	12.00	4.5 feet from one end.	3,271	182
11	7.00	7.00	2.00	At centre.	8,748	486
12	7.00	1.75	6.75	" "	7,562	420
13	6.67	3.00	9.00	" "	4,931	274
14	6.67	3.00	9.00	At 4 points, 16 ins. apart.	4,961	276
15	16.00	3.90	12.00	4.5 feet from one end.	5,218	289

previous articles on tension and compression of timber. The results for such small selected pieces are far too high for the full-size beams which are employed in engineering practice and they should be used only after having been subjected to a suitable reduction. It will be noticed that the first four values in the table belong to a condition in which the amount of moisture is 15% by weight of the timber, while the remaining values are for a 12% moisture.

The results shown in Table VIII possess much value for the reason that they belong largely to full-size beams of Oregon Pine, although the span in no case exceeds 84 inches.

The tests were made under the supervision of Mr. Wm. Hood, chief engineer of the Southern Pacific R. R. Co. at San Francisco, Cal. in 1895. The material was fairly well air-seasoned, corresponding, perhaps, to 15 or 18 per cent. of moisture, and of good merchantable quality not selected. Naturally the small sticks two or three inches only in depth with 24-inch spans were better in quality than the larger beams, since knots and other defects not sufficient to cause the rejection of the latter might easily exclude the former from acceptance.

TABLE VII.

JOHNSON'S TESTS.

No.	Species.	Average Highest 10 Per Cent. of Tests.	Average Lowest 10 Per Cent. of Tests.	Average of All Tests.
		Pounds per Square Inch in Extreme Fibre.		
	Reduced to 15% Moisture.			
1	Longleaf pine	14,200	8,800	10,900
2	Cuban pine	14,600	8,800	11,900
3	Shortleaf pine	12,400	7,000	9,200
4	Loblolly pine	13,100	8,100	10,100
	Reduced to 12% Moisture.			
5	White pine	10,100	5,000	7,900
6	Red pine	12,300	4,900	9,100
7	Spruce pine	13,600	5,800	10,000
8	Bald cypress	11,700	5,000	7,900
9	White cedar	8,400	4,000	6,300
10	Douglas spruce, *a*	12,000	4,100	7,900
11	White oak	18,500	7,600	13,100
12	Overcup oak	14,900	6,300	11,300
13	Post oak	15,300	7,400	12,300
14	Cow oak	12,500	6,500	11,500
15	Red oak	15,400	9,100	11,400
16	Texan oak	16,900	10,000	13,100
17	Yellow oak	14,600	5,700	10,800
18	Water oak	15,700	7,200	12,400
19	Willow oak	13,800	5,400	10,400
20	Spanish oak	15,600	6,900	12,000
21	Shagbark hickory	20,300	9,400	16,000
22	Mockernut hickory	19,700	7,900	15,200
23	Water hickory	17,300	5,400	12,500
24	Bitternut hickory	19,300	8,700	15,000
25	Nutmeg hickory	15,600	8,100	12,500
26	Pecan hickory	18,100	10,300	15,300
27	Pignut hickory	24,300	11,500	18,700
28	White elm	13,600	7,300	10,300
29	Cedar elm	17,300	8,500	13,500
30	White ash	14,200	6,300	10,800
31	Green ash	16,000	5,100	11,600
32	Sweet gum	12,700	6,000	9,500

a Actual tests on "dry" material not reduced for moisture.

The values in Table VIII illustrate in a clear manner the variations to be expected between large and small beams. The lower half of the table belongs to practically full-size beams and the upper portion to small beams. The coefficients of elasticity and elastic-limit stresses in extreme fibres for the smallest beams are about 75 per cent. greater than for the largest full-size beams, while the ultimate stress in the extreme fibres is about twice as great. This difference is not accidental, but it is seen to vary with some approach to regularity throughout the table.

TABLE VIII.

TESTS OF OREGON PINE BEAMS, UNTREATED, NOV., DEC., 1895.

Summary of Values Calculated from Tests.

	Span in Inches.	Breadth in Inches.	Depth in Inches.	Ratio of Length to Depth.	Modulus of Elasticity E, Pounds per Square Inch.		
					Least.	Greatest.	Average.
I	24	3	2	12	1,651,155	2,159,738	1,793,761
II	24	2	3	8	1,168,366	1,618,438	1,473,334
III	36	3	4	9	1,330,841	1,871,446	1,620,333
IV	48	4	6	8	1,443,880	1,814,433	1,590,239
V	84	13	7	12	1,677,670	1,867,363	1,784,945
VI	84	6	8	10.5	1,347,448	1,860,960	1,642,521
VII	84	8	10	8.4	1,281,167	1,435,930	1,356,035
VIII	84	6	12	7	1,105,217	1,496,196	1,347,364
IX	84	4 and 7	16	5.25	959,046	1,181,250	1,074,334

	Outer Fibre Stress at Elastic Limit, Pounds per Square Inch.			Outer Fibre Stress at Failure, Pounds per Square Inch.		
	Least.	Greatest.	Average.	Least.	Greatest.	Average.
I	6,600	7,282	6,953	10,574	12,677	11,215
II	4,489	8,415	5,887	9,887	12,538	11,358
III	6,040	7,251	6,818	10,193	13,145	11,678
IV	5,675	8,127	6,467	10,840	11,715	11,205
V	4,076	5,340	4,859	7,486	10,049	8,294
VI	3,890	4,986	4,560	7,616	9,912	8,575
VII	3,123	4,120	3,679	5,477	7,782	6,856
VIII	3,617	4,772	3,937	5,406	6,824	6,198
IX	2,446	5,673	3,769	4,104	6,961	5,592

Table IX contains ultimate resistances and working stresses for the extreme fibres of full-size beams as compiled by the Committee of the Association of Railway Superintendents of Bridges and Buildings, which has already been drawn upon in previous articles on the tension and compression of timber.

TABLE IX.

ULTIMATE AND WORKING STRESSES OF TIMBER BEAMS.

Timber.	Extreme Fibre Stress.	Coefficients of Elasticity.	Working Fibre Stress.
	Pounds per Square Inch.		
White oak	6,000	1,100,000	1,000
White pine	4,000	1,000,000	700
Southern long-leaf or Georgia yellow pine	7,000	1,700,000	1,200
Douglas, Oregon, and yellow fir	6,500	1,400,000	1,100
Washington fir or pine (red fir)	5,000	——	800
Northern or short-leaf yellow pine	6,000	1,200,000	1,000
Red pine	5,000	1,200,000	800
Norway pine	4,000	1,200,000	700
Canadian (Ottawa) white pine	——	——	——
Canadian (Ontario) red pine	5,000	1,400,000	800
Spruce and Eastern fir	4,000	1,200,000	700
Hemlock	3,500	900,000	600
Cypress	5,000	900,000	800
Cedar	5,000	700,000	800
Chestnut	5,000	1,000,000	800
California redwood	4,500	700,000	750
California spruce	5,000	1,200,000	800

The working stresses given in the fourth column are those regularly employed in engineering practice in the design and construction of full-size timber beams under ordinary circumstances. It is evident that the working stress must vary to some extent with the duty to which the beam is to be subjected. The coefficients of elasticity are such as may reasonably be employed in the cases where it is desired to compute the deflections of beams, but it is to be remembered also in making such computations that the elastic properties of timber are extremely variable and that the results may not prove to be as accurate as might be expected.

Failure of Timber Beams by Shearing Along the Neutral Surface.

In the preceding treatment of timber beams, it has been assumed that when broken under test the extreme fibres will fail, either in tension or compression. As a matter of fact, failure of such beams usually takes place at some weak spot, as a knot, point of incipient or active decay, or at some other point where abnormal weakness is developed. This latter observation holds true whether the failure of the beam takes place by tension or compression in the extreme fibres or by shearing in the neutral surface.

In Art. 10 it was shown that the greatest intensity of either transverse or longitudinal shear in any normal section of a beam takes place at the neutral surface, and hence that the tendency of the fibres there is to separate by longitudinal movement over each other. This is precisely the kind of failure which actually takes place in some short timber beams. If the total transverse shear at any normal section of the beam is S, eq. (8) of Art. 10 shows that the maximum intensity, s, of shear in the neutral surface is

$$s=\frac{3}{2}\frac{S}{bd}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

In this equation, b is the breadth or width of the beam and d the depth, usually taken in inches.

If W is a single weight or load at the centre of span of a beam simply supported at each end, the shear s, as far as that single load is concerned, is constant throughout the entire length of the beam with the value

$$s = \frac{3W}{4bd}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

If, again, the beam is uniformly loaded with the total load W', the intensity of shear s in the neutral surface has a value which varies from zero at the centre of span to the value given by eq. (21) after making $W = W'$. Whenever the value of the intensity s exceeds the ultimate intensity of shear along the fibres lying in the neutral surface, the beam will fail by the separation of its two halves or parts at the neutral surface.

TABLE X.

FAILURE OF TIMBER BEAMS BY SHEARING IN THE NEUTRAL SURFACE.

Timber.	No of Tests.	Ultimate Shearing Resistance in Pounds per Square Inch in Neutral Surface.		
		Max.	Mean.	Least.
Spruce....................	14	273	200	117
Yellow pine...............	10	289	224	151
White oak................	2	379	266	152
White pine...............	3	180	151	119

These beams were generally about 12 inches deep, with spans from 7 to 20 feet.

Professor G. Lanza has made a considerable number of tests of full-size beams of spruce, yellow pine, white pine, and white oak at his laboratory in the Massachusetts Institute of Technology at Boston, a number of which failed by shearing in the neutral surface. Table X (Professor Lanza's "Applied Mechanics," page 697) exhibits a summary of those tests and shows the intensities of the ultimate shearing resistance found in full-size beams. Those values are seen to be low, so that in the design of timber beams it is essential that there should be a proper margin of safety against failure in this way. While this precautionary computation should be made, the ordinary procedures of design of timber beams are based upon eqs. 16 to 19, in which the failure of extreme fibres by tension or compression are the controlling influences.

It has been found in many cases, and possibly in nearly all, that wind-shakes, season cracks, and other influences which induce at least partial separation of the fibres at the neutral surface, are the sources of incipient failure by shearing in the neutral surface. The presence of those elements of the case probably accounts for the lowvalues of the shearing intensities parallel to the fibres, shown in Table X.

Influence of Time on the Strains of Timber Beams.

It has been found by actual observation that if a timber beam is loaded to no greater extent than one fourth of its ultimate load, the resulting deflection will continue to increase under continued loading for a long period of time. Sufficient investigations have not yet been made to express these results quantitatively with much accuracy. Enough has been ascertained, however, to show that the influence of time is most important in determining the deflection of timber beams under loads applied for a considerable period of time, and that when the loading becomes a large portion of the ultimate, i.e., perhaps 75 per cent., the beam may fail if the application be sufficiently continued. Indeed, some experiments indicate that failure may possibly take place at .6 or .7 of the ultimate of a single application, if that amount be imposed a sufficient length of time.

It should be understood, therefore, that in using the coefficients of elasticity given in this article for the purpose of computing deflections, such computations are applicable only when loads are applied for short periods of time. For long periods of application it may be necessary in some cases to take only one half of those coefficients of elasticity for the purpose of computing desired deflections.

Timber Beams of Natural and Prepared Wood.

Table XI contains the results of some experiments by A. M. Wellington, C.E. ("R. R. Gazette," Dec. 17, 1880), on small specimens 1¼ inches square and 15 inches between supports. "All the woods, except as specified, had been cut six to eight months and were partially seasoned."

TABLE XI.

SPECIMENS, 1.25 INCHES SQUARE, 15 INCHES LONG.

Kind of Timber.	Natural.		Prepared.		Loss Per Cent.
	W, in Lbs.	*B*, in Lbs.	*W*, in Lbs.	*B*, in Lbs.	
White oak, well seasoned....	989	633	——	——	——
White ash................	926	593	825	527	11.2
Beech....................	864	553	801	513	7.2
Elm......................	763	489	763	489	0.0
Pin oak..................	941	602	755	482	20.0
White oak, green..........	747	479	——	——	——
Soft maple................	742	476	643	411	13.7
Black ash.................	685	439	640	409	6.9
Sycamore..................	628	401	550	332	17.2

The "prepared" specimens had been treated by the Thilmeny (sulphate of baryta) process; and all specimens of the same kind of wood were cut from the same stick.

The column "*Loss*" is the per cent. of loss caused by the preservative process employed.

The results of testing creosoted Oregon pine beams by Mr. William Hood, Chief Engineer of the Southern Pacific Railroad Co., in 1895, are shown in Table XII. Corresponding values for the same, but untreated timber, are given in Table VIII. The amount of creosote in the timber was about 11 pounds per cubic foot.

TABLE XII.

OREGON PINE BEAMS—CREOSOTED

Span in Inches.	Breadth in Inches.	Depth in Inches.	Ratio of Length to Depth.	Coefficient of Elasticity E.					
				Pounds per Square Inch.			Ratio of Creosoted to Untreated.		
				Greatest.	Mean.	Least.	Gr'test.	Mean.	Least.
36	6	4	9	1,607,813	1,531,250	1,378,125	1.036	0.945	0.859
48	4	6	8	1,190,972	979,940	676,974	0.656	0.616	0.469
84	10	8	10.5	1,543,113	1,412,167	1,262,175	0.829	0.860	0.937
84	8	10	8.4	1,715,788	1,319,536	896,399	1.195	0.973	0.700
84	4 and 7	16	5.25	1,195,183	975,754	770,772	1.011	0.970	0.842

Span in Inches.	Breadth in Inches	Depth in Inches.	Ratio of Length to Depth.	Extreme Fibre Stress at Elastic Limit.					
				Pounds per Square Inch.			Ratio of Creosoted to Untreated.		
				Greatest.	Mean.	Least.	Gr'test.	Mean.	Least.
36	6	4	9	4,601	4,382	3,944	0.653	0.643	0.635
48	4	6	8	4,380	3,505	2,629	0.542	0.539	0.463
84	10	8	10.5	4,422	4,341	4,282	1.101	0.952	0.887
84	8	10	8.4	3,776	3,038	2,334	0.917	0.826	0.747
84	4 and 7	16	5.25	2,498	1,998	1,484	0.607	0.530	0.440

Span in Inches.	Breadth in Inches.	Depth in Inches.	Ratio of Length to Depth.	Outer Fibre Stress at Failure.					
				Pounds per Square Inch.			Ratio of Creosoted to Untreated.		
				Greatest.	Mean.	Least.	Gr'test.	Mean.	Least.
36	6	4	9	6,570	6,351	5,913	0.580	0.544	0.500
48	4	6	8	5,692	4,671	3,942	0.486	0.417	0.364
84	10	8	10.5	5,572	5,262	4,712	0.619	0.614	0.562
84	8	10	8.4	4,802	4,158	3,793	0.693	0.617	0.606
84	4 and 7	16	5.25	4,707	4,157	3,634	0.884	0.774	0.678

Note.—4-inch and 6-inch creosoted beams were cut from interior of 8-inch by 10-inch beams.

With four exceptions (three in connection with the coefficient of elasticity) the creosoted timber gave less values of the coefficient of elasticity and of the extreme fibre stresses than the untreated and in most cases much less. This might reasonably be anticipated in consequence of the operations to which the timber is subjected in the process of creosoting. Preservative treatments appear to reduce the resistance of the timber.

Concrete Beams.

When a concrete or a natural stone beam is subjected to transverse loading it fails by tearing apart on the tension side. The failure of the beams, therefore, indicates to some extent the ultimate tensile resistance of the material. Obviously, in the case of concrete beams the ultimate carrying capacity will depend upon a number of elements, such as the kind and quality of cement, sand and broken stone used, and the proportions of the mixture. Table XIII contains results of tests of a considerable number of concrete beams 6 ins. by 6 ins. in cross-section and six months of age. For three months these beams were frequently wetted though kept in air. During the remaining three months they were kept in air without wetting. The length of span for some of these beams was 42 ins. and 18 ins. for the remainder. Within the limits of the tests this difference in span appeared to make no essential difference in the ultimate intensities of stress in the extreme fibres. With the cross-sections of the beams, i.e., 6 ins. wide and 6 ins. deep, the ratio of span length divided by the depth was either 7 or 3, making the beams very short. The different columns of the table show the character of the ingredients of the concrete as well as the greatest, mean, and least values of the intensities of extreme fibre stress K. As would be anticipated, the values

TABLE XIII.

CONCRETE BEAMS SIX MONTHS OLD.

	Concrete.	Size of Stone in Inches.	No. of Tests.	Ultimate Stress in Extreme Fibres, Lbs. per Sq. In.		
				Max.	Mean.	Min.
	c. s. br.					
B'klyn Bridge Rosendale......	1–2–4	0–2½	6	140	103	76
" " "	1–3–5	"	5	128	80	33
" " "	1–2–4	1–2½	4	153	128	109
" " "	1–3–5	"	3	140	136	134
" " "	1–2–4	0–1	6	134	125	120
" " "	1–3–5	"	3	1.8	126	122
Atlas Portland...............	1–2–4	0–2½	6	647	526	460
" "	1–3–5.	"	6	516	449	360
" "	1–2–4	1–2½	6	510	452	335
" "	1–3–5	"	6	458	402	360
" "	1–2–4	0–1	6	560	503	458
" "	1–3–5	"	6	516	420	355
Silica Portland...............	1–2–4	0–2½	5	385	349	282
" "	1–3–5	"	6	329	283	238
" "	1–2–4	1–2½	6	554	424	326
" "	1–3–5	"	6	297	268	224
" "	1–2–4	0–1	6	423	377	297
" "	1–3–5	"	5	329	272	238
Alsen Portland...............	1–2–4	0–2½	6	560	472	404
" "	1–3–5	"	6	491	404	341
" "	1–2–4	1–2½	6	574	493	453
" "	1–3–5	"	6	460	419	391
" "	1–2–4	0–1	6	654	566	466
" "	1–3–5	"	6	541	484	414
B'klyn Bridge Rosendale	1–2–4	Gravel.	6	192	171	147
" " "	1–3–5	"	1	—	157	—
Atlas Portland...............	1–2–4	"	6	554	481	414
" "	1–3–5	"	6	417	352	285
Silica Portland...............	1–2–4	"	6	379	344	312
" "	1–3–5	"	6	279	245	195
Alsen Portland	1–2–4	"	5	460	382	326
" "	1–3–5	"	6	373	314	266

"*c.*" indicates cement; "*s.*" indicates sand; "*br.*" indicates broken stone or gravel. An excellent limestone was used for broken stone.

for the Portland cement beams are much higher than those for the Rosendale cement. The table exhibits the usual variations in the results for such material, but on the whole those for gravel are seen to be somewhat less than those for broken stone, the proportions of mixture being the same for the two materials. Even with Portland cement, and with as rich a mixture as 1–2–4, the results show that working values of the greatest intensity in extreme fibres should not exceed 40 to 60 pounds per square inch.

The investigations from which the results in Table XIII have been taken were conducted by Messrs. George C. Saunders and Herbert D. Brown, graduating students in the class of Civil Engineering of Columbia University in 1898.

The results of tests of twelve Giant Portland cement concrete beams with 30- and 68-inch spans are given in the "U. S. Report of Tests of Metals and Other Materials" for 1900, and they are shown in Table XIV.

TABLE XIV.

TRANSVERSE TESTS OF GIANT PORTLAND-CEMENT CONCRETE BEAMS.

Composition: 1 c., 3 s., 5 br. st.

Span, Inches.	Breadth, Inches.	Depth, Inches.	No. of Tests.	Ultimate Stress, K, in Extreme Fibres, Pounds per Square Inch.		
				Max.	Mean.	Min.
68	6	6	1	—	472	—
30	6 and 4	6	7	564	493	348
30	6 and 4	6	4	454	415	367

The age of these beams was made up of 2 days in air, 2 months in water, and then 1 one month in air, making a total of 3 months and 2 days. The broken stone included all sizes passing a 2½-inch ring, and retained on a sieve with ½-inch meshes. In these tests the ratio of length over depth was 5, except in the first where it was 11. There seems to be little difference in the values of K for the two ratios, but the number is much too small to yield any law of variation.

The values of the ultimate extreme fibre stresses (K), shown in Table XV, are the results of testing to failure short Portland cement concrete beams by Mr. H. Von Schon, Chief Engineer of the Michigan Lake Superior Power Company, at Sault Ste. Marie, Mich., and they are taken from his paper in the "Transactions of the American Society of Civil Engineers" for December 1899. The beams were 6 inches by 6 inches in cross-section, with a span of 18 inches. The ratio of length over depth, therefore, was 3.

TABLE XV.

PORTLAND-CEMENT CONCRETE BEAMS, 6 INS. BY 6 INS. SECTION, 18 INS. SPAN.

Cement.	Broken Stone.	Mixture.	No. of Tests.	Ultimate Fibre Stress, K, Pounds per Square Inch.		
				Max.	Mean.	Min.
E	Sandstone	*A*	2	178	176	174
"	"	*B*	2	225	217	209
"	"	*C*	2	288	280	272
"	"	*D*	2	329	325	321
"	"	*E*	2	108	102	97
"	Boulder stone	*A*	2	354	326	298
"	"	*B*	2	358	328	299
"	"	*C*	2	390	373	356
"	"	*D*	2	420	410	400
"	"	*E*	2	350	330	310
R	Sandstone	*A*	2	181	169	158
"	"	*B*	2	183	175	167
"	"	*C*	2	266	262	258
"	"	*D*	2	328	308	288
"	"	*E*	2	195	182	169
"	Boulder stone	*A*	2	390	347	204
"	"	*B*	2	423	406	390
"	"	*C*	2	410	392	374
"	"	*D*	2	411	393	375
"	"	*E*	2	332	322	312

Mixture *A*. . . . 1 cement, 2.4 sand, 5.3 broken stone.
" *B*. . . . 1 " 2.4 " 4.8 " "
" *C*. . . . 1 " 2.4 " 4.4 " "
" *D*. . . . 1 " 2.4 " 4 " "
" *E*. . . . 1 " 0.3 lime, 3.1 sand, 5.3 broken stone.

The beams were left from two to eight days in their forms or moulds after being made, and then tested at the age of 60 days in air.

The chief elements in the composition of the Portland cements indicated by *E* and *R* in the Table were as follows:

	Cement *E*.	Cement *R*.
Lime	62.38	63.55
Silica	23.08	21.70
Alumina	5.69	8.76
Magnesia	1.21	2.96
Iron oxide	5.35	1.27
Potash and soda	1.66	1.12

The sand used in Mr. Von Schon's tests was from St. Mary's River, the broken sandstone was the native Potsdam variety, while the broken boulder stone was granitic in character. All broken stone would pass through a 1½-inch ring and be retained on a 1-inch ring; the material was, therefore, little balanced.

In the constructions executed under the supervision of the Boston Transit Commission, large amounts of concrete were needed, and in the report of the Commission for the year ending June 30, 1902, there are exhibited a large number of tests of Portland-cement concrete beams 6 inches by 6 inches in cross-section with 30-inch spans. The ratio of length of span over depth of beam in this case is, therefore, 5. Table XVI gives the greatest, average, and least results of these tests with the number of beams broken.

TABLE XVI.

PORTLAND-CEMENT CONCRETE BEAMS, 6 INS. BY 6 INS. SECTION, 30 INS. SPAN.

Composition by Volume.			Hours in Compressed Air.	Air Pressure, Lbs. per Sq. In.	No. of Tests.	Ultimate Fibre Stress, *K*, Lbs. per Sq. In.		
Cement.	Stone Dust.*	Broken Stone.*				Max.	Mean.	Min.
1	1.7	2.75	24	7–12	12	999	851	677
1	1.9	2.6	24	12–18	50	924	850	590
1	2	2.4	48	18–25	30	904	731	622
1	2	2.4	28–30 days	20–25	100	900	728	523

* Approximate volumes.

The concrete was machine mixed and Vulcanite-Portland cement was used. The stone dust, to which reference is made in the table, was finely crushed stone varying from impalpable powder up to ⅛ inch diameter, the broken stone, on the other hand, being of ordinary size. It will be noticed that these beams were kept a part of the time in compressed air at pressures varying from 7 to 25 pounds, presumably for the reason that some of the material was to be used under such conditions.

Table XVII contains results of a number of tests of concrete beams 6 inches by 6 inches in cross-section and with 30-inch spans, made for the purpose of comparing the resistances of concretes made with stone dust and sand. This table is also taken from the Report of the Boston Transit Commission for the year ending June 30, 1902.

TABLE XVII.

PORTLAND-CEMENT CONCRETE BEAMS, 6 INS. BY 6 INS. SECTION, 30 INS. SPAN.

Composition by Volume (Approximate).				No. of Tests.	Ultimate Fibre Stress, K, Pounds per Square Inch.		
Cement.	Sand.	Stone Dust.	Broken Stone.		Max.	Mean.	Min.
1	—	2	2.4	4	947	848	760
1	.9	.9	2.7	4	846	784	704
1	1.6	—	3	4	773	711	656
1	.9	.9	2.7	4	862	806	759

This concrete was also made with Vulcanite-Portland cement and the mixing was done by hand. The beams were kept in air for the first 24 hours and then 29 days in damp earth.

The results both as to coefficient of elasticity and extreme fibre stress, given in Table XVIII, were determined at the mechanical laboratory of the Department of Civil

Engineering of Columbia University in 1902 by Mr. Myron S. Falk.* They have special value from the age of the beams, which was about seven years. These beams were originally made under the supervision of Mr. A. Black, Instructor in Civil Engineering, Columbia University, for the purpose of determining thermal linear expansion. They were kept well moistened for several months after being made, but subsequently until tested they were kept under cover without moistening. The gravel used was rounded, varying in size from ½ to 2½ inches.

TABLE XVIII.

PORTLAND-CEMENT MORTAR AND CONCRETE BEAMS BROKEN BY CENTRE WEIGHT.

Bar.	Age, Years.	Span in Inches.	Section of Bar in Inches.		Coefficient of Elasticity, Pounds per Sq. In.	Extreme Fibre Stress, Pounds per Square Inch
			Depth.	Width.		
A	7·4	36	4.12	4.06	——	——
A_1	7·4	16	"	"	1,591,000	278
A_2	7·4	16	"	"	1,102,000	315
B	7	36	"	4.00	2,122,000	606
B_1	7	16	"	"	2,440,000	636
B_2	7	16	"	"	1,220,000	530
C	7	36	"	4.05	1,315,000	247
C_1	7	16	"	"	387,000	229
C_2	7	16	"	"	1,023,000	208
D	7·3	36	4.10	4.15	1,165,000	294
D_1	7·3	16	"	"	597,000	415
D_2	7·3	16	"	"	597,000	346

Bars A, 1 Aalborg cement, 2 sand, 4 gravel.
" B, 1 Atlas " 3 "
" C, 1 Alsen " 3 " 5 gravel.
" D, 1 " " 2 "

Some of the coefficients of elasticity are abnormally low, those belonging to the beams B, B_1, and B_2 are fairly representative of what may be expected with such material in flexure.

---

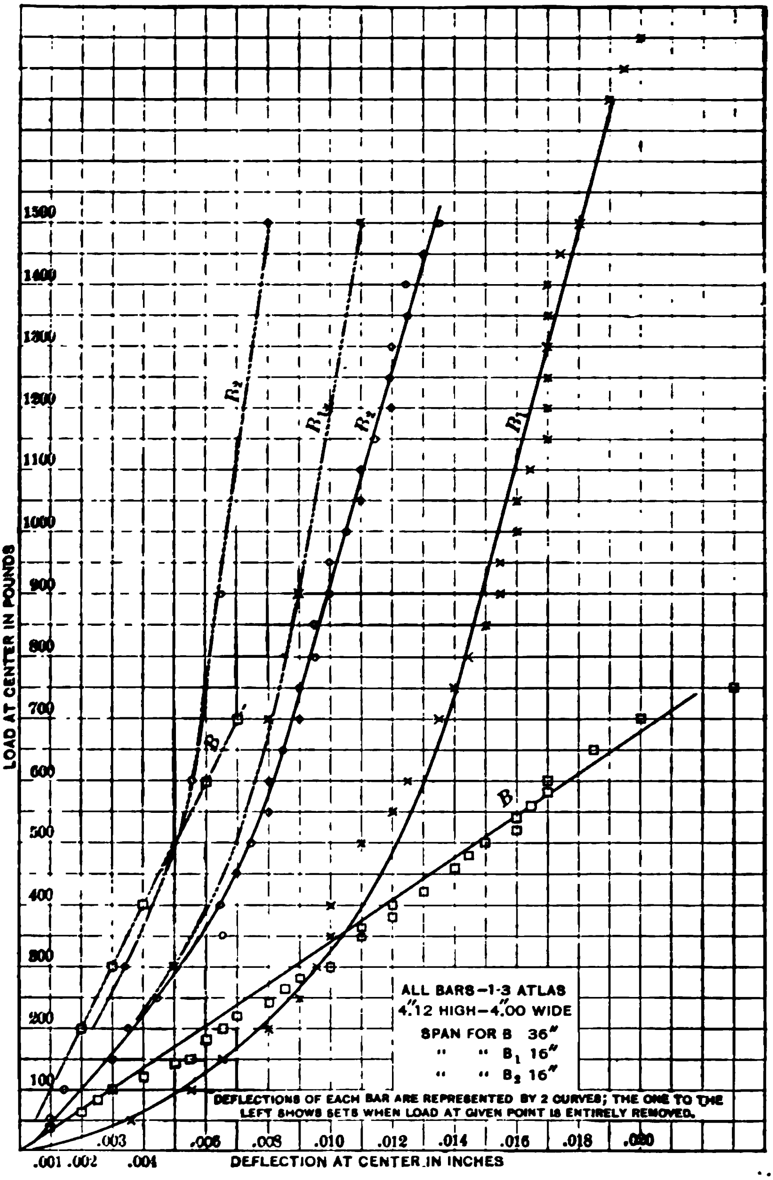
* See Proc. Am. Soc. C. E., February, 1903.

Plate A represents graphically the results of the tests of the preceding three bars *B*. As usual, the strain of deflection consisted of two parts in all cases, one permanent, at least for the time being, and one elastic, which disappeared on the removal of the load. This feature is shown by two lines, in each case indicated by the same letter and subscript. The difference between the total and permanent strain or deflection varied very nearly as the centre load, and that difference being the elastic deflection was used in computing the coefficients of elasticity given in Table XVIII. No coefficient of elasticity was computed for a centre loading less than about 200 pounds. For the purpose of computing deflections under ordinary working stresses from a condition of little or no loading, it would be best to take the coefficient of elasticity at not more than one half of the values given in the Table, in order to allow for that part of the deflection which does not disappear immediately upon the removal of the loading.

Reviewing all the preceding values of the ultimate stress in the extreme fibres of concrete and mortar beams, the working intensities of stress in extreme fibres can probably not be properly taken higher than 50 to 75 pounds per square inch when Portland cement is used for well-balanced mixtures not less rich than 1 cement, 2 sand, and 4 broken stone, or possibly, where exceptionally well made, 1 cement, 3 sand, and 5 broken stone. If gravel is employed, some reduction should be made, depending upon its character, and a similar observation must be applied to mixtures less rich in cement than the preceding.

For natural cements, values of working stress greater than one fourth of the preceding probably should not be used. Indeed, it may be a serious question whether

PLATE A.


LOAD AT CENTER IN POUNDS
100
200
300
400
500
600
700
800
900
1000
1100
1200
1300
1400
1500
DEFLECTION AT CENTER IN INCHES
.001 .002 .003 .004 .006 .008 .010 .012 .014 .016 .018 .020
B_2
B_1
B_2
B_1
B
B
ALL BARS-1-3 ATLAS
4."12 HIGH-4."00 WIDE
SPAN FOR B 36"
" " B_1 16"
" " B_2 16"
DEFLECTIONS OF EACH BAR ARE REPRESENTED BY 2 CURVES; THE ONE TO THE LEFT SHOWS SETS WHEN LOAD AT GIVEN POINT IS ENTIRELY REMOVED.

natural cement should be used at all where concrete or mortar may be subjected to flexure.

TABLE XIX.

BRICK-MASONRY BEAMS.

(Age of beams about equally 5 months, 8 days, and 6 months.)

ROSENDALE-CEMENT MORTAR: 1 C., 2 S.

Span, Inches.	$\frac{l}{d}$.	Stress in Extreme Fibre, Pounds per Square Inch.			No. of Tests.
		Max.	Mean.	Min.	
96	7.4	67	54	38	5
78	6	—	18	—	—
66	5.1	81	56	23	4
42	3.2	91	73	54	8

PORTLAND-CEMENT MORTAR: 1 C., 3 S.

Span, Inches.	$\frac{l}{d}$.	Max.	Mean.	Min.	No. of Tests.
96	7.4	173	144	124	4
66	5.1	145	120	96	4
42	3.2	229	166	94	10

Table XIX exhibits some interesting results of the tests of brick-masonry beams. These investigations were made by Messrs. A. W. Gill and Frederick Coykendall, graduating students in Civil Engineering in Columbia University in 1897. Fig. 1 shows the manner of laying up the brick to form the beams which were tested. The breadth of each beam was about 12 ins. and the depth 13 ins. The spans varied from 8 ft. down to 3 ft. 6 ins., with the ratios of length over depth of beam given in the column headed $\frac{l}{d}$. This column of ratios shows that the beams should be considered short.

The Rosendale-cement mortar was mixed with one volume of cement to two volumes of sand, while the Portland-cement mortar was mixed with one volume of cement to three volumes of sand. During the first three months the beams were kept well wetted, but less so during the last three months. At no time were they dry. The Table gives all the results of tests and shows that the beams had very little resisting capacity, although possibly 15 to 20 pounds per square inch might be justified as working values in the extreme fibres of the beams built with Portland-cement mortar. The bricks were laid by ordinary masons with such care as could be impressed upon them, although the experimenters stated that the brickwork was of very indifferent quality and hence that the results are lower than they should be.

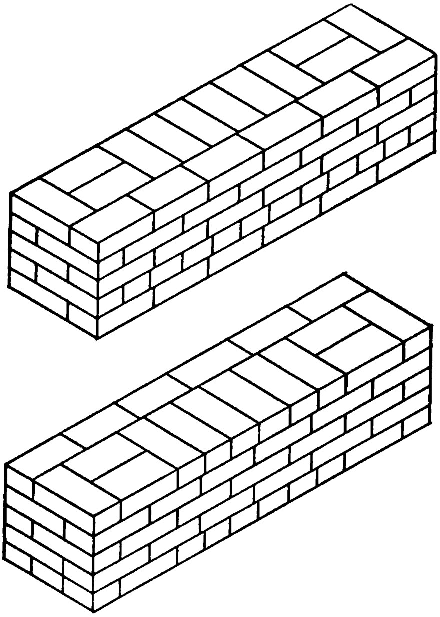


FIG. 1.

Natural-stone Beams.

Table XX exhibits results found by the same experimenters as in the case of Table XIX with a number of natural-stone beams, the spans for which varied from 36 ins. down to 12 ins. The first figure in the second column of the table headed "Section" gives the depth of each beam,

Table XX.

NATURAL-STONE BEAMS.

Span, Inches.	Section, Inches.	$\frac{l}{d}$	Stress in Extreme Fibre, Pounds per Square Inch.			No. of Tests.
			Max.	Mean.	Min.	
			BLUESTONE.			
24	4×6	6.15	3,958	3,512	3,054	5
36	6×8	6.2	3,288	2,797	2,906	3
12	4×6	3	4,112	3,237	2,282	11
24	8×6	3	3,929	3,547	2,715	6
			GRANITE.			
24	4×6	6	2,321	2,250	2,178	3
36	6×8	6	1,861	1,798	1,766	3
12	4×6	3	2,714	2,487	2,086	9
			SANDSTONE.			
24	4×6	6	1,575	1,354	1,237	3
36	6×4	6	1,204	945	637	3
12	4×6	3	1,907	1,539	1,267	9
			MARBLE.			
24	4×6	6	2,036	1,880	1,617	3
36	6×8	6	1,683	1,548	1,354	3
12	4×6	3	2,455	2,026	1,696	9

while the second figure gives the width. It will be observed from the ratios of $\frac{l}{d}$ given in the third column that the beams were very short. The extreme fibre stresses are seen to run comparatively high for the bluestone, granite, and marble. Indeed, working values of intensities may reasonably be taken as follows:

For blue tone.............	250 to 400	pounds per square inch.
" granite...............	200 to 300	" " "
" marbl................	17 to 225	" " " "
" sandstone.............	100 to 150	" " " "

In the use of sandstone it should be understood that the preceding values apply only to the best qualities of that particular stone.

CHAPTER XI.

CONCRETE-STEEL MEMBERS.

Art. 70.—Composite Beams or Other Members of Concrete and Steel.

CONCRETE, like other masonry, is admirably adapted to resist compression. Its capacity of resistance to tension is much less than its ultimate compressive resistance, although if the concrete is well made the tensile resistance may have considerable value. The purpose of the concrete-steel combination is the production of a beam or other member almost entirely of concrete, but which shall have a high capacity to resist tension in those portions which may be subjected to tensile stresses. This result is accomplished by embedding steel bars of desired shape and of suitable cross-sectional area in the proper parts of the concrete. While no general rule can be given for the area of the steel section in comparison with the concrete, it may be stated approximately that the steel section is usually between ¾ and 1½ per cent. of the area of a normal section of the concrete. Inasmuch as the presence of the steel is for the purpose of giving tensile resistance to the member it is evident that the re-enforcing steel bars will always be found in those portions of the concrete mass which may be subjected to tension. In such concrete-steel construction as arches the steel re-enforcement is frequently used both on the tension and compression sides of the concrete.

In the case of concrete-steel beams or other similar members, as the steel is entirely embedded in the concrete, the loads and reactions must obviously be applied directly to the latter. When the concrete takes its stress, therefore, at least a portion of that stress must be conveyed to the steel, and that requires that the adhesive joint or bond between the steel and concrete shall be as strong as possible. Hence in laying the steel bars in the concrete it is necessary that the contact between the two materials shall be intimate and essentially continuous. Various means are employed to accomplish these ends. Square bars are frequently twisted, while round bars may be nicked and flat ones either twisted continuously in one direction or have alternate portions twisted in opposite directions, or, finally, rolled with alternately enlarged and contracted sections. Again, where built-up members are embedded in concrete, rivet-heads and other details of construction serve the same general purposes. The efficiency of the concrete-steel construction depends wholly upon the resistance of this bond, and the design must always be such that the adhesive shear, so to speak, or the stress of sliding along the steel surface, shall never exceed per square unit the ultimate resistance of the bond.

In the analysis and computations which follow it is assumed, as it must be, that the bond between the steel and concrete is such as to make the entire mass act as a unit, so that the combination of the two heterogeneous elements shall act as a single whole.

Art. 71.—Physical Features of the Concrete-steel Combination in Beams.

It will be shown later on that so far as can be determined from physical data now available the coefficient of elasticity for concrete in compression for the operations ordinarily employed in designing engineering structures and for mixtures not less rich in cement than 1 cement, 3 sand, and 6 gravel or broken stone, at ages of one to six months, may range from about 2,000,000 pounds per square inch to more than 5,000,000 pounds per square inch, while for concrete beams the coefficient or modulus may range from about 1,500,000 pounds per square inch for comparatively shallow beams to more than 3,000,000 pounds per square inch for beams of comparatively great depths. Values for the coefficient of elasticity for concrete in tension can be found in Art. 46. Further tests for the determination of this quantity are much to be desired, but enough has been done to establish at least closely approximate values. Some authorities assume the tensile coefficient to be only one half of the coefficient of elasticity for concrete or mortar in compression. As a matter of fact, the tests of a Monier arch of 75 feet span by a committee of the Austrian Society of Engineers and Architects, which made its report in 1895, showed in that particular case the coefficient of elasticity of concrete in tension to be nearly one fifth greater than the coefficient for compression, although it should be stated that the age of the tensile specimens was materially greater than that of the compression material. The values in Art. 46 indicate that the tensile coefficient is at least equal to the compressive. It is possible that subsequent investigations may show that the tensile coefficient of elasticity is less than that for compression, but at the present time there appears to be practically no basis for that assumption. It seems to be reasonable and safe, as it is more simple to take the two coefficients equal to each other until further investigations have conclusively established a different ratio.

A clear understanding of the behavior of mortars and concrete is necessary to the correspondingly clear view of what takes place in the concrete-steel beam under loading. In many cases of concrete under compression of varying intensities a careful measurement of the resulting strains shows that a permanent deformation or compression remains at least for the time being after the removal of load, even when the latter is sometimes not more than 100 or 200 pounds per square inch. This permanent set is dependent upon the age of the material and usually, perhaps always, decreases as age increases. In many other cases a permanent set is observable only under intensities of stress as high as 1000 or 1200 pounds per square inch, or even considerably more. When these sets occur they are frequently found far below what may probably be termed the elastic limit of the material, and in some quarters they have given the impression that mortar and concrete have little or no true elastic behavior. This, however, is an erroneous view, as in the testing of concrete and mortar cubes equal increments of stress intensities quite uniformly give equal increments of strain or deformation over a considerable range. Although the upper limit of this essentially constant ratio between stress and strain is usually not very clearly defined, it is so defined in a considerable percentage of cases, and in almost all tests of well-made concrete and mortar that limit may readily be assigned near enough for all practical purposes.

A large amount of data bearing upon these points will be found in the " Report of Tests of Metals and Other Materials" at the Watertown Arsenal for 1899. Twelve-inch cubes with a great variety of proportions of constituent elements ranging from a few days up to six months in age were employed in those investigations. Figs. 1 and 2 exhibit graphically the results of twelve of those tests so taken as to be fairly representative of all. The vertical ordinates of the curves represent compressive stress intensities up to failure, while the horizontal ordinates represent the total compressive strains or deformation under the corresponding stresses also up to the point of failure. These strains are shown in the figures one hundred times their actual amounts.

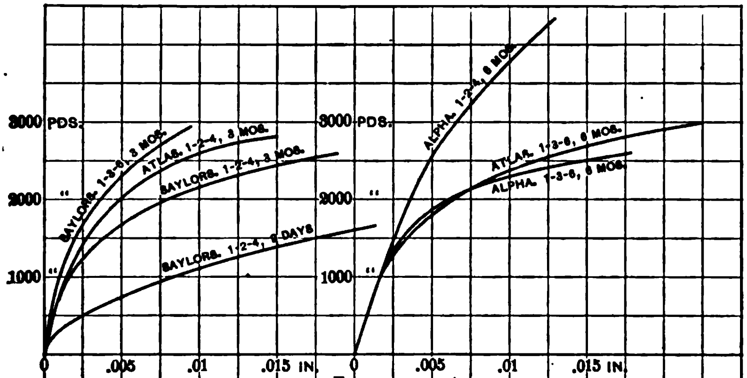


FIG. 1

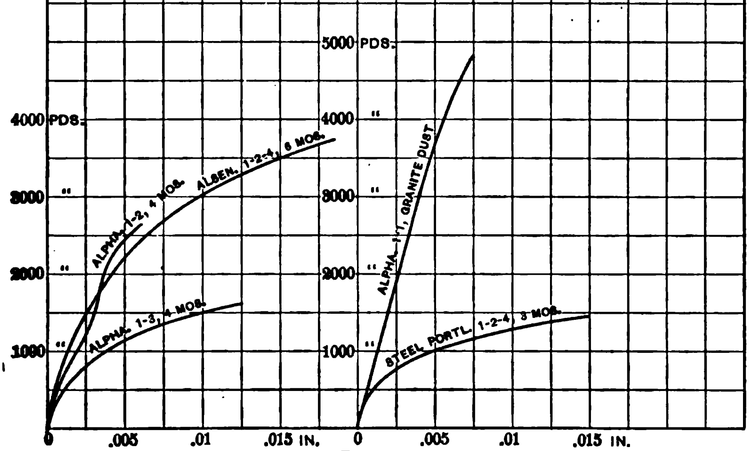


FIG. 2

In Fig. 1 the concrete nine days old shows only little resisting power and a low coefficient of elasticity, as would be expected. In nearly all the other cases, on the other hand, the ratio between stress and strain is reasonably constant up to nearly 1000 pounds per square inch. The two exceptions are found in Fig. 2, belonging to 1 to 3 Portland-cement mortar and to 1, 2, and 4 steel-cement concrete, the former four months old and the latter three months old. On the other hand, the 1, 2, and 4 concrete six months old in the right-hand group of Fig. 1 discloses constant proportionality between stress and strain up to 2000 pounds per square inch, and the same observation may apply to a similar concrete represented by one of the curves in the left-hand group of Fig. 2. Again the 1 to 1 granite-dust mortar four months old represented by one of the curves in the right-hand group of Fig. 2 shows a constant ratio up to nearly 4000 pounds per square inch. Indeed, the whole group of curves probably shows a more satisfactory approach to a constant ratio between stress and strain than do similar curves for cast iron. It should be stated, as will be observed by referring to the report cited, that some of the curves shown in Fig. 1 and Fig. 2 belong to groups for which small permanent sets were observed below elastic limits, while others belong to those which show no such permanent set. This observation does not appear from the test records to be applicable to any particular character of curves, but sometimes to those which are more nearly straight and sometimes to those which are less so.

The results deduced from the tests of cubes covered by the 1899 and other "Reports of Tests of Metals and Other Materials" are confirmed by the investigations of such foreign authorities as M. Considére, Melan, Brik, and others. They show conclusively that it is reasonable and safe to apply to concrete and concrete-steel beams the formulæ established by the common theory of flexure after introducing into them empirical quantities established by experiment precisely as is done with iron and steel beams.

It is important to state in this connection that the results of tests with concrete-steel beams, so far as they have been made, indicate that the elastic or semi-elastic behavior of concrete under stress will in the main characterize the behavior of the same material when under loading in the composite beam of concrete and steel; so that the coefficients of elasticity determined for concrete alone may be used in the composite member.

There is one important respect in which the action of concrete alone is quite different from that which takes place when it is combined with steel. In the latter case the concrete will stretch under a stress nearly or quite equal to its ultimate resistance a comparatively large amount. It is customary to state that under such conditions the coefficient of tensile elasticity of the concrete is practically zero, but there is just as much ground, or more, for making the same observation in connection with such ductile materials as structural steel. What is actually meant is simply that the concrete will stretch before parting much more when its deformation is controlled by the corresponding deformation of the steel reinforcement than when it acts by itself or without such reinforcement. This feature of the action under stress of concrete in the composite beam has a most important bearing upon some rather peculiar phenomena connected with the testing of such beams to failure. M. Considére has stated ("Comptes Rendus Académie des Sciences," Paris, Dec. 12, 1898) that mortar will stretch twenty times as much when combined with steel as when unaided by that combination. He further states that the concrete stretches uniformly with uniform increments of bending moment up to about four tenths of the ultimate moment.

As the coefficient of elasticity for concrete is a small fraction only of that of steel the tendency of the concrete in composite beams is to stretch or compress more than the steel embedded in it. Hence the concrete immediately adjacent to the steel tends to slide along the latter, but that tendency is resisted by the adhesive shear at the joint, in consequence of which the steel acquires its stress whether of tension or compression. The normal section of the unloaded beam, therefore, will not remain normal after flexure, but there will be either a cup-shaped depression around the steel or a similar shaped elevation. This is illustrated in Fig. 3.

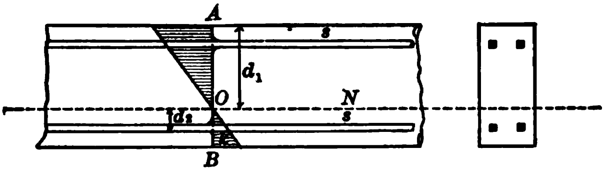


FIG. 3.

In that figure the intensity of stress on either side of the neutral axis is assumed to vary directly as the distance from the axis, but in a subsequent analysis a different law of variation will be assumed in order that the treatment may be complete, although the author is not of opinion that the assumption of any law of variation different from that of the common theory of flexure is at the present time justified. It will further be assumed in the analysis which follows that normal sections of the unloaded beam will remain normal under loading. This is a common procedure, and it is not believed that the amount of variation from a plane section under stress, described above, is sufficient to make the assumption sensibly in error.

Art. 72.—Rate at Which Steel Reinforcement Acquires Stress.

The determination of the rate at which the concrete gives stress to the steel is not of great importance in ordinary design work or in most other practical relations; yet it is desirable in some cases, and it is an element of the action of internal stresses in a composite beam which should be understood as clearly as practicable.. The following analysis offers a means of determining that rate as nearly as it can be done at the present time. The notation used is shown also in Fig. 3 on the opposite page.

The intensity of stress in the concrete at the distance d_2, the distance of the steel reinforcement, from the neutral axis is k. Then if I represent the moment of inertia of the entire composite section about its neutral axis (located by d_1, determined hereafter), there may be written

$$M=\frac{kI}{d_2};\quad \therefore dM=\frac{dk\,.I}{d_2}. \quad . \quad . \quad . \quad . \quad (1)$$

If S is the total transverse shear in the normal section in question at the distance x from one end of the beam,

$$dM=Sdx=\frac{dk\,.I}{d_2}. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Let p be the total perimeter of section of the steel reinforcement at the section located by x.

Let A_2 be the area of steel section with perimeter p.

Let s be the intensity of adhesive shear at the surface or joint between the steel and concrete.

Let k_2 be the intensity of stress in the steel.

The variation of k_2 for the indefinitely small distance dx is dk_2. From what has preceded there may be written

$$p\,.dx\,.s=A_2dk_2;\quad \therefore\ dx=\frac{A_2dk_2}{ps}. \quad . \quad . \quad . \quad (3)$$

Inserting the value of dx from eq. (3) in eq. (2),

$$SA_2\frac{dk_2}{ps}=\frac{dk}{d_2}I.$$

By solving this equation for s and remembering that

$$\frac{dk_2}{dk}=\frac{E_2}{E_1},$$

$$s=SA_2\frac{d_2}{Ip}\frac{dk_2}{dk}=S\frac{E_2}{E_1}\frac{d_2}{I}\frac{A_2}{p}. \quad . \quad . \quad . \quad . \quad (4)$$

This value of s must never exceed the ultimate adhesive resistance between the steel and concrete.

There are but few records of tests to determine this adhesive shear between steel and concrete, but it has sometimes been taken as .16 to .2 the ultimate compressive resistance of the concrete. Bauschinger found it in some experiments to have a value of 640 pounds per square inch. It seems to depend largely upon the amount of water used in the concrete mixture and the care with which the mortar of the concrete is brought into contact with the steel or iron bars. The continuity of this contact is much more effectively secured by a comparatively wet concrete than with a dry mixture, and this accounts for the advantageous results secured in using a comparatively large amount of water in the mixture. Considére found this value in some of his experiments to range from 171 to 71 pounds per square inch, according to the amount of water used in mixing the material, the smallest value corresponding to the least percentage of water. Prof. W. K. Hatt gives some determinations of this quantity in a paper entitled "Tests of Reinforced Concrete Beams," read before the American Society for Testing Materials at its annual meeting, 1902. He pulled $\frac{7}{16}$- and $\frac{5}{8}$-inch wrought-iron rods out of concrete around which it was formed to a depth of about 6 inches and with ages of 32 and 35 days. The averages of three tests each were as follows:

Diameter of rod.	$\frac{7}{16}$ in.	$\frac{5}{8}$ in.
Age.	32 days	35 days
Maximum lbs. per sq. in.	735	780
Average lbs. per sq. in.	636	756
Minimum lbs. per sq. in.	470	714

For suitable mixtures it is probably safe to assume the ultimate value of this adhesive shear to range from at least 500 to 800 pounds per square inch. In these tests Prof. Hatt found that after the rod was once started so that the shear changed to friction, the latter had values running from 50 to 70 per cent. of the adhesive shear. It was also found in these experiments that the contact between the concrete and the iron rods was not continuous but quite irregular. The working value of this shear for ordinary design work should be taken about ⅛ to ⅒ of its ultimate value. Where shock exists ⅒ is probably high enough, but where shock is absent it is probably safe to use ⅙ to ⅛ the ultimate value.

Anything which roughens or makes uneven the exterior surface of the steel reinforcement will obviously increase the resistance of the bond between it and the concrete. The preceding values are for unroughened steel surfaces. When square bars are twisted, or other efficient means are employed to make the bond materially stronger, correspondingly larger working values of the adhesive shear may be used.

Art. 73.—Ultimate and Working Values of Empirical Quantities for Concrete-steel Beams.

It is necessary for the practical use of the preceding and following analyses that a number of empirical quantities be determined, chiefly for the concrete. The coefficient of elasticity for wrought iron for this purpose may be taken at 28,000,000 pounds per square inch, and 30,000,000 pounds per square inch for structural steel, which is now invariably used in the reinforcement of concrete-steel beams.

In Art. 54 there will be found set forth in considerable detail some of the latest and most reliable determinations of the coefficient of elasticity for concrete in compression for different mixtures and at ages ranging from a few days up to six months. The following table is a condensed statement of average values taken from those data. It is found in a paper by Mr. Edwin Thacher, prepared from the original Reports, and was given by him in a lecture before the College of Civil Engineering of Cornell University, 1902.

This table exhibits as reasonable values for the coefficient of elasticity in compression as can be determined at the present time. The value to be selected for any particular case will depend upon the proportions of mixture and upon the degree of balancing of the sand and gravel or broken stone, although the influence of the latter cannot be definitely stated. It is not improbable that a considerable portion at least of the variations in the results of the table are due to the varying degrees of natural balancing in the different test blocks. The value will also depend upon the age of the concrete. For all ordinary engineering constructions it is reasonable to take the coefficient of compressive elasticity at 2,500,000 to 3,000,000 pounds per square inch for a concrete mixture of 1 cement, 2 sand, and 4 gravel or broken stone. This table shows that practically the same value may be taken for a concrete of 1 cement, 3 sand, and 6 gravel or broken stone, especially if the materials are well selected and balanced. If the concrete is mixed in the proportions of 1 cement, 6 sand, and 12 gravel or broken stone, the coefficient of elasticity is seen to decrease materially and should not be taken higher than 1,500,000 pounds per square inch. Suitable quantities for mixtures other than those named in the table can be reasonably and safely selected from those afforded in it.

These values show that the ratio of the coefficient of elasticity for steel over that for concrete may range from 10 to 20 for the varying conditions described. In the

TABLE I.

COEFFICIENTS OF ELASTICITY OF PORTLAND-CEMENT CONCRETE IN COMPRESSION FROM TESTS MADE AT THE UNITED STATES ARSENAL, WATERTOWN, MASS., FOR GEORGE A. KIMBALL, CHIEF ENGINEER, 1899. THE CONCRETE WAS HAND-MIXED. EACH RESULT IS A MEAN OF FIVE OR MORE TESTS.

Cement.	Age.	Concrete, 1, 2, 4, between Loads of			Concrete, 1, 3, 6, between Loads of			Concrete, 1, 6, 12, between Loads of		
		100 and 600.	100 and 1000.	1000 and 2000.	100 and 600.	100 and 1000.	1000 and 2000.	100 and 600.	100 and 1000.	1000 and 2000.
Atlas........	7 days	2,778,000	1,875,000	———	1,667,000	———	———	———	———	———
Germania....	"	2,500,000	2,143,000	1,351,000	2,273,000	1,607,000	———	———	———	———
Alsen........	"	2,500,000	2,143,000	———	1,667,000	1,452,000	———	———	———	———
Mean........	"	2,592,000	2,033,000	1,351,000	1,869,000	1,529,000	———	———	———	———
Atlas........	1 mo.	3,125,000	2,812,000	1,724,000	3,125,000	2,647,000	———	1,316,000	———	———
Alpha........	"	2,083,000	1,875,000	1,190,000	2,083,000	2,143,000	———	1,667,000	1,607,000	———
Germania....	"	———	———	———	2,273,000	1,875,000	1,219,000	961,000	———	———
Alsen........	"	2,778,000	2,647,000	1,471,000	2,273,000	1,875,000	———	1,562,000	———	———
Mean........	"	2,662,000	2,444,000	1,462,000	2,438,000	2,135,000	1,219,000	1,376,000	*1,140,000	———
Atlas........	3 mos.	4,167,000	3,214,000	2,083,000	2,778,000	2,500,000	1,471,000	1,136,000	833,000	———
Alpha........	"	4,167,000	3,750,000	2,778,000	3,571,000	2,812,000	2,000,000	1,786,000	1,800,000	———
Germania....	"	3,571,000	3,214,000	2,381,000	2,778,000	2,812,000	2,083,000	2,083,000	1,667,000	———
Alsen........	"	2,778,000	2,500,000	1,389,000	2,778,000	2,500,000	1,667,000	1,562,000	1,154,000	———
Mean........	"	3,670,000	3,170,000	2,157,000	2,976,000	2,656,000	1,805,000	1,642,000	1,363,000	———
Atlas........	6 mos.	3,125,000	3,214,000	2,273,000	3,571,000	3,461,000	2,174,000	1,786,000	1,607,000	———
Alpha........	"	3,125,000	3,214,000	3,571,000	4,167,000	4,091,000	1,724,000	1,923,000	1,667,000	———
Germania....	"	4,167,000	3,750,000	1,852,000	3,125,000	3,000,000	1,852,000	1,786,000	1,364,000	———
Alsen........	"	4,167,000	4,091,000	2,631,000	3,571,000	3,461,000	1,724,000	1,786,000	1,452,000	———
Mean........	"	3,646,000	3,567,000	2,581,000	3,608,000	3,503,000	1,868,000	1,820,000	1,522,000	———

* Probable.

paper of Prof. Hatt, to which allusion was made in Art. 72, he gives values of the coefficient of elasticity for 1, 2, and 4 concrete in compression ranging from 3,680,000 pounds to 4,702,000 pounds per square inch, and there are some foreign as well as domestic tests which show results even higher. It is probable that these extremely high values are due to special conditions of mixture or other influences which do not characterize actual concrete work, and as they are few in number in comparison with the great mass of concrete testing work done at the Watertown Arsenal it is believed that the results given in the table are better adapted for engineering practice.

The ultimate resistances of mortar and concrete in tension and compression will be found in Arts. 46 and 54. These values will also depend upon the proportions and character of mixture or upon the age. The records of tests and experience which have thus far accumulated in connection with concrete-steel construction show that the compressive working stress of concrete in beams, where the mixture is in the proportions of 1 cement, 2 sand, and 4 gravel or broken stone, may probably be taken as high as 500 pounds per square inch. It should be remembered that this intensity will exist in the extreme fibres of the beam only. Mixtures of less strength would require a corresponding reduction in the maximum working intensity of compression. A mixture, for example, of 1 cement, 2½ sand, and 5 broken stone, unless the materials were well balanced, might justify a reduction of the greatest working stress to 400 pounds per square inch.

Some foreign authorities have prescribed two degrees of safety, in the first of which the maximum working stress of compression of 427 pounds per square inch is allowed, and 711 pounds per square inch for safety of the second degree. Structures in which the duty of the concrete is severe might be designed with the smallest of those values, but where the duty is materially less severe, with the larger.

It is not unusual at the present time in the design of concrete-steel arches to allow a maximum intensity of compression of 500 pounds per square inch and 50 to 75 pounds per square inch for the maximum intensity of tension.

Tensile tests of concrete show that where proportions of 1 cement, 2 sand, and 4 gravel or broken stone are used a maximum intensity of tension of 50 to 70 pounds per square inch is about $\frac{1}{6}$ to $\frac{1}{8}$ the ultimate tensile resistance at the age of three to six months. These values are reasonable and may be employed in concrete beams where it is desired to avail of the tensile resistance of concrete. In much of the best engineering practice at the present time, however, the tensile resistance of the concrete is neglected in the interests of additional safety in concrete-steel beam construction. Inasmuch as fine cracks may appear in concrete from other agencies than tensile stress, it is undoubtedly advisable in most cases certainly to omit the bending resistance of the concrete in tension, especially as that omission does not sensibly increase the weight or cost of the beam when properly designed.

Art. 74.—General Formulæ and Notation for the Theory of Concrete-steel Beams According to the Common Theory of Flexure.

The application of the common theory of flexure to the bending of concrete-steel beams is in reality the development of the theory of flexure for composite beams of any two materials. The notation to be used and the general formulæ will first be written, therefore, and then the special formulæ for concrete-steel beams will be established in the succeeding articles. These general formulæ, it should be observed, apply to beams of any shapes of cross-section of either material or for any relative areas of cross-section of those materials, although in concrete-steel beams the area of cross-section of the steel is frequently or perhaps usually but one to one and a half per cent. of the area of the concrete.

Again, the formulæ will be so written as to make practicable the use of different coefficients of elasticity for concrete in tension and compression if that should be desired.

The notation to be used in the succeeding articles is chiefly the following:

E_2 = coefficient of elasticity of the steel.

E_1 = " " " " " concrete in compression.

nE_1 = " " " " " concrete in tension.

A_1 and A_2 are the areas of normal section of the concrete and steel respectively.

I_1 and I_2 are the moments of inertia of A_1 and A_2 respectively about the neutral axis of the normal section.

k_1 = greatest intensity of bending compression in the concrete.

k' = greatest intensity of bending tension in the concrete.

c = greatest intensity of bending compression in the steel.

t = greatest intensity of bending tension in the steel.

b = breadth of the concrete.

h_1 and h_2' are total depths of the concrete and steel respectively.

h_2 = vertical distance between the centres of the steel reinforcing members.

d_1 = distance of extreme compression "fibre" of the concrete from the neutral axis.

d_2 = distance of the centre of the compression steel reinforcing member from the neutral axis.

d_2' = distance from extreme compression fibre of the steel to the neutral axis.

a = distance of the centre of the compression steel reinforcing member from exterior compression surface of concrete.

o = distance of the centre of the tension steel reinforcing member from exterior tension surface of concrete.

rA_2 = area of normal section of reinforcing steel in tension.

$(1-r)A_2$ = area of normal section of reinforcing steel in compression.

p_1 = intensity of compressive stress in the concrete at distance z from the neutral axis.

p' = intensity of tensile stress in the concrete at distance z from the neutral axis.

p_2 = intensity of stress in the steel at distance z from the neutral axis.

u = tensile or compressive strain in unit length of "fibre" at unit distance from the neutral axis.

In all the theory of bending of concrete-steel beams it is assumed, as in the common theory of flexure, that any plane, normal section of the beam, before bending takes place, will remain plane (and normal) while the beam is subjected to bending. Hence

$$p_1 = E_1uz, \quad p' = nE_1uz, \quad \text{and} \quad p_2 = E_2uz. \quad . \quad . \quad (1)$$

Inasmuch as all the loading carried by concrete-steel beams is supposed to act in a direction normal to the axes of the beams, as is usual in the common theory of flexure, the total stresses of tension and compression in any normal section of a beam induced by the bending must be equal to zero. The expression of this sum, written by the aid of eqs. (1) and by which the neutral axis of the composite section is determined, is the following:

$$E_1u\left\{\int_0^{d_1} zdA_1 - n\int_{h_1-d_1}^{0} zdA_1\right\} + E_2u\int_{d_2'-h_2'}^{d_2'} zdA_2 = 0. \quad (2)$$

Or

$$\int_0^{d_1} zdA_1 - n\int_{h_1-d_1}^{0} zdA_1 + \frac{E_2}{E_1}\int_{d_2'-h_2'}^{d_2'} zdA_2 = 0. \quad . \quad (3)$$

Eq. (3) is perfectly general, and the position of the neutral axis can always be located by it whatever may be the shape of cross-section of either the concrete or steel.

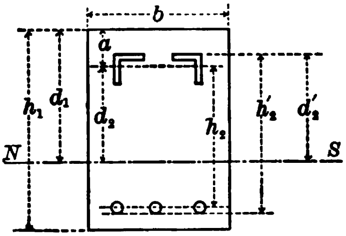


Fig. 1.

Fig. 1 may be taken as an arbitrary typical composite section showing the preceding system of notation applied to it. The outline of the concrete is rectangular, as in the ordinary concrete-steel beam. The steel in compression is represented as two steel angles, while three round rods constitute the steel in tension. In the next article the application of the general eq. (3) to the special case of the ordinary concrete-steel beam will be made.

The general value of the bending moment of the stresses induced in any normal section of a composite beam can be at once written by the aid of eqs. (1). The typical expression of the differential moment is

$$p_1dA_1z = E_1uz^2dA_1.$$

Hence the value of the moment is

$$M = E_1u\int_0^{d_1} z^2dA_1 + nE_1u\int_{h_1-d_1}^{0} z^2dA_1 + E_2u\int_{h_2'-d_2'}^{d_2'} z^2dA_2. \quad (4)$$

This equation is also completely general whatever may be the shape of section of either material. It will be developed for the ordinary form of concrete-steel beams in Art. 76.

Eqs. (3) and (4) cover completely the theory of bending or flexure of composite beams of two materials, one of them having different values for the coefficients of elasticity in tension and compression. It will be observed that the position of the neutral axis of any section of the beam, as located by eq. (3), is affected by the values of E_1, E_2, and n, and that it does not in general pass through the centre of gravity of the section.

Art. 75.—Position of Neutral Axis in Ordinary Rectangular Concrete-steel Beams by Common Theory of Flexure.

The ordinary reinforced or concrete-steel beam is formed with a rectangular normal cross-section with a constant depth d_1 and a constant width b, the outline of the concrete section being rectangular and the steel reinforcement, or bars, being embedded in the concrete, all as typically shown in the figure on the opposite page. The area of the steel section is small in comparison with that of the concrete, usually from about one to one and one half per cent. of the latter, so that there is no sensible error in making $dA_1 = bdz$. The first two terms of the first member of eq. (3) of the preceding article then takes the form

$$b\int_0^{d_1} zdz - nb\int_{h_1-d_1}^{0} zdz = \frac{bd_1^2}{2} - \frac{nb(h_1-d_1)^2}{2}$$
$$= (1-n)\frac{bd_1^2}{2} + nbd_1h_1 - n\frac{bh_1^2}{2}. \quad (1)$$

As the steel section is always small it will be essentially correct to consider each portion of it concentrated at its centre of gravity, so that there may be written

$$\int_{h_2'-d_2'}^{d_2'} z\,dA_2 = (1-r)A_2d_2 - rA_2(h_2-d_2) = A_2(d_2-rh_2). \quad (2)$$

By introducing the values of eqs. (1) and (2) in eq. (3) of the preceding article the latter will become

$$\frac{b(1-n)}{2}d_1^2 + nbh_1d_1 - n\frac{bh_1^2}{2} + \frac{E_2}{E_1}A_2(d_2-rh_2) = 0.$$

Or, since $d_2 = d_1 - a$,

$$d_1^2 + \frac{2\left(nh_1 + \frac{E_2}{E_1}\frac{A_2}{b}\right)}{1-n}d_1 = \frac{nh_1^2}{1-n} + \frac{2\frac{E_2}{E_1}A_2(a+rh_2)}{b(1-n)}.$$

The solution of this quadratic equation gives

$$d_1 = -\frac{nh_1 + \frac{E_2}{E_1}\frac{A_2}{b}}{1-n} \pm\sqrt{\frac{nh_1^2}{1-n} + \frac{2\frac{E_2}{E_1}\frac{A_2}{b}(a+rh_2)}{1-n} + \frac{\left(nh_1 + \frac{E_2}{E_1}\frac{A_2}{b}\right)^2}{(1-n)^2}}. \quad (3)$$

Eq. (3) will locate the neutral axis of the section under consideration by giving its distance d_1 from the upper surface. All quantities appearing in its second member are known either in the case of an actual beam or in the process of designing. In the one case a and h_2 will be known from given dimensions and they will be prescribed in the other. The same observation applies to the sectional area A_2, although a prescribed or assumed area may be subject to revision in the operation of designing, thus necessitating possibly more than one application of the formula.

If the entire steel reinforcement is on the tension side of the beam as shown in Fig. 1 and as often occurs, r will be unity in eq. (3), or $1-r=0$. The only effect produced by this substitution in eq. (3) is to make

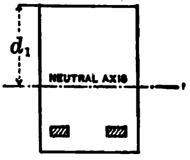


FIG. 1.

$a+rh_2=a+h_2=$ *distance from centre of steel bars to upper surface of beam.*

The use of eq. (3) is by no means so formidable or involved as may appear at first sight. If, as has sometimes been taken, $n=\frac{1}{2}$ and $\frac{E_2}{E_1}=10$, that equation becomes

$$d_1=-h_1-20\frac{A_2}{b}\pm\sqrt{h_1{}^2+40\frac{A_2}{b}(a+rh_2)+\left(h_1+20\frac{A_2}{b}\right)^2} \quad (4)$$

If the coefficients of elasticity of concrete in tension and compression are taken the same, so that $n=1$, it will be only needful to revert to the equation immediately preceding eq. (3).

After first multiplying both members of that equation by $(1-n)$ and then placing $1-n=0$, there will result

$$2\left(h_1+\frac{E_2}{E_1}\frac{A_2}{b}\right)d_1=h_1{}^2+2\frac{E_2}{E_1}\frac{A_2}{b}(a+rh_2);$$

$$\therefore\ d_1=\frac{\frac{1}{2}h_1{}^2+\frac{E_2}{E_1}\frac{A_2}{b}(a+rh_2)}{h_1+\frac{E_2}{E_1}\frac{A_2}{b}} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

As in the preceding case, when the steel reinforcement is wholly on the tension side of the beam it is necessary only to place $r=1$ in eq. (5).

Neglect of the Tensile Resistance of Concrete.

Many civil engineers consider it advisable to neglect the bending resistance of that part of the concrete in tension and to estimate the carrying capacity of the reinforced beam as if the compressive bending resistance constitutes the entire bending resistance of the concrete. In that case the stress conditions assumed would be exhibited by Fig. 2. These stress conditions usually exist before rupture in consequence of the concrete in tension failing by cracking. Again, cracks may sometimes be formed, or at least be started, by a number of extraneous causes independent of the loading. For these reasons it will usually, or at any rate frequently, be the part of prudence and good practice to ignore the tensile bending resistance of the concrete in concrete-steel beams.

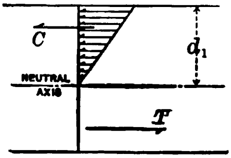


Fig. 2.

In order to find the neutral axis for this condition of stresses it is only necessary to omit the second integral in the first member of eq. (3) of the preceding article. This is equivalent to omitting from eq. (3) of this article every term in which n enters, or, in other words, to making n equal to zero. By completing these omissions the value of d_1 at once takes the form

$$d_1 = -\frac{E_2}{E_1}\frac{A_2}{b} \pm \sqrt{\left(\frac{E_2}{E_1}\frac{A_2}{b}\right)^2 + 2\frac{E_2}{E_1}\frac{A_2}{b}(a+rh_2)}. \quad . \quad (6)$$

The value of d_1 in eq. (6) gives the distance of the neutral axis below the upper or compressive surface of the beam when the tensile resistance of the concrete is neglected.

When the steel reinforcement is wholly on the tension side of the beam, $r=1$, and $a+rh_2=a+h_2$.

Art. 76.—Bending Moments in Concrete-steel Beams by Common Theory of Flexure.

The complete expressions for the bending moments of concrete-steel beams may now be written and their values for any particular case estimated. By introducing the notation already employed into the ordinary formulæ of the common theory of flexure the following general expression for the bending moment will result:

$$M = E_1 u \int_0^{d_1} z^2 dA_1 + nE_1 u \int_{h_1 - d_1}^{0} z^2 dA_1 + E_2 u \int_{h_2' - d_2'}^{d_2'} z^2 dA_2. \quad (1)$$

This equation obviously is based on the condition that the intensities of the tensile and compression bending stresses vary directly as the distances from the neutral axis. The bending moment based on the assumption that those intensities vary according to the parabolic curve will be determined in a later article.

The first two integral expressions in the second member of eq. (1) are moments of inertia of the two portions of A_1, about the neutral axis of the section, while the third is the moment of inertia of A_2 about the same axis. That equation may then be written

$$M = E_1 uI' + nE_1 uI'' + E_2 uI_2. \quad . \quad . \quad . \quad . \quad (2)$$

If I_1 is the moment of inertia of A_1 about the neutral axis, then $I' + I'' = I_1$.

Again, if k_1 is the greatest intensity of compression in the concrete and k' the greatest intensity of tension, while c is the intensity of compression stress in the steel at the distance d_2' from the neutral axis, then

$$E_1 u = \frac{k_1}{d_1}, \quad nE_1 u = \frac{k'}{h_1 - d_1}, \quad \text{and} \quad E_2 u = \frac{c}{d_2}. \quad . \quad . \quad (3)$$

By introducing these values in eq. (2),

$$M = \frac{k_1}{d_1} I' + \frac{k'}{h_1 - d_1} I'' + \frac{c}{d_2} I_2. \quad . \quad . \quad . \quad . \quad (4)$$

This is the general value of the bending moment for any form of section or any two materials. The rectangular normal section will be employed for the outline of the concrete and steel bars of any desirable shape, but of comparatively small section, for concrete-steel beams. The corresponding quantities are then to be introduced into eq. (4). For this case

$$I' = \frac{bd_1^3}{3} \quad \text{and} \quad I'' = \frac{b(h_1 - d_1)^3}{3}.$$

Also

$$I_2 = (1 - r)A_2 d_2^2 + rA_2(h_2 - d_2)^2.$$

The value of I_2 may be written in this form for the reason that the steel sections (A_2) are small.

The insertion of these quantities into eq. (4) will give

$$M = k_1 \frac{bd_1^2}{3} + k' \frac{b(h_1 - d_1)^2}{3} + c\left[(1 - r)A_2 d_2 + rA_2 d_2\left(\frac{h_2}{d_2} - 1\right)^2\right]. \quad (5)$$

The use of this equation for the value of the bending moment can be made more convenient if its second member be changed in form so as to involve but one intensity of stress, and that one may preferably be the greatest compression, k_1, in the concrete.

From eq. (3),

$$k' = n\frac{h_1 - d_1}{d_1} k_1 \quad \text{and} \quad c = \frac{E_2}{E_1} \frac{d_2}{d_1} k_1. \quad . \quad . \quad . \quad (6)$$

By introducing these values in eq. (5),

$$M = k_1\left(\frac{bd_1^2}{3} + \frac{b}{d_1} \frac{n(h_1 - d_1)^3}{3} + \frac{E_2}{E_1} \frac{d_2^2}{d_1} A_2\left[(1 - r) + r\left(\frac{h_2}{d_2} - 1\right)^2\right]\right). \quad (7)$$

This value of the bending moment M requires only the greatest intensity of compression, k_1, in the concrete to be given, all other quantities in the second member being either given or assignable.

The greatest intensity of tension in the steel will be

$$t=\frac{h_2-d_2}{d_2}c=\frac{E_2}{E_1}\frac{h_2-d_2}{d_1}k_1. \quad . \quad . \quad . \quad . \qquad (8)$$

Thus all the intensities in the extreme fibres may be expressed in terms of any one. As a practical matter it is probably preferable to use the greatest compression in the concrete for that purpose, as will appear later.

If the two coefficients of elasticity for the concrete are taken to be the same, $n=1$ in eq. (7).

If the steel reinforcement is wholly on the tension side of the beam, $r=1$ and eq. (7) will take the form

$$M=k_1\left(\frac{bd_1^2}{3}+\frac{b}{d_1}\frac{n(h_1-d_1)^3}{3}+\frac{E_2}{E_1}\frac{A_2}{d_1}(h_2-d_2)^2\right). \quad . \qquad (9)$$

Neglect of the Tensile Resistance of Concrete.

If the bending resistance of the concrete on the tension side of the neutral axis be neglected it is necessary only to omit the second term of the second member of eq. (7) in which n appears. That equation will then take the form

$$M=k_1\left(\frac{bd_1^2}{3}+\frac{E_2}{E_1}\frac{d_2^2}{d_1}A_2\left[(1-r)+r\left(\frac{h_2}{d_2}-1\right)^2\right]\right). \quad . \qquad (10)$$

In case the steel reinforcement is wholly on the tension side of the beam, $r=1$ and the bending moment becomes

$$M=k_1\left(\frac{bd_1^2}{3}+\frac{E_2}{E_1}\frac{A_2}{d_1}(h_2-d_2)^2\right). \quad . \quad . \quad . \qquad (11)$$

In all the preceding formulæ the distance d_1 of the neutral axis from the exterior compression surface of the beam is to be found by the appropriate equation in the preceding article, d_1 from eq. (3) of that article being required for use in the general eq. (7) of this.

The preceding equations complete all that is necessary in the treatment of practical questions of design or of ultimate carrying capacity. Their actual application to engineering work will be illustrated in considerable detail.

Art. 77.—Position of Neutral Axis in Ordinary Rectangular Concrete-steel Beams if Intensities of Stress Vary by the Parabolic Curve.

If the intensities of the bending stresses on the two sides of the neutral axis vary as the abscissæ of a parabola, the condition of stress may be represented by Fig. 1. The vertex of the parabola on the compression side is at V and at v on the tension side. The branches of the two parabolas start from the common point O at the neutral axis. In this case the position of that axis can readily be located by placing the sum of all the tensile stresses in the normal section of the beam equal to the sum of all the compressive stresses in the usual manner. In Fig. 1 the upper part of the beam is supposed to be in compression with the maximum intensity k_1, the lower part being in

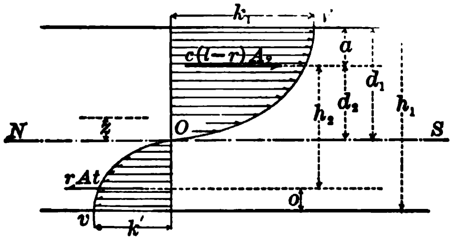


FIG. 1.

tension with the greatest intensity k'. The neutral surface of the beam is NS. The stress in the steel reinforcement on the compression side is $c(1-r)A_2 = C$ and $trA_2 = T$ on the tension side, c and t being the intensities.

The width of the beam being b, the total compressive stress in the concrete is $\frac{2}{3}bd_1k_1$ and the total tension $\frac{2}{3}b(h_1-d_1)k'$, the relatively small area A_2 of the steel section being neglected. Hence the desired equation will be

$$\frac{2}{3}bd_1k_1 + c(1-r)A_2 = \frac{2}{3}b(h_1-d_1)k' + trA_2. \quad . \quad . \quad (1)$$

If it be assumed that plane normal sections of the beam remain plane after flexure, as is usually done, the rate of strain u at unit's distance from the neutral axis may be written, as in the common theory of flexure, as follows, E being the coefficient of elasticity for the concrete in compression and nE_1 in tension:

$$u = \frac{1}{E_1}\frac{k_1}{d_1} = \frac{1}{nE_1}\frac{k'}{(h_1-d_1)} = \frac{1}{E_2}\frac{c}{d_2} = \frac{1}{E_2}\frac{t}{(h_2-d_2)}. \quad . \quad (2)$$

If the values of k_1, k', c, and t each in terms of u and its coefficient of elasticity be taken from eq. (2) and inserted in eq. (1), the latter will become, after uniting its terms,

$$\frac{2}{3}bE_1(1-n)d_1^2 + \frac{4}{3}bh_1nE_1d_1 = \frac{2}{3}bnE_1h_1^2 + E_2A_2(rh_2-d_2).$$

But $d_2 = d_1 - a$; hence

$$d_1^2 + \frac{2h_1n + \frac{3}{2}\frac{E_2}{E_1}\frac{A_2}{b}}{1-n}d_1 = \frac{h_1^2n}{1-n} + \frac{\frac{3}{2}\frac{E_2}{E_1}\frac{A_2}{b}}{1-n}(rh_2+a);$$

$$\therefore\ d_1 = -\frac{h_1n + \frac{3}{4}\frac{E_2}{E_1}\frac{A_2}{b}}{1-n}$$

$$\pm\sqrt{\frac{h_1^2n}{1-n} + \frac{\frac{3}{2}\frac{E_2}{E_1}\frac{A_2}{b}}{1-n}(rh_2+a) + \frac{\left(h_1n + \frac{3}{4}\frac{E_2}{E_1}\frac{A_2}{b}\right)^2}{(1-n)^2}}. \quad (3)$$

All the quantities appearing in the second member of eq. (3) are known, and the position of the neutral axis will be determined by finding the value of d_1.

By comparing eq. (3) with eq. (3) of the second preceding article it will be observed that the two values of d_1 are nearly the same, but it should be noticed that the analysis of this article involves some incongruity. In passing from eq. (1) to eq. (2) the intensities of bending stresses in the concrete are written in the usual manner by employing the coefficients of elasticity and the uniformly varying strains required by the assumption that the normal sections remain plane after flexure. This analytic operation makes the intensities of bending stresses in the concrete vary directly as the distances from the neutral axis, whereas eq. (1) was written on the assumption that those intensities vary according to the parabolic curve shown in Fig. 1. Eq. (3) is thus analytically faulty and possesses no sensible advantage over the other and more rational value of d_1.

If the steel is reinforced entirely on the tension side of the beam r becomes equal to unity, and hence $rh_2+a = h_2+a$, the distance of the centre of the steel bars from the upper surface of the beam.

In case the coefficients of elasticity for tension and compression in the concrete are taken equal, $n=1$ or $1-n=0$. By multiplying both members of the equation immediately preceding eq. (3) by $1-n$, then making $n=1$, there will result

$$d_1 = \frac{h_1^2 + \frac{3}{2}\frac{E_2}{E_1}\frac{A_2}{b}(rh_2+a)}{2h_1 + \frac{3}{2}\frac{E_2}{E_1}\frac{A_2}{b}}. \quad . \quad . \quad . \quad . \quad (4)$$

When the steel reinforcement is wholly on the tension side of the beam r is to be made equal to 1 in eq. (4).

Neglect of the Tensile Resistance of the Concrete.

The conditions of stress for this case are shown in Fig. 2. The stress T in the steel reinforcement is the only tensile stress assumed to act on the tension side of the beam. In eq. (1), therefore, it will be necessary to omit the first term of the second member, leaving the equation in the form,

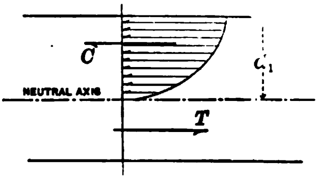


FIG. 2.

$$\tfrac{2}{3}bd_1k_1+c(1-r)A_2=trA_2. \quad . \quad . \quad . \quad . \quad (5)$$

The effect of this change is equivalent to making $n=0$ in eq. (3). The value of d_1 will then become

$$d_1=-\frac{3}{4}\frac{E_2}{E_1}\frac{A_2}{b}\pm\sqrt{\left(\frac{3}{4}\frac{E_2}{E_1}\frac{A_2}{b}\right)^2+\frac{3}{2}\frac{E_2}{E_1}\frac{A_2}{b}(rh_2+a)}. \quad . \quad (6)$$

The distance of the neutral axis below the upper surface of the beam is thus determined for this particular case. Where the steel reinforcement is wholly on the tension side of the beam $r=1$ and $rh_2+a=h_2+a$.

Art. 78.—Bending Moments in Concrete-steel Beams if Intensities of Stress Vary by the Parabolic Curve.

If the intensities of tension and compression in the concrete of the composite beam vary as the horizontal coordinates of the two branches of the parabolas shown in Fig. 1 of the preceding article, the vertices of the parabolas being in the upper and lower surfaces of the beam, the moments of the internal stresses may readily be found. The centres of gravity of the two parabolic areas will be $\frac{5}{8}d_1$ and $\frac{5}{8}(h_1-d_1)$ from the neutral axis, and these areas will be $\frac{2}{3}k_1d_1$ and $\frac{2}{3}k'(h_1-d_1)$. Hence the total bending moment of the stresses of tension and compression in the concrete will be

$$\tfrac{5}{12}d_1{}^2k_1+\tfrac{5}{12}(h_1-d_1)^2k'.$$

The total bending moment of all the stresses, including those of the reinforcing steel, will then be, remembering that the breadth of the concrete is b,

$$M=\tfrac{5}{12}(d_1{}^2k_1+(h_1-d_1)^2k')b+c(1-r)A_2d_2+trA_2(h_2-d_2). \quad (1)$$

As E_1 is the coefficient of elasticity of the concrete in compression and nE_1 in tension, there may be written

$$k'=n\frac{(h_1-d_1)}{d_1}k_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Also

$$c=\frac{E_2}{E_1}\left(1-\frac{a^2}{d_1{}^2}\right)k_1, \quad \text{and} \quad t=\frac{E_2}{nE_1}\left(1-\frac{o^2}{(h_1-d_1)^2}\right)k'. \quad (3)$$

Or, by using the preceding expression for k' in the second of eqs. (3),

$$t=\frac{E_2}{E_1}\frac{(h_1-d_1)}{d_1}\left(1-\frac{o^2}{(h_1-d_1)^2}\right)k_1. \quad . \quad . \quad . \quad (4)$$

Thus all the intensities of stresses k', c, and t are expressed in terms of k_1. The values of those intensities given in eqs. (2), (3), and (4) may, if desired, be inserted in eq. (1), so that the bending moment M would be expressed in terms of k_1. As those intensities must, however, usually be determined in any event, it is scarcely necessary to write the value of M in such an extended and complicated equation.

If the two coefficients of elasticity for concrete are taken equal to each other the ratio n becomes unity in eqs. (2) and (3).

If the steel reinforcement is found wholly on the tension side of the beam, $r=1$, and the term $c(1-r)A_2d_2$ becomes equal to zero in eq. (1).

Finally, if the resistance of the concrete to tension be neglected, $k'=o$, and the term $\frac{5}{12}(h_1-d_1)^2k'$ in the second member of eq. (1) disappears.

In case the latter two conditions concur, i.e., if there is tension steel reinforcement only, and if the tensile resistance of concrete is neglected, the value of the bending moment M may be written in terms of k_1 as follows:

$$M=\left(\frac{5}{12}bd_1^2+\frac{E_2}{E_1}rA_2(h_2-d_2)\left(\frac{h_1-d_1}{d_1}-\frac{o^2}{d_1(h_1-d_1)}\right)\right)k_1. \quad (5)$$

Art. 79.—Division of Loading Between the Concrete and Steel Under the Common Theory of Flexure.

It is occasionally desirable to determine the portion of the total loading of either a concrete-steel beam or arch carried by the steel and concrete parts of the member. In making this determination the formula established in the preceding articles in accordance with the common theory of flexure will be employed. It will be convenient also for this purpose to represent the intensity of stress in the extreme fibre of the steel, whether tension or compression, by k_2, the distance of that extreme fibre from the neutral axis of the composite section, established in Art. 75, being represented by d_2. It will further be supposed that the coefficients of elasticity for concrete in tension and compression are the same. Eq. (4) of Art. 76, representing the resisting moment of the internal stresses in a normal section of a composite member, may then be written

$$M = \frac{k_1 I_1}{d_1} + \frac{k_2 I_2}{d_2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Let the total load on the composite beam or arch be represented by W, while W_1 and W_2 represent the portions of W carried by the steel and concrete respectively. Also let q_1 and q_2 be so taken that $q_1 = \frac{W_1}{W}$ and $q_2 = \frac{W_2}{W}$. The remaining notation will be that given in Art. 74.

Since the bending moments in the portions A_1 and A_2 are proportional to the loads which those portions carry, remembering that $\frac{k_1}{d_1}$ and $\frac{k_2}{d_2}$ are equal to $E_1 u$ and $E_2 u$ respectively, there may be written, as indicated by eq. (1),

$$\left.\begin{array}{l} q_1 = \frac{W_1}{W} = \frac{E_1 I_1}{E_1 I_1 + E_2 I_2} \quad \text{and} \quad q_2 = \frac{W_2}{W} = \frac{E_2 I_2}{E_1 I_1 + E_2 I_2} \\ \text{Also} \\ q_1 = \frac{I_1}{I_1 + e I_2} \quad \text{and} \quad q_2 = \frac{e I_2}{I_1 + e I_2} \end{array}\right\} . \quad (2)$$

Also, if $n = \frac{W_1}{W_2}$,

$$q_1 = \frac{n}{n+1} \quad \text{and} \quad q_2 = \frac{1}{n+1}. \quad . \quad . \quad . \quad . \quad (3)$$

Then, since $M_1 = q_1 M$ and $M_2 = q_2 M$,

$$k_1 = \frac{q_1 M d_1}{I_1} \quad \text{and} \quad k_2 = \frac{q_2 M d_2}{I_2}. \quad . \quad . \quad . \quad (4)$$

Eqs. (2), (3), and (4) show the portions of loading carried by the two materials and the greatest intensities of stresses in their extreme fibres.

It is sometimes necessary to combine a bending moment with the direct compression (or tension) produced by a force P acting along or parallel to the axis of a beam or arch. Let p_1 and p_2 represent the intensities of stress produced in the two portions A_1 and A_2 by such a direct force. Since equal unit longitudinal strains exist in the two materials, the intensities of stress in the portions A_1 and A_2 will be proportional to their coefficients of elasticities. Hence

$$\frac{p_1}{p_2}=\frac{E_1}{E_2} \quad \text{and} \quad p_2=\frac{E_2}{E_1}p_1. \quad . \quad . \quad . \quad . \quad (5)$$

Hence

$$p_1A_1+ep_1A_2=P; \quad \therefore \; p_1=\frac{P}{A_1+eA_2}. \quad . \quad . \quad (6)$$

In the case of an elastic arch like those of combined concrete and steel, the thrust P is in general exerted along the axis of the arch ring but at some distance, l, from it. In such a case the bending moment is

$$M=Pl; \quad \text{hence} \quad M_1=q_1Pl \quad \text{and} \quad M_2=q_2Pl. \quad . \quad (7)$$

The values of the bending moments are to be placed in eq. (4), in order to determine the intensities k_1 and k_2.

In determining the resultant of stress for any section of an arch ring, if the conditions under which eqs. (2) were written be employed, the thrust on the portion A_1 will be q_1P, and q_2P on A_2, since the thrusts on the two portions will be proportional to the loads which they carry. Hence, if k_1 and k_2 again be used to represent the greatest intensities of stress in the two portions, there may at once be written

$$k_1 = q_1\left(\frac{P}{A_1} \pm \frac{Md_1}{I_1}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

$$k_2 = q_2\left(\frac{P}{A_2} \pm \frac{Md_2}{I_2}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

In eqs. (8) and (9), $M = Pl$.

If, again, the last members of eqs. (5) and (6) be used in connection with eqs. (2) and (4) the resultant values of k_1 and k_2 will be

$$k_1 = p_1 + \frac{q_1 Md_1}{I_1} = \frac{P}{A_1 + eA_2} + \frac{Md_1}{I_1 + eI_2}, \quad . \quad . \quad . \quad (10)$$

$$k_2 = p_2 + \frac{q_2 Md_2}{I_2} = e\left(\frac{P}{A_1 + eA_2} + \frac{Md_2}{I_1 + eI_2}\right). \quad . \quad . \quad (11)$$

In the use of all these equations, care must be taken to give the proper sign to the bending moment M.

These equations comprise all that are necessary in order to ascertain the distribution of the loading between the steel and the concrete, or any other two materials, whether the case may be one of pure bending or a combination of bending and direct stress.

Art. 80.—Practical Application of the Formulæ for Concrete-steel Beams.

The practical application of the formulæ of the preceding articles either in the operations of design or in the testing of actual beams is a simple matter. This will be illustrated by determining the distance d_1 from the top surface of the beam to the neutral axis, and the extreme fibre stresses in the concrete and steel of a number of composite beams the results of whose tests to failure are given by Edwin Thacher, C.E., in the "Transactions of the Association of Civil Engineers of Cornell University," 1902, Vol. X.

TABLE I.

TESTS OF CONCRETE-STEEL BEAMS.

Experimenter.	Span, Ins.	Depth of Beam, Ins.	Br'dth of Beam, Ins.	Concrete.		Metal.		No. of Tests.
				Mixture.	Age, Days.	No. and Size.	Total Area, Sq. Ins.	
Thacher. .	138	11	5	1st 1 c., 3 s., 6 b. s. 2d 1 c., 2 s., 4 b. s.	78 81 & 83	3—⅜″ ○	.331	3
" . . .	72	4	10	1st 1 c., 3 s., 6 b. s. 2d 1 c., 2 s., 4 b. s.	79 84	2—⅜″ ○	.221	3
Kirkaldy. .	60	12	12	1 c., 2 crushed brick	60–90	1*—2½ × ¼″	.625	9
Ransome.	49.8	4	60	1 c., 3 s., 6 b. s.	21	10—⅜″ □	1.406	1

Centre of Steel Rods.	Neutral Surface.	Breaking Weight at Centre of Span, Pounds.	Maximum Fibre Stress in Pounds per Square Inch.						Manner of Failure.
From Top Surface of Beam, Ins.			Concrete.			Steel.			
			Max.	Mean.	Least.	Max.	Mean.	Least.	
10	3.28	4,985 to 5,095	2,420	2,385	2,375	59,500	58,600	58,400	Rods broke.
3.25	1.07	2,091 to 2,237	2,583	2,520	2,414	63,151	61,600	59,020	Rods broke.
10	2.7	16,761–21,967	2,230	1,935	1,705	72,300	62,700	55,300	Rods broke and concrete failed.
3.75	1.2	21,220	——	1,320	——	——	33,700	——	Concrete failed; not old enough.

* Wrought-iron bars.

Table I contains the elements of those tests and their results. The beams were all of rectangular section and of sufficient dimensions to make them full size. *The tensile resistance of the concrete is neglected in all of them and the steel or iron reinforcement is concentrated on the tension side of the beam in every case.*

In order to locate the neutral axis, therefore, eq. (6) of Art. 75 must be employed. Although the coefficient of elasticity E_2 is generally a little larger for the 1 cement, 2 sand, and 4 broken-stone concrete than for the 1 : 3 : 6

mixture, the difference is small and the value 2,500,000 will be taken for each. As the coefficient for steel is 30,000,000, the ratio $\frac{E_2}{E_1}=12$. Eq. (6) of Art. 75 will then take the form

$$d_1=-12\frac{A_2}{b}\pm4.9\sqrt{6\left(\frac{A_2}{b}\right)^2+\frac{A_2}{b}(a+h_2)}. \quad . \quad . \quad (1)$$

In the first set of three tests in the table $A_2=.331$ sq. in., and $b=5$ inches. Also $a+h_2=10$ inches. The insertion of these quantities in eq. (1) gives

$$d_1=3.28 \text{ inches.}$$

In the same manner, the quantities to be inserted in eq. (1) for the second set of three values (second line of table) are

$A_2=.22$ sq. in., $b=10$ inches, and $a+h_2=3.25$ inches.

Hence

$$d_1=1.07 \text{ inches.}$$

The simple repetition of these operations for the Kirkaldy and Ransome tests will give the values of 2.7 inches and 1.2 inches shown in the eleventh column of the table.

The determination of the stresses in the extreme fibres of the concrete and steel requires the use first of eq. (11), Art. 76, and then eq. (8) of Art. 76. The first of those equations gives, remembering that h_2-d_2 is the distance from the neutral surface to the centre of the steel or iron reinforcing members, and that $\frac{E_2}{E_1}=12$,

$$k_1=\frac{3M}{bd_1^2+36\frac{A_2}{d_1}(h_2-d_2)^2}. \quad . \quad . \quad . \quad . \quad (2)$$

All the beams were broken by a centre load, but the centre breaking loads given in the table include the half-weights of the beams. If, therefore, W represent any centre breaking weight in the table and l the span in inches, the moment M in eq. (2) will be

$$M = \frac{Wl}{4}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

In the first set of three tests of the table, represented by the first or upper line, the following numerical quantities are found for the mean results:

$$W = 5022 \text{ pounds}; \quad l = 138 \text{ inches};$$

$$\therefore \ M = W\frac{l}{4} = 173{,}259 \text{ in.-lbs.}$$

Also, $h_2 - d_2 = 10 - 3.28 = 6.72$ inches;

$b = 5$ inches; $d_1 = 3.28$ inches; $A_2 = .331$ sq. in.

The substitution of these quantities in eq. (2) will give

$$k_1 = 2385 \text{ pounds per sq. in.}$$

Eq. (8) of Art. 76 now gives

$$t = 12 \frac{6.72}{3.28} 2385 = 58{,}600 \text{ pounds per sq. in.}$$

Again, in the second set of three tests the smallest centre breaking weight, including half the weight of the beam, is 2091 pounds, the span being 72 inches. Hence

$$M = \frac{Wl}{4} = \frac{2091 \times 72}{4} = 37{,}638 \text{ in.-lbs.}$$

Also, $h_2 - d_2 = 2.18$ inches; $d_1 = 1.07$ inches;

$$A_2 = .221 \text{ sq. in.}; \quad b = 10 \text{ inches.}$$

Employing these numerical quantities in eq. (2),

$$k_1 = 2414 \text{ pounds per sq. in.}$$

Eq. (8) of Art. 76 will then give

$$t = 12 \frac{2.18}{1.07} 2414 = 63,151 \text{ pounds per sq. in.}$$

Both of the preceding values of t are partly representative of the ultimate tensile resistance of low steel.

Eq. (2) shows that where there are a number of tests of beams of the same size and span M is the only variable. In other words, k_1, and hence t, will vary as M or, what is equivalent, as the centre breaking weight. Using the data of the last of the preceding examples, the greatest centre breaking load was 2237 pounds. Hence the greatest k_1 is

$$k_1 = \frac{2237}{2091} 2414 = 2583 \text{ pounds per sq. in.}$$

Also, for the stress in the steel,

$$t = \frac{2237}{2091} 59,020 = 63,151 \text{ pounds per sq. in.}$$

The remaining values of k_1 and t in the table were found in precisely the same manner by employing the data yielded by the tests, also exhibited in the table.

Hatt's Tests of Reinforced Beams.

The results of tests of reinforced and plain concrete beams given in Table II have been computed from data in a paper presented before the American Society for Testing Materials at its annual meeting in June, 1902, by Prof. W. Kendrick Hatt. As the section of every beam was 8 in. by 8 in., those beams having a reinforcement of 1 per cent. of wrought iron contained bars whose aggregate sectional area was 0.64 sq. in. Similarly the 2 per cent. and 3 per cent. reinforcements indicate 1.28 sq. in. and 1.92 sq. in. total sectional areas of wrought iron respectively.

TABLE II.

TESTS OF REINFORCED AND PLAIN PORTLAND-CEMENT CONCRETE BEAMS, 8 INS. × 8 INS. IN SECTION AND 80 INS. SPAN, BROKEN BY CENTRE LOADS.

No.	Per Cent. of Iron Section.	Iron from Bottom of Beam, Ins.	Age in Days.	Centre Breaking Load, Pounds.	Extreme Fibre Stress in Concrete, Pounds per Sq. In.	Stress in Iron, Pounds per Sq. In.	Remarks.
1	1	2	25	5,735	2,170	34,720	Broken-stone concrete: 1 cem., 2 sand, 4 b. s.
			28	6,485	2,450	39,200	
2	1	2	7	6,235	2,360	37,760	Do. Do. Do.
			9	5,735	2,170	34,720	
3	2	2	25	10,485	3,280	33,456	Do. Do. Do.
			27	10,485	3,280	33,456	
4	1	1	25	7,735	2,840	55,700	Do. Do. Do.
			23	7,535	2,760	54,180	
5	2	1	27	12,235	2,930	32,820	Do. Do. Do.
			25	12,035	2,880	32,260	
6	1	1½	30	6,735	2,230	37,700	Do. Do. Do.
			30	6,635	2,195	37,100	
7	3	1	6	11,735	1,795	20,590	Gravel concrete: 1 cement, 5 gravel.
8	2	1	17	5,410	1,150	15,180	Cinder concrete.
			17	2,480	530	7,000	
9	—	—	28	2,635	618	——	Plain broken-stone concrete.
			28	2,435	571	——	
10	—	—	7	3,520*	425	——	Do. Do. Do.
			8	1,735	407	——	
11	—	—	14	760	176	——	Plain cinder concrete.
12	—	—	11	1,260*	165	——	" " "
			11	1,260*	165	——	" " "

* These beams tested on a span of 40 inches instead of 80 inches.

The third column of the table gives the distances of the centres of the wrought-iron reinforcements from the bottom surfaces of the beams. It is particularly to be observed that the wrought-iron reinforcement is placed entirely in the lower or tension portion of the beams, and that the tensile resistance of the concrete is ignored.

The tests gave the centre breaking loads shown in the fifth column of the table, each of which includes half the own weight of the beam for an 80-inch span, except for the tests indicated by stars, in which the span was but 40 inches and the half-weight of beam 118 pounds.

The manner of computing the intensities of extreme fibre stresses given in the sixth column of the table is the same for reinforced beams as that set forth in connection with Table I, and it will be illustrated by one value of Table II only.

In the upper of the two number 4 tests the centre breaking load was found to be 7500 pounds, which with the half-weight, 235 pounds, of the beam makes a total of 7735 pounds. The centre breaking moment is $M = \frac{7735 \times 80}{4} = 154,700$ in.-lbs. The area A_2 of the wrought-iron section is 1 per cent. of 64 sq. in. = 0.64 sq. in. The remaining data are

$$b = h_1 = 8 \text{ inches}; \quad a + h_2 = 7 \text{ inches};$$

$$E_2 = 29,000,000; \quad E_1 = 2,400,000; \quad \therefore \; \frac{E_2}{E_1} = 12 \text{ (nearly).}$$

These quantities inserted in eq. (6), Art. 75, give

$$d_1 = -.96 \pm 3.615 = 2.655 \text{ inches};$$

$$\therefore \; h_2 - d_2 = 7 - 2.655 = 4.345 \text{ inches.}$$

By using the preceding numerical values in eq. (11), Art. 76, put in the form of eq. (2) of this article,

$$k_1 = \frac{M}{54.56} = \frac{154,700}{54.56} = 2840 \text{ lbs. per sq. in.}$$

This is the intensity of the greatest stress in the extreme fibres of the concrete.

Eq. (8) of Art. 76 gives for the greatest intensity of tensile stress in the wrought iron

$$t = 12\frac{h_2 - d_2}{d_1}k_1 = 55,700 \text{ lbs. per sq. in.}$$

These operations illustrate sufficiently the methods of finding all the values in the sixth and seventh columns of the table for the reinforced beams.

Tests Nos. 9, 10, 11, and 12 are of plain concrete beams, the concrete being either of broken stone or cinder without reinforcement. Eq. (3) of Art. 69 gives for such beams

$$K = k_1 = \frac{6M}{bh^2} = .2344W.$$

In this equation *W* is the centre breaking load in pounds, including half the weight of the beam. The values of the centre breaking loads in the fifth column of the table substituted in the preceding equation will give the greatest tensile intensities (k_1) of stress shown in the sixth column. These plain concrete beams failed, as would be expected, on the tensile side.

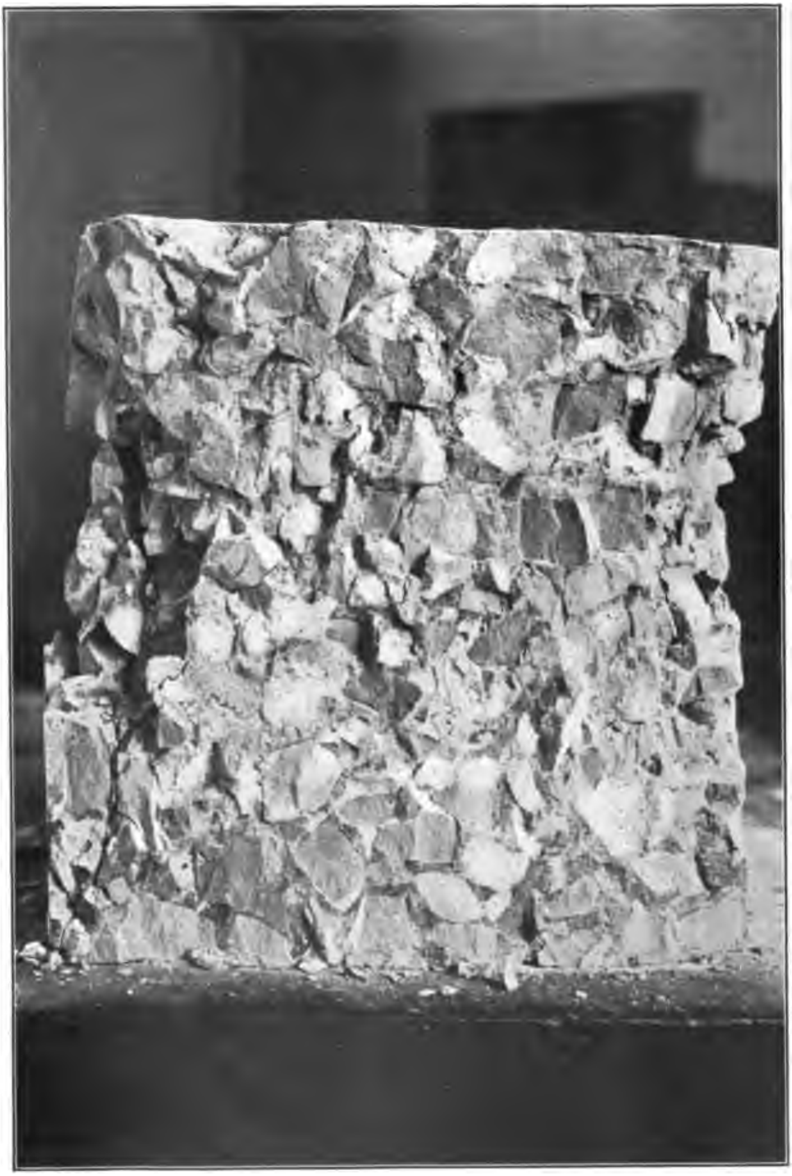
The reinforced broken-stone concrete beams to which Table II applies failed by the breaking of the reinforcement, the concrete in compression showing no signs of failure, although the greatest age of the concrete was but 30 days and in two cases 7 and 9 days only.

The cinder-concrete beams, both reinforced and plain, show ultimate resistances scarcely half those of the broken-stone concrete.

A coefficient of elasticity of 1,450,000 was taken for cinder concrete, making the ratio $E_2 \div E_1 = 20$ in the two tests No. 8.

In 1901 ten beams of concrete and concrete reinforced with steel rods were broken at the mechanical laboratory of the Massachusetts Institute of Technology, Boston, with the results given in the following table. These beams were all 8 ins. wide and 12 ins. deep, the effective span being 11 ft. Beam No. 1 was of concrete without the reinforcement of steel rods. The other nine beams carried either one or two twisted square steel rods of the Ransome type of the sizes shown in the table. These rods were placed with their centres 2 ins. from the outside bottom or top surface of the beam. The breaking load was applied at the centre of the span for seven of these beams, but for the other three the breaking load was applied at two points 44 ins. apart and equidistant from the centre of the span. In the latter cases, therefore, the two equal loads were applied at the one-third points of the span. These elements of the tests are all clearly shown in Table III.

The concrete was composed of one volume Alpha Portland cement, three volumes of (Plum Island) sand, and six volumes of crushed stone, the latter being composed of one third trap rock with the greatest dimension of ½ in. and two thirds of the same rock of 1 in. greatest dimension. Briquettes of the mortar used in this concrete gave an ultimate tensile resistance of 250 lbs. per sq. in. at the age of 17 days and 300 lbs. per sq. in. at the age of 28 days.



A view exhibiting the failure under compression of a 12-in. concrete cube. The composition is 1 Portland cement, 1 sand, and 4.5 broken stone. The age of the concrete was 1 year, 8 months, 23 days, and the ultimate compressive resistance attained was 4481 lbs. per sq. in.

(*To face page* 660.)

TABLE III.

No. of Beam.	Age, Days.	Reinforcing Rods: Size and Position.	Loading.	Final Stresses, Lbs. per Sq. In.		Manner of Failure.
				Concrete, Compression.	Steel, Tension.	
1	40	None	Centre	776	0	
2	39	One—½ inch.....B	Two points	2,816	117,700	Rod drew out.
3	38	One—¾ " .....B	Centre	3,427	95,360	Con. crushed.
4	50	One—⅞ " .....B	"	3,910	84,470	Rod drew out.
5	50	One—1 " .....B	"	3,770	68,400	Concrete split off at bottom.
6	41	One—1¼ " .....B	"	2,802	37,612	Rod drew out.
7	41	Two—1 " .....B	"	3,655	41,949	" " "
8	42	Two—1¼ " .....B	"	4,780	32,360	" " "
9	30	Two—1¼ " .....B	Two points	4,053	22,750	" " "
10	31	Two—1½ " .....B	" "	3,412	27,020	Rod drew out, con. crushed.

B indicates bottom.

The ultimate tensile resistance of the ¼-in. square steel rods was 82,560 lbs. per sq. in., while the ½-in. rods gave 99,880 and 102,000 lbs. per sq. in. The ¾-in., ⅞-in., and 1-in. rods gave ultimate resistances running from 82,000 down to 70,800 lbs. per sq. in., the lowest values belonging to the 1-in. square rods.

The ultimate compressive resistances of the concrete and the greatest tensile stresses in the steel rods have been computed by the preceding formulæ, as already illustrated in the other tables of this article, and hence the detailed operations need not be repeated here.

Reviewing all the preceding results, it may be stated that the greatest working intensity of compression in the extreme fibres of the concrete of reinforced beams may properly be taken at 400 to 500 pounds per sq. in. for 1 : 2 : 4 broken-stone concrete, or even for 1 : 3 : 5 mixtures if the materials are well balanced, care being taken in all cases to produce the best grade of concrete.

PROBLEMS.

One or two applications to design work will show how simple is the practical use of the formulæ of the preceding articles.

Prob. 1. Design a concrete-steel beam for a clear span of 15 feet to carry a load of 380 pounds per linear foot in addition to its own weight, the maximum compression k_1 in the extreme fibres of the concrete to be 400 pounds per sq. in.

Let it be supposed that the concrete mixture is 1 cement, 3 sand, and 5 broken stone, for which the coefficient of elasticity is 2,500,000, the coefficient for steel being 30,000,000. Assume a trial beam 18 inches deep and 12 inches wide as shown in Fig. 1. Also assume the entire steel reinforcement, on the tension side of the beam, to be composed of four ¾-inch square steel bars with their centres 1½ inches from the tension surface of the beam. The tensile resistance of the concrete will be neglected. The following data will then result, remembering that the weight of the beam per linear foot will be taken as $150+75=225$ lbs.:

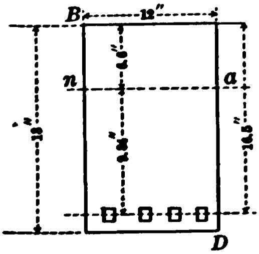


FIG. 1.

$$\text{Total load} = (380+225)16 = 9680 \text{ lbs.}$$

The effective length of span is taken at 16 feet.

$$A_2 = 2.25 \text{ sq. ins.}; \quad b = 12 \text{ ins.}; \quad k_1 = 400 \text{ lbs.};$$

$$l = 192 \text{ ins.}; \quad \frac{E_2}{E_1} = 12; \quad a + h_2 = 16.5 \text{ ins.}$$

As the two coefficients of elasticity for concrete are taken equal to each other, $n=1$.

These data substituted in eq. (6) of Art. 75 give for the distance of the neutral axis from the upper or compressive surface of the beam

$$d_1 = -2.256 + 8.918 = 6.66 \text{ ins.}$$

The depth of the beam is fixed at 18 inches, but the width or breadth may vary. Eq. (11) of Art. 76 gives

$$b = \frac{3\frac{M}{k_1} - 36\frac{A_2}{d_1}(h_2 - d_2)^2}{d_1^2}.$$

Using the preceding data,

$$M = \frac{9680 \times 192}{8} = 232{,}320 \text{ in.-lbs.,} \quad \text{and} \quad h_2 - d_2 = 9.84 \text{ ins.}$$

Substituting in the above equation for b, there will result

$$b = 12.6 \text{ ins.}$$

The beam should therefore be made 12.6 inches wide. This is sufficiently near the assumed width to need no recomputation. If the difference were considerable, so as to vary sensibly the weight of beam and, hence, the moment M, a revision of the values of d_1 and b would be necessary.

Eq. (8) of Art. 76 gives the intensity of tensile stress in the steel as

$$t = 12\,\frac{9.84}{6.66}\,400 = 7104 \text{ lbs.}$$

This is smaller than necessary, as t might safely be taken at 14,000 pounds per sq. in. A revision should therefore be made if the beam were to be built, so as to reduce properly the size of the steel bars.

Prob. 2. Let it be desired to design a concrete-steel floor-beam or plate between steel I beams placed 6 feet between centres, the load to be carried being 100 pounds per sq. ft., in addition to the weight of the concrete-steel plate itself. A thickness or depth of concrete of 4 inches is usually sufficient for such a span, and that depth will be assumed. In order to illustrate the application of the formulæ, small steel bars or rods will be assumed in the compression part of the concrete, although they would not ordinarily be employed. The arrangement of a 12-inch section of the plate is shown in Fig. 2. A row of ¼-inch square steel rods 4 inches between centres is placed .9 inch from the upper surface of the concrete, and a row of ⅜-inch square steel bars at the same distance from the lower surface. The distance between centres of the upper and lower steel bars is, therefore, 2.2 inches. The tensile resistance of the concrete will be neglected, and the coefficients of tensile and compressive elasticity of the same material will be given the same value, i.e., 2,500,000 pounds. The coefficient of elasticity for steel will be taken at 30,000,000 pounds. The following data will therefore result for a 12-inch width of the concrete-steel plate:

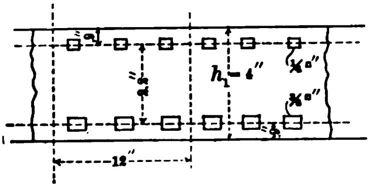


FIG. 2.

Sectional area of three ¼-inch square bars...... .188 sq. in.
" " " " ⅜- " " " ...... .42 " "

Hence.............. $A_2 = .608$ " "

$r = .7$; $b = 12$ ins.; $h_1 = 4$ ins.; $a = .9$ in.;
$h_2 = 2.2$ ins.; $rh_2 = 1.54$ ins.; $a + rh_2 = 2.44$ ins.;
$\frac{E_2}{E_1} = 12$; $\frac{A_2}{b} = .051$.

The substitution of these values in eq. (6) of Art. 75 will give the distance of the neutral surface from the top surface of the beam as

$$d_1 = -.612 + 1.834 = 1''.22.$$

Hence

$$d_2 = 1.22 - .9 = .32.$$

The greatest permissible compressive working stress, k_1, in the extreme fibres of the concrete will be taken at 440 pounds per sq. in. The 4-inch plate of concrete with its steel bars will be taken to weigh 65 pounds per sq. ft. As the length of span is 6 feet, i.e., 72 inches, the total bending moment M is

$$M = \frac{165 \times 6 \times 72}{8} = 8910 \text{ in.-lbs.}$$

If the numerical values found above be inserted in eq. (10) of Art. 76, there will result

$$M = 440(5.96 + 14.63) = 9060 \text{ in.-lbs.}$$

This is sufficiently near to 8910 in.-lbs. to justify accepting as final the depth of concrete, arrangement, and size of steel bars assumed.

Eq. (10) of Art. 76 might have been used to test the trial values by finding in that equation either k_1 or b after substituting the given value of M and then comparing the results with those assumed. Either procedure most convenient may be employed.

Eq. (8) of Art. 76 gives the intensity of tensile stress in the steel as

$$t = 12 \frac{1.88}{1.22} 440 = 8131 \text{ lbs. per sq. in.,}$$

and the compressive intensity

$$c = \frac{.32}{1.88} 8131 = 1384 \text{ lbs. per sq. in.}$$

The low value of c shows that the steel bars in compression could well be omitted, as they usually are in such cases.

The important duty performed by the concrete in compression, as shown by these practical applications, indicates that a strong concrete should be employed for this class of construction.

CHAPTER XII.

ROLLED AND CAST FLANGED BEAMS.

Art. 81.—Flanged Beams in General.

In the beams which are to follow, the material is distributed in a much more advantageous manner, in respect of its resisting moment, than in the solid beams which have been heretofore treated. In these beams it will be found, in almost all cases, that the ultimate intensity of bending stress, at the point which first ruptures, is equal either to the ultimate resistance to tension or compression. In this respect at least, therefore, the ultimate load for flanged beams is more easily and exactly determined than for solid ones.

In Fig. 1 of Art. 41 is shown a "flanged beam." The "flanges" are the two horizontal parts above and below; the "web" is the vertical part uniting the two flanges so as to form the perfect beam.

In order that there may be economy of material in the beam, neither flange must begin to fail before the other; in other words, the two exterior layers of fibres, above and below, must begin to fail at the same time.

The intensities, then, in these two layers must, at the instant of rupture, equal the ultimate resistances to tension and compression in bending.

These general considerations show that the flange stresses in the flanged beam correspond largely to the direct tensile and compressive stresses of the solid beam, and if the bending resistance of the web be neglected they correspond exactly.

If the bending resistance of the web be recognized, the moment of the internal stresses of a flanged beam has the same expression as for the solid beam,

$$M=\frac{KI}{h_1} \quad \text{or} \quad M=\frac{KI}{h}, \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

according as the failure is in one flange or the other, I being the moment of inertia of the entire section about its neutral axis, and K the intensity of stress in the extreme fibres at the distance h_1 or h. If, on the other hand, the bending resistance of the web is to be neglected, as is frequently done in plate girders, the expression for the moment of the internal stresses takes a much simpler form. The stresses of tension and compression in the flanges are then supposed to be uniformly distributed over the flange sections, so that the centre of stress in each flange is the centre of gravity of that flange section. The normal or vertical distance between those centres of gravity is called the effective depth of the beam or girder.

Let a be the sectional area of the tension flange and T the uniform intensity of stress in it, and let a' and C be the corresponding values for the compression flange, while d is the effective depth. Then, since $aT=a'C$, the moment of the internal stresses will be

$$M=aTd=a'Cd. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

The use of both eqs. (1) and (2) will be illustrated by numerous practical applications.

It is clear from what has preceded that the chief function of the flanges is to resist the bending proper, while the main function of the web is to resist the transverse shear.

Art. 82.—Flanged Beams with Unequal Flanges.

Although the tensile and compressive coefficients of elasticity for all the structural metals may be taken the same for each metal in engineering design, the increasing use of beams of a composite nature makes it desirable to treat flanged beams with unequal flanges first with equal coefficients of elasticity and then with unequal.

Equal Coefficients of Elasticity.

By the common theory of flexure, if the two coefficients of elasticity are equal, it has been shown that if C, Fig. 1, is the centre of gravity of the cross-section, the neutral axis of the section will pass through that point. Let it now be supposed that the lower flange is in tension, while the upper is in compression. Also let T represent the ultimate resistance to tension in bending, and let C represent the same quantity for compression in bending. Then since intensities vary directly as distances from the neutral axis,

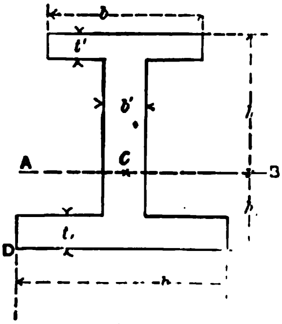


Fig. 1.

$$\frac{h_1}{h} = \frac{T}{C}; \quad \therefore \; h_1 = h\frac{T}{C} = n'h. \quad . \quad . \quad . \quad . \quad (1)$$

The ratio between ultimate intensities is represented by n'. If $d=h+h_1$ is the total depth of the beam, and hence $h=d-h_1$,

$$h_1 = n\,(d-h_1) = \frac{dn'}{1+n} = \frac{\frac{T}{C}d}{1+\frac{T}{C}} \quad . \quad . \quad . \quad . \quad (2)$$

If, as an example, for structural steel there be taken

$$n' = \frac{T}{C} = .9, \quad h_1 = \frac{9}{19}d.$$

The relation between h and h_1 shown in eq. (2) is entirely independent of the form of cross-section. But according to the principles just given, in order that economy of material shall obtain, *the cross-section should be so designed that h and h_1 shall represent the distances of the centre of gravity from the exterior fibres.*

The analytical expression for the distance of the centre of gravity from DF is

$$x_1 = \frac{\frac{1}{2}b'a^2 + (b-b')t'(d-\frac{1}{2}t') + \frac{1}{2}(b_1-b')t_1^2}{b\,d + (b-b')t' + (b_1-b')t_1} \quad . \quad . \quad . \quad (3)$$

The meaning of the letters used is fully shown in the figure. In order that the beam shall be equally strong in the two flanges, the various dimensions of the beam must be so designed that

$$x_1 = h_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

It would probably be found far more convenient to cut sections out of stiff manila paper and balance them upon a knife-edge.

The moment of inertia about the axis AB, thus determined, is

$$I = \tfrac{1}{3}[bh^3 + b_1h_1^3 - (b-b')(h-t')^3 - (b_1-b')(h_1-t_1)^3].$$

This value is to be substituted in eq. (2) of Art. 68, now changed to

$$M = \frac{CI}{h} = \frac{TI}{h_1}.$$

For various beams whose lengths are l and total load W the greatest value of M becomes:

Cantilever uniformly loaded,

$$M = \frac{Wl}{2}.$$

Cantilever loaded at end,

$$M = Wl.$$

Beam supported at each end and uniformly loaded,

$$M = \frac{W}{8} = \frac{pl^2}{8}.$$

Beam supported at each end and loaded at centre,

$$M = \frac{Wl}{4}.$$

The last two cases combined,

$$M = \frac{l}{4}\left(W + \frac{pl}{2}\right).$$

Sometimes the resistance of the web is omitted from consideration. In such a case the intensity of stress in each flange is assumed to be uniform and equal to either T or C. At the same time the lever-arms of the different fibres are taken to be uniform, and equal to h for one flange and h_1 for the other, *h and h_1 now representing the vertical distances from the neutral axis to the centres of gravity of the flanges, while $d=h+h_1$.*

On these assumptions, if f is the area of the upper flange and f' that of the lower, there will result

$$M=fC.h+f'T.h_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

But since the case is one of pure flexure,

$$fC=f'T. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$\therefore \ M=fC(h+h_1)=fCd=f'Td. \quad . \quad . \quad . \quad (7)$$

Also, from eq. (6),

$$\frac{f}{f'}=\frac{T}{C}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Or, *the areas of the flanges are inversely as the ultimate resistances.*

Unequal Coefficients of Elasticity.

All the preceding results presuppose equality between the coefficients of elasticity for tension and compression. In some cases this presumption is not permissible. Resort must then be made to the formulæ of Art. 26.

The neutral surface must first be located. If d is the total depth of the beam, $h_1=d-h$; h, then, must be found. Eq. (5) of Art. 26, when applied to Fig. 1, becomes

$$E'\left[\frac{bh^2}{2}-\frac{(b-b')(h-t')^2}{2}\right]=E\left[\frac{b_1(d-h)^2}{2}-\frac{(b_1-b')(d-h-t_1)^2}{2}\right]$$

E' representing the coefficient of elasticity for compression, and E that quantity for tension.

Performing the operations indicated and reducing, writing n for $E' \div E$,

$$(n-1)b'h^2 + 2[nt'(b-b') + t_1(b_1-b') + b'd]h \\ = nt'^2(b-b') + (2d-t_1)(b_1-b')t_1 + b'd^2. \quad (9)$$

h is to be measured on the compression side of the beam.

This is a quadratic equation of condition for the determination of h. It is best to leave it as it is until the numerical substitutions are made and then to solve it. h_1 immediately results from the equation $h_1 = d - h$.

Frequently there is no compression flange, the section being like that shown in Fig. 2. In such case b is equal to b', or t' is equal to zero; hence the two terms $nt'(b-b')$ and $nt'^2(b-b')$ in eq. (9) disappear. No other change occurs.

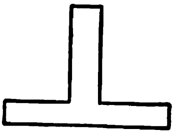


FIG. 2.

Eq. (1) of Art. 26 then gives the following resisting moment of the section:

$$M = \frac{C}{3h}\left(bh^3 - (b-b')(h-t')^3 + \frac{b_1h_1^3}{n} - \frac{(b_1-b')(h_1-t_1)^3}{n}\right). \quad (10)$$

C is the greatest intensity of stress in the section of the same kind as E'.

If the section is like Fig. 2, b again equals b' and the term $(b-b')(h-t')^3$ in eq. (10) disappears, but nothing else is changed.

If T is the greatest stress on the other side of the neutral surface from C,

$$M = \frac{T}{3h_1}[nbh^3 - n(b-b')(h-t')^3 + b_1h_1^3 - (b_1-b')(h_1-t_1)^3]. \quad (11)$$

In order that the beam may be equally strong in the two flanges, the ratio between h and h_1, as determined by eq. (9), should be the same as that determined by the following process. If u is the rate of strain at units' distance from the neutral surface,

$$\left.\begin{array}{l} E'uh = C, \\ Euh_1 = T; \end{array}\right\} \therefore \frac{h}{h_1} = \frac{CE}{TE'}. \quad . \quad . \quad . \quad . \quad (12)$$

If there is no waste of material, the cross-section must be so designed that the ratios given by eqs. (9) and (12) will be the same.

If the thicknesses of the flanges t' and t_1 are small compared with the depth d of the beam, and if b' also is small, i.e., if the flanges are assumed to give the *whole* resistance to bending, while the web takes up the shea eqs. (10) and (11) may be much simplified.

Making, therefore $b' = 0$ in eq. (10), putting $\dfrac{C}{hn} = \dfrac{T}{h_1}$ and then expanding the quantities $(h - t')^3$ and $(h_1 - t_1)^3$,

$$M = Cbt'\left(h - t' + \frac{t'^2}{3h}\right) + Tb_1t_1\left(h_1 - t_1 + \frac{t_1^2}{3h_1}\right).$$

Under the conditions taken, $Cbt' = Tb_1t_1$; also, $\dfrac{t'^2}{3h}$ and $\dfrac{t_1^2}{3h_1}$ are very small and may be neglected. Hence

$$M = Cb \ (d - t' - t_1) = Tb_1t_1(d - t' - t_1). \quad . \quad . \quad (13)$$

But both of these approximations have made M *too small*. As an approximate compensation, therefore, $-\left(\dfrac{t' + t_1}{2}\right)$ may be written for $-(t' + t_1)$. The moment then becomes

$$M = Cbt'\left(d - \frac{t' + t_1}{2}\right). \quad . \quad . \quad . \quad . \quad . \quad (14)$$

The quantity within the parenthesis of the second member of this equation is evidently the distance between the centres of gravity of the flanges, while the quantity $Cbt' = Tb_1t_1$ is simply the flange stress. Eq. (34) is, then, identical with eq. (7), as was to be anticipated. The equality of flange stresses gives

$$\frac{bt'}{b_1t_1} = \frac{T}{C},$$

a relation identical with eq. (8).

If desirable, an approximate correction for the neglect of the web may be introduced in eq. (14). It has been seen that that equation is precisely the same as if E' were equal to E, i.e., as if the two coefficients of elasticity were equal. Now it will be shown in the next article that if $E' = E$, the resistance of the web to bending is equal to that of one sixth of its area of normal section concentrated in each flange. Hence, if a is the area of the normal section of the web, since bt' and b_1t_1 are areas of the normal sections of the upper and lower flanges, there may be approximately written

$$M = C\left(bt' + \frac{a}{6}\right)\left(d - \frac{t' + t_1}{2}\right) = T\left(b_1t_1 + \frac{a}{6}\right)\left(d - \frac{t' + t_1}{2}\right). \quad (15)$$

Values of C and T may be determined by experiment.

In the case of solid beams, it has been seen that if r and r' are certain ratios, $K = rT$ or $r'C$. Hence, since the web of a flanged beam is really a solid beam subjected to flexure, eq. (15) may be written

$$M=TD\left(a'+\frac{ra}{6}\right)=CD\left(a''+\frac{r'a}{6}\right), \quad . \quad . \quad (16)$$

in which

$D=d-\frac{t'+t_1}{2}$ = depth between flange centres;

a = area of section of web;

$a'=b_1t_1$ = area of bottom flange;

$a''=bt'$ = area of top flange;

$r=K\div T$, and $r'=K\div C$.

Cast-iron Flanged Beams.

In Art. 69 it has been seen that the ratio r is equal to about 2 for a solid bar with square cross-section. This would make $r\div 6=\frac{1}{3}$. A few imperfect experimental indications, however, seem to indicate a decrease of r for a greater ratio of depth to breadth. Let, therefore, $r\div 6=0.25$. Eq. (16) then becomes

$$M=TD\left(a'+\frac{a}{4}\right). \quad . \quad . \quad . \quad . \quad . \quad (17)$$

If W = centre breaking load in pounds,

W_1 = total uniform breaking load in pounds,

l = span in feet,

$12l$ = span in inches,

$$\frac{W.12l}{4}=3Wl=TD\left(a'+\frac{a}{4}\right);$$

$$\therefore W=\frac{TD\left(a'+\frac{a}{4}\right)}{3l}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

In the same manner

$$W_1 = 2\frac{TD\left(a' + \frac{a}{4}\right)}{3l}. \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Or, if pl is the weight of the beam, supposed uniformly distributed,

$$\left(W + \frac{pl}{2}\right) = \frac{TD\left(a' + \frac{a}{4}\right)}{3l}. \quad . \quad . \quad . \quad . \quad (20)$$

It has been shown under the head of "Tension" that T varies from 15,000 pounds per square inch, for ordinary castings, to 30,000 for those of extra quality. In eqs. (18), (19), and (20),

D must be taken in inches;
a and a' in square inches; and
l in feet.

Those equations have been verified in a most satisfactory manner by the numerous English experiments of Hodgkinson and Cubitt ("Experimental Researches," etc., by Eaton Hodgkinson, F.R.S., 1846), and Berkley ("Proc. Inst. of Civil Engineers," Vol. XXX), as shown by the tables in those publications.

Cast-iron Beams without Upper Flange.

When the beam has the section shown in Fig. 2 eq. (10) should be used in all such cases where anything more than a very loose approximation is desired. In that equation n may be taken equal to unity, the great irregularities in the ratio of the two coefficients of elasticity precluding any other value. Since, in this case (see Fig. 1), $b=b'$, eq. (10) becomes

$$M=\frac{C}{3h}[bh^3+b_1h_1^3-(b_1-b')(h_1-t_1)^3], \quad . \quad . \quad (21)$$

The distance of the centre of gravity of the beam's section from its lower side DF, Fig. 1, is found by making $b=b'$ or $t'=0$ in eq. (3) and writing h_1 for x_1,

$$h_1=\frac{b'd^2+(b_1-b')t_1^2}{2[b'd+(b_1-b')t_1]}. \quad . \quad . \quad . \quad . \quad (22)$$

The total depth of the beam being d, there may at once be written

$$h=d-h_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

By the aid of eqs. (22) and (23) h and h_1 at once become known and eq. (21) may be employed in practical applications.

In designing beams of this class the preceding equations must be used to some extent in a trial manner.

If the weight of the beam is taken into consideration, as in eq. (20),

$$M=\left(W+\frac{pl}{2}\right)3l.$$

A mean of three of Mr Hodgkinson's beams of 4.5 feet span, 5.125 inches depth, gave

$$W+\frac{pl}{2}=8766 \text{ lbs.} \quad \text{and} \quad C=45{,}700 \text{ lbs.}$$

One of Mr. Cubitt's beams of 15 feet span and 14 inches depth gave

$$W+\frac{pl}{2}=28{,}100 \text{ lbs.} \quad \text{and} \quad C=30{,}850 \text{ lbs.}$$

The bottom flange of this beam was unsound.

C must necessarily depend upon the span, since that portion of the web which is subjected to compression is somewhat in the condition of a long column. This, indeed, is true of the compression flange of any flanged beam, but the effects resulting from such a condition are much more marked in the class of beams shown in Fig. 2.

If, then, W is the centre breaking weight and W_1 the total uniform breaking load (not including the weight of the beam), eq. (21) becomes

$$W = \frac{W_1}{2} = \frac{C}{9lh}[bh^3 + b_1h_1^3 - (b_1 - b')(h_1 - t_1)^3] - \frac{pl}{2}. \qquad (24)$$

In this equation l must be taken in feet and other dimensions in inches.

For 5 feet span C may be taken at 45,000 lbs.

For 15 feet span C may be taken at 35,000 lbs.

Design of Beams.

In order that a beam with top and bottom flanges may give the best result, i.e., reach its ultimate resistance in each flange at the same time, Mr. Hodgkinson found that the area of the lower flange section should equal about six times that of the upper. That relation has been anticipated in eq. (8).

In general, if it is desired to design a cast-iron flanged beam to carry any load whatever producing the bending moment M, in inch-pounds, eq. (17) shows that the sum of flange and web areas in square inches must be

$$a' + \frac{a}{4} = \frac{M}{TD}. \quad \ldots \ldots \quad (25)$$

For ordinary castings $T = 15,000$ to 18,000 pounds per square inch for ultimate resistance or 2000 to 3000 pounds per square inch for intensity of working resistance in designing.

With l expressed in feet, if pl is the weight of the beam itself, eq. (20) gives for a load W at the centre of span

$$a' + \frac{a}{4} = \frac{3l}{TD}\left(W + \frac{pl}{2}\right). \quad . \quad . \quad . \quad . \quad (26)$$

If the load is uniformly distributed and represented by W_1, including the weight of the beam itself, eq. (19) gives

$$a' + \frac{a}{4} = \frac{3l}{2TD}W_1. \quad . \quad . \quad . \quad . \quad . \quad (27)$$

PROBLEMS.

Prob. 1. A cast-iron flanged beam like Fig. 1 and of 8 feet effective span must be designed to carry a uniform load of 23,000 pounds in addition to its own weight. Find the dimensions of the cross-section. Eq. (27) is to be used. The depth D between centres of flanges will be taken at 10 inches, and a trial weight of beam of 600 pounds will be assumed. The working stress T will be assumed at 2000 pounds per square inch. Eq. (27) then gives

$$a' + \frac{a}{4} = \frac{3 \times 8}{2 \times 2000 \times 10} 23,600 = 14.15 \text{ sq. ins.}$$

If the lower flange be made $10 \times 1\frac{1}{4}$ inches and the web $9 \times \frac{3}{4}$ inches,

$$a' = 12.5 \text{ sq. ins.} \quad \text{and} \quad \frac{a}{4} = 9 \times \frac{3}{4} \times \frac{1}{4} = 1.69 \text{ sq. ins.}$$

The dimensions satisfy the above equation and make good proportions. The upper flange may then be taken at $3 \times \frac{3}{4}$ inches, giving an area a little more than one sixth of the lowe flange area. The total area of beam section will then be

$$9 \times \tfrac{3}{4} + 3 \times \tfrac{3}{4} + 10 \times 1\tfrac{1}{4} = 21.5 \text{ sq. ins.}$$

Prob. 2 It is required to design a cast-iron flanged beam of 5 ft. effective span to carry a load of 1800 lbs. applied at the centre of span, the section of the beam to be like that shown in Fig. 2, i.e., without upper flange. The greatest permitted working stress in compression will be 8000 lbs. per sq. in., and the total depth of the beam is to be taken at 9 inches.

Referring to eq. (21), eq. (22), and Fig. 1 for the notation, the given data and the dimensions to be assumed for trial will be as follows: $d = 9$ ins.; $b = b' = \frac{7}{8}$ in.; $b_1 = 8$ ins.; $t_1 = 1$ in.; $l = 5$ ft.; and $C = 8000$. The introduction of these values into eq. (22) will give for the distance of the centre of gravity above the bottom surface of the beam

$$h_1 = 2.6 \text{ ins.} \quad \text{and} \quad h = d - h_1 = 6.4 \text{ ins.}$$

The preceding trial dimensions will make the beam weigh about 50 lbs. per lineal foot. If all the preceding values are substituted in eq. (22), which is simply another form of eq. (21), there will be found

$$W = 1994 - 125 = 1869 \text{ lbs.}$$

The trial dimensions, therefore, give the centre-load capacity of the beam 69 pounds greater than required, which may be considered sufficiently near to show that the assumed dimensions are satisfactory.

Art. 83.—Deflection of Cast-iron Flanged Beams.

If W is the centre load in pounds, l and w the span and centre deflection, respectively, in inches, and I the moment of inertia of the cross-section, eq. (28) of Art. 22 gives

$$E=\frac{Wl^3}{48wI}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

Or, if l is in feet, which is more convenient,

$$E=\frac{36Wl^3}{wI}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

A mean of two ot Mr. Berkley's beams gave

$l=4.5$ feet; $w=0.284$ inch; $W=20,160$ lbs.;
$I=18.74$. Hence $E=12,424,600$ lbs.

A mean of two of Mr. Cubitt's beams gave

$l=15$ feet; $w=0.465$ inch; $W=11,200$ lbs.;
$I=227.03$. Hence $E=12,886,720$ lbs.

The four preceding beams had top and bottom flanges, as in Fig. 1, Art. 82. Another of Mr. Cubitt's beams, without top flange, as in Fig. 2 of the same article, gave

$l=15$ feet; $w=0.41$ inch; $W=13,440$ lbs.;
$I=373$. Hence $E=10,679,400$ lbs.

This last beam had a defective bottom flange, hence there may be taken without essential error

$$E=12,000,000 \text{ lbs.}$$

Taking l in feet, eq. (2) now gives for the centre deflection

$$w = \frac{3W'l^3}{1,000,000I}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

in which W' is either the centre load, or five eighths (⅝ths) the total uniform load, as the case may be.

The formula by which I is to be computed is the one which immediately follows eq. (4) of Art. 82.

Art. 84.—Flanged Beams with Equal Flanges.

Nearly all the flanged beams used in engineering practice are composed of a web and two equal flanges. It has already been seen that the ultimate resistances, T and C, of structural steel and wrought iron to tension and compression are essentially equal to each other; the same may be said also of their coefficients of elasticity for tension and compression. These conditions require equal flanges for both steel and wrought-iron rolled beams.

In Fig. 1 is represented the normal cross-section of an equal-flanged beam. It also approximately represents what may be taken as the section of any wrought-iron or steel I beam, the exact forms with the corresponding moments of inertia being given in Art. 58. Although the thickness t' of the flanges of such beams is not uniform, such a mean value may be taken as will cause the transformed section of Fig. 1 to be of the same area as the original section.

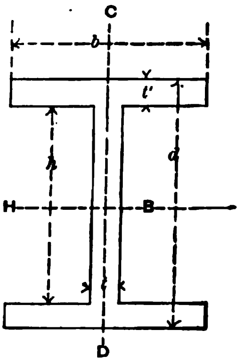


FIG. 1.

Unless in exceptional cases where local circumstances compel otherwise, the beam is always placed with the web vertical, since the resistance to bending is much greater in that position. The neutral axis *HB* will then be at half the depth of the beam. Taking the dimensions as shown in Fig. 1, the moment of inertia of the cross-section about the axis *HB* is

$$I=\frac{bd^3-(b-t)h^3}{12}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

while the moment of inertia about *CD* has the value

$$I_1=\frac{2t'b^3+ht^3}{12}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

With these values of the moment of inertia, the general formula, $M=\frac{KI}{d_1}$, becomes (remembering that $d_1=\frac{d}{2}$ or $\frac{b}{2}$)

$$M=C\frac{bd^3-(b-t)h^3}{6d}, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

or

$$M'=C\frac{2t'b^3+ht^3}{6b}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

C is written for *K*, since $K=T=C$.

Eq. (3) is the only formula of much real value. It will be found useful in making comparisons with the results of a simpler formula to be immediately developed.

Let $d_1 =\frac{1}{2}(d+h)$. Since t' is small compared with $\frac{d}{2}$, the intensity of stress may be considered constant in each flange without much error. In such a case the total stress in each flange will be $Cbt'=Tbt'$, and each of those forces will act with the lever-arm $\frac{1}{2}d_1$. Hence the moment of resistance of both flanges will be

$$Cbt'.\ d_1.$$

The moment of inertia of the web will be $\frac{th^3}{12}$. Consequently its moment of resistance will have very nearly the value

$$\frac{Cth^2}{6}.$$

The resisting moment of the whole beam will then be

$$M = C\left(bt'd_1 + \frac{th^2}{6}\right). \quad . \quad . \quad . \quad . \quad . \quad (5)$$

A further approximation is frequently made by writing d_1h for h^2; then if each flange area $bt' = f$, eq. (5) takes the form

$$M = Cd_1\left(f + \frac{th}{6}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Eq. (6) shows that *the resistance of the web is equivalent to that of one sixth the same amount concentrated in each flange.*

If the web is very thin, so that its resistance may be neglected,

$$M = Cfd_1 = Cbt'd_1, \quad . \quad . \quad . \quad . \quad . \quad (7)$$

or

$$f = \frac{M}{Cd_1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Cases in which these formulæ are admissible will be given hereafter. It virtually involves the assumption that the web is used wholly in resisting the shear, while the flanges resist the whole bending and nothing else. In other words, the web is assumed to take the place of the neutral surface in the solid beam, while the direct resistance to tension and compression of the longitudinal fibres of the lätter is entirely supplied by the flanges.

Again recapitulating the greatest moments in the more commonly occurring cases:

Cantilever uniformly loaded,

$$M=\frac{Wl}{2}=\frac{pl^2}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (9)$$

Cantilever loaded at the end,

$$M=Wl. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (10)$$

Beam supported at each end and uniformly loaded,

$$M=\frac{Wl}{8}=\frac{pl^2}{8}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (11)$$

Beam supported at each end and loaded at centre,

$$M=\frac{Wl}{4}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (12)$$

Beam supported at each end and loaded both uniformly and at centre,

$$M=\frac{l}{4}\left(W+\frac{pl}{2}\right). \quad . \quad . \quad . \quad . \quad . \quad . \qquad (13)$$

In all cases W is the total load or single load, while p, as usual, is the intensity of uniform load, and l the length of the beam.

Art. 85.—Rolled Steel Flanged Beams.

The resisting moments of all rolled steel beams subjected to bending are computed by the exact formula

$$M=\frac{KI}{d_1}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

K being the greatest intensity of stress (i.e., in the extreme fibres) at the distance d_1 from the neutral axis about which the moment of inertia I is taken. In all ordinary cases the webs of beams are vertical so that the axis for I is horizontal; but it sometimes is necessary to use the moment of inertia I computed about the axis passing through the centre of gravity of section and parallel to the web. The latter is frequently employed in considering the lateral bending effect of the compression in the upper flange.

The upper or compression flange of a rolled beam under transverse load, unless it is laterally supported, is somewhat in the condition of a long column and, hence, tends to bend or deflect in a lateral direction. This tendency depends to some extent on the ratio of the length of flange (l) to the radius of gyration (r) of the section about the axis parallel to the web, as will be shown in detail in a later article. It will be found there that the ultimate compression flange stress decreases as the ratio $l \div r$ increases. Hence in Table I there will be found values of $l \div r$ for the different beams tested.

The results of tests given in Table I were found by Mr. James Christie, Supt. of the Pencoyd Iron Co., and they are taken from a paper by him in the "Trans. Am. Soc. C. E." for 1884. All beams, both I and bulb, were loaded at the centre of span. Hence the moment of the centre load, W, and the uniform weight of the beam itself, pl, will be, as shown in eq. (13) of Art. 84,

$$M = \frac{l}{4}\left(W + \frac{pl}{2}\right) = \frac{KI}{d_1}. \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

Hence

$$K = \frac{d_1 l}{4KI}\left(W + \frac{pl}{2}\right). \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

The known data of each test will give all the quantities in the second member of eq. (3). The two columns of elastic and ultimate values of K in the table were computed by eq. (3). The positions of the bulb beams (i.e., the bulb either up or down) in the tests are shown by the skeleton sections in the second column.

The coefficients of elasticity E were computed from the data of the tests taken below the elastic limit by the aid of eq. (21), Art. 22:

$$w_1 = \frac{l^3}{48EI}\{W + \tfrac{5}{8}pl\}, \quad . \quad . \quad . \quad . \quad . \quad (4)$$

W being the centre load and pl the weight of the beam, the length of span l being given in inches.

All beams were rolled at the Pencoyd Iron Works. The "mild steel" contained from 0.11 to 0.15 per cent. of carbon, and the "high steel" about 0.36 per cent. of carbon. These steels are the same as those referred to in Art. 60.

No. 14 is the only test of a "high" steel beam; all the remaining tests being with mild-steel shapes. Tests 3 to 9 inclusive were of deck or bulb beams, as the skeleton sections show.

Beams 3 and 4 were rolled from the same ingot, as were also 6 and 7, as were also 10, 12, and 13, and as were also 16, 17, 18, and 19. All beams were unsupported laterally in either flange. The moments of inertia were computed from the actual beam sections. The length of span is represented by l, while r is the radius of gyration of each beam section about an axis through its centre of gravity and parallel to its web. The values of r were as follows:

5 inch I . . . . $r=0.54$ inch.
6 " " . . . . $r=0.63$ "
7 " " . . . . $r=0.71$ "
9 " " . . . . $r=0.83$ "

3 inch I . . . . $r=0.59$ inch.
8 " " . . . . $r=0.88$ "
10 " " . . . . $r=0.95$ "
12 " " . . . . $r=1.01$ "

TABLE I.

TRANSVERSE TESTS OF STEEL BEAMS.

No.	Kind of Beam.	Span in Ins.	$\frac{l}{r}$.	Moment of Inertia.	Final Centre Load in Pounds.	*K* in Pounds per Square Inch at		Coefficient of Elasticity *E*, in Pounds per Square Inch.
						Elastic.	Ultimate	
1	Mild 3″ I	59	100	2.76	5,500	41,100	45,200	30,890,000
2	" 3″ "	39	66	2.76	8,300	40,800	45,100	25,011,000
3	" 5″ ⫯	108	200	12	8,800	50,000	55,000	27,718,000
4	" 5″ ⫰	108	200	12	8,400	46,900	52,500	25,489,000
5	" 6″ ⫯	96	152	22	14,860	51,200	54,300	23,692,000
6	" 7″ "	69	97	37.6	34,000	47,100	59,300	18,765,000
7	" 7″ ⫰	69	97	37.6	34,000	47,100	59,300	23,040,000
8	" 9″ ⫯	240	290	84.8	14,500	46,000	51,300	29,923,000
9	" 9″ ⫰	240	290	82.9	13,500	39,800	48,800	30,209,000
10	" 8″ I	240	273	70.2	13,000	37,600	44,400	28,889,000
11	" 8″ "	240	273	70.3	12,930	37,500	44,100	29,055,000
12	" 8″ "	144	164	70.2	19,480	32,800	39,900	31,313,000
13	" 8″ "	96	109	70.2	31,300	40,300	42,800	23,689,000
14	High 3″ "	39	——	2.74	11,500	54,300	——	27,515,000
15	Mild 10″ "	156	164	150.5	22,500	35,000	——	28,414,000
16	" 10″ "	168	177	150.5	21,000	35,200	——	27,182,000
17	" 10″ "	180	189	150.5	19,500	35,000	——	29,160,000
18	" 10″ "	192	202	150.5	18,000	34,400	——	29,727,000
19	" 12″ "	240	238	264.7	24,500	33,400	——	30,749,000
20	" 12″ "	240	238	267.6	24,200	32,500	——	29,568,000
21	" 12″ "	228	226	273.8	22,000	27,500	——	29,164,000
22	" 12″ "	216	214	263.7	29,000	35,600	——	30,219,000
23	" 12″ "	204	202	256.7	27,000	32,100	——	30,030,000
24	" 12″ "	192	190	257.8	34,000	38,000	——	29,709,000
25	" 12′ "	192	190	262.6	34,000	37,300	——	28,234,000
26	" 12″ "	180	178	262.4	36,700	37,700	——	27,717,000
27	" 12″ "	168	166	264.0	38,000	36,300	——	28,784,000
28	" 12″ "	156	154	261.7	43,000	38,400	——	27,818,000

The values of *K* both for the elastic limit and the ultimate are erratic, and the range of results in the table is not sufficient to establish any law, but on the whole the small ratios $l \div r$ accompany the larger values of *K*. The bulb or deck beams also appear to give larger values of *K* than the I beams.

The results of these tests indicate that the greatest working intensities of stress in the flanges of rolled steel beams may be taken from 12,000 to 16,000 pounds per square inch if the length of unsupported compression flange does not exceed $150r$ to $200r$.

In the work of design, the quantity $I \div d_1$ used in eq. (2), called the "section modulus," is much employed, and it can be taken directly from the Cambria Steel Company's tables at the end of the book, as can the moment of inertia I. Eq. (2) shows that

$$\frac{I}{d_1}=\frac{M}{K}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Hence the moment of the loading in inch-pounds divided by the allowed greatest flange stress in pounds per square inch must be equal or approximately equal to the section modulus of the required beam.

Prob. 1. It is required to design a rolled steel beam for an effective span of 20 ft. to carry a uniform load of 725 lbs. per linear foot in addition to the weight of the beam itself, the circumstances being such that it is not advisable to use a greater total depth of beam than 12 ins. The greatest permitted extreme fibre stress K will be taken at 12,000 lbs. per sq. in. It will be assumed for trial purposes that the beam itself will weigh 35 lbs. per linear foot, so that the total uniform load will be 760 lbs. per linear foot. The centre moment in inch-pounds will, therefore, be

$$M=\frac{760\times 20\times 20\times 12}{8}=456,000 \text{ in.-lbs.}$$

By eq. (5) the section modulus will be 456,000 ÷ 12,000 = 38. By referring to the Cambria Steel Company's tables at the end of the book it will be found that this section modulus belongs to a 12-in. 35-lb. steel rolled beam, and that beam fulfils the requirements of the problem.

Prob. 2. It is required to design a rolled steel beam for a 32-ft. effective span to carry a load of 1280 lbs. per linear foot in addition to the weight of the beam, and a concentrated load of 11,000 lbs. at a point 11 ft. distant from one end of the span. The greatest permitted working stress in the extreme fibres of the beam is 16,000 lbs. per sq. in.

It will be assumed for trial purposes that a 24-in. beam weighing 95 lbs. per linear foot will be required so that the total uniform load per linear foot will be 1375 pounds. It will then be necessary to ascertain at what point in the span the maximum bending moment occurs, i.e., at what point the transverse shear is equal to zero. Let a be the distance of the concentrated weight from the nearest end of the span, i.e., $a=11$ ft. Then et P be the single weight, p the total uniform load per linear foot, and l the length of span. The following equation representing the condition that the transverse shear must be equal to zero may be written

$$\frac{pl}{2}-px+\frac{Pa}{l}=0.$$

Hence

$$x=\frac{l}{2}+\frac{Pa}{pl}.$$

In the above equation x is obviously the distance from that end of the span farthest from P to the section of greatest bending moment. Substituting the above numerical values in the equation for x, there will result

$$x=16+2.75=18.75 \text{ ft.}$$

Since $32-18.75=13.75$, the following will be the value of the greatest bending moment in inch-pounds:

$$M=\left(\frac{13.75\times 18.75}{2}\times 13.25+\frac{11{,}000\times 11}{32}\times 18.75\right)12$$
$$=2{,}900{,}363 \text{ inch-pounds.}$$

The section modulus of the beam required is by eq. (5) $2,900,363 \div 16,000 = 181$. The section modulus of a 24.-in. steel beam weighing 85 lbs. per linear foot is 180.7, as will be found by referring to the tables at the end of the book. Hence that beam will be assumed for the correct solution of the problem. The fact that the beam weighs 10 lbs. per linear foot less than the assumed weight has too small an effect upon the greatest bending moment to call for any revision.

Prob. 3. A steel tee beam of 8 ft. span is to be used as a purlin to carry a uniform load of 125 lbs. per linear foot with the web of the tee in a vertical position. The greatest permitted intensity of stress in the extreme fibre of the tee is 14,000 lbs. per sq. in. It is required to find the dimensions of the tee. By referring to eq. (5) the section modulus will be written

$$S = \frac{1000 \times 96}{8 \times 14,000} = .86 \text{ in.}$$

By referring to the Cambria Steel Company's tables it will be found that a $3 \times 3 \times \frac{5}{16}$ in. steel tee weighing 6.6 lbs. per lin. ft. has just the section modulus required. That tee therefore fulfils the requirements of the problem.

Prob. 4. It is required to support a single weight of 12,000 lbs. at the centre of a span of 13 ft. 6 ins. on two rolled steel channels with their webs in a vertical position and separated back to back by a distance of 3 ins., the greatest permitted intensity of stress in the extreme fibre of the flanges being 15,000 lbs. Find the size of channels required.

Art. 86.—The Deflection of Rolled Steel Beams.

The deflections of rolled steel beams may readily be computed by the formulæ of Art. 22. The general procedure will be illustrated by using the equations for a non-continuous beam simply supported at each end and loaded by a weight at the centre of span, or uniformly, or in both ways concurrently. Eq. (20) will give the deflection at any point located by the coordinate x, while eq. (21) will give the centre deflection only. The tangent of the inclination of the neutral surface at any point located by x will be given by the value of $\frac{dw}{dx}$ found in eq. (19).

Prob. 1. Let the centre deflection of the rolled-steel beam of Prob. 1 of Art. 85 be required. Referring to eq. (21) of Art. 22,

$$W=0; \quad l=20 \text{ feet} = 240 \text{ inches}; \quad p=760 \text{ pounds};$$
$$I=228.3; \quad \text{and} \quad E \text{ may be taken at } 29,000,000.$$

Hence the centre deflection is

$$w_1 = \frac{240 \times 240 \times 240 \times 5 \times 760 \times 20}{48 \times 8 \times 29,000,000 \times 228.3} = .414 \text{ inch.}$$

If half the external uniform load of 725 pounds per linear foot had been concentrated at the centre of span,

$$W = \frac{725 \times 20}{2} = 7250 \text{ pounds}; \quad p=35$$

and $\quad l=20$ ft. $=240$ ins. Also $pl=700$ pounds.

Hence the centre deflection would be

$$w_1 = \frac{240 \times 240 \times 240 \times (7250+437.5)}{48 \times 29,000,000 \times 228.3} = .333 \text{ inch.}$$

Prob. 2. In Prob. 2 of Art. 85 place the 11,000-pound weight at the centre of span, then find the inclination of the neutral surface and the deflection of the 24-inch 85-

pound steel beam at the centre and quarter points of the 32-foot span, taking $E=29,000,000$ pounds.

Art. 87.—Wrought-iron Rolled Beams.

Although wrought-iron rolled beams are not now manufactured, being completely displaced by steel beams, yet many are still in use. Hence it is advisable to exhibit the empirical quantities required to design them and to determine their safe carrying capacities as well as their deflections under loading.

It has been observed in Art. 74 that the upper or compression flange of a loaded flanged beam will deflect or tend to deflect laterally at a lower intensity of compressive stress as the unsupported length of such a flange is increased. The experimental results given in Table I exhibit the values of the intensity of stress K in the extreme fibres of the beam both at the elastic and ultimate limits, the usual formula for bending resistance being used,

$$M=\frac{KI}{d_1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

In the autumn of 1883 an extensive series of tests of wrought-iron rolled beams, subjected to bending by centre loads, was made by the author, assisted by G. H. Elmore, C.E., at the mechanical laboratory of the Rensselaer Polytechnic Institute. The object of these tests was to discover, if possible, the law connecting the value of K for this class of beams with the length of span when the beam is *entirely without lateral support*. The means by which the latter end was accomplished, and a full detailed account of the tests will be found in Vol. I, No. 1, "Selected Papers of the Rensselaer Society of Engineers." The main results of the tests are given in Table I. All the tests were made on 6-inch I beams with the same area of normal cross-

TABLE I.

No	Span, Feet.	Final Centre Weight, Pounds.	$\frac{l}{r}$.	K Elastic Limit, Pounds.	K Ultimate, Pounds.	Perm'nent Vertical Deflection, Inches.	Perm'nent Lateral Deflection, Inches.	E Pounds per Square Inch.
1	20	4,060	400	27,726	31,094	0.14	——	24,170,000
2		4,200	400	29,623	32,885	0.30	——	26,374,000
3	18	4,390	360	28,264	30,791	0.2	0.5	24,520,000
4		4,570	360	28,264	32,020	0.18	0.4	24,313,000
5	16	4,770	320	26,564	29,579	0.28	1.00	25,771,000
6		5,270	320	29,596	32,632	0.48	1.25	25,003,000
7	14	6,130	280	31,191	33,049	0.30	1.20	26,082,000
8		6,125	280	31,164	33,023	0.30	1.10	23,373,000
9	12	7,161	240	30,221	32,907	0.35	1.08	25,287,000
10		7,350	240	31,314	33,817	0.33	1.09	24,022,000
11	10	9,255	200	33,082	35,358	0.39	1.08	25,115,000
12		9,655	200	33,082	37,064	0.50	1.50	24,218,000
13	8	11,485	160	29,736	35,010	0.30	0.90	21,611,000
14		11,980	160	31,936	36,527	0.29	1.05	21,987,000
15	6	18,300	120	35,497	41,737	0.605	1.53	23,040,000
16		18,145	120	36,617	41,396	0.67	1.88	20,935,000
17	5	22,870	100	34,136	43,434	0.67	1.75	22,023,000
18		23,065	100	34,136	43,813	0.67	1.75	25,272,000
19	4	29,985	80	32,619	45,532	0.96	1.70	24,315,000
20		28,585	80	32,619	44,744	0.60	1.86	21,275,000

section of 4.35 square inches. Actual measurement showed the depth d of the beams to be 6.16 inches. The moment of inertia of the beam section about a line through its centre and normal to the web was $I=24.336$. The radius of gyration of the same section in reference to a line through its centre and *parallel* to the web was $r=0.6$ inch. l was the length of span in inches.

If M is the bending moment in inch-pounds, W the total centre load (including weight of beam), and K the stress per square inch in extreme fibre, the following formulæ result:

$$K=\frac{Md}{2I} \quad \text{and} \quad M=\frac{Wl}{4}. \quad . \quad . \quad . \quad . \quad (2)$$

$$\therefore K=\frac{Wld}{8I}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

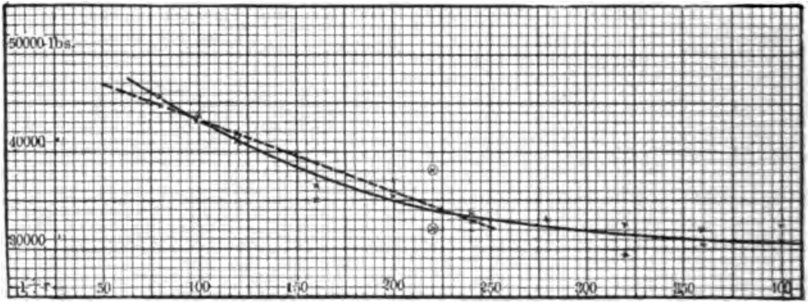
The experimental values of W, l, d, and I inserted in the above formula give the values of K shown in the table. The coefficient of elasticity, E, was found by the usual formula,

$$E = \frac{Wl^3}{48wI}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

in which w is the deflection caused by W.

The full line is the graphical representation of the values of K given in Table I. Since K must clearly decrease with the length of span, and increase with the radius of gyration of the section about an axis through its centre and parallel to the web (the latter, of course, being vertical), K has been plotted in reference to $l \div r$ as shown. No simple formula will closely represent this curve, but the broken line covers all lengths of span used in ordinary engineering practice, and is represented by the formula

Plate I.



$$K = 51{,}000 - 75\frac{l}{r}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

For railway structures the greatest allowable stress per square inch in the extreme fibres of rolled beams may be taken at

$$k = 10{,}000 - 15\frac{l}{r}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Values of k taken from a large scale plate, like Plate I, are, however, far preferable to those given by any formula.

The ultimate values of K given in Table I are fairly representative of the best wrought-iron I beams. The coefficients of elasticity E range from about 22,000,000 to about 25,000,000 pounds; the average may be taken about 24,000,000 pounds.

The deflection of wrought-iron beams may be computed by the formula

$$w = \frac{Wl^3}{48EI}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

when the load W is at the centre of the beam. In the general case of a beam carrying the centre load W and the uniform load pl, the quantity $(W + \frac{5}{8}pl)$ must displace W in eq. (7). If the beam carry only the uniform load pl, W in eq. (7) must be displaced by $\frac{5}{8}pl$.

If it is desired to apply the law expressed in eqs. (5) and (6) to mild-steel beams, the second members of those equations may be multiplied by $\frac{6}{5}$ to $\frac{5}{4}$ for close approximations.

CHAPTER XIII.

CONNECTIONS.

Art. 88.—Riveted Joints.

ALTHOUGH riveted joints possess certain characteristics under all circumstances, yet those adapted to boiler and similar work differ to some extent from those found in the best riveted trusses. The former must be steam- and water-tight, while such considerations do not influence the design of the latter, consequently far greater pitch may be found in riveted-truss work than in boilers. Again, the peculiar requirements of bridge and roof work frequently demand a greater overlap at joints and different distribution of rivets than would be permissible in boilers.

Kinds of Joints.

Some of the principal kinds of joints are shown in Figs. 1 to 6. Fig. 1 is a "lap-joint" single-riveted; Fig. 2 is a "lap-joint" double-riveted; Fig. 3 is a "butt-joint" with a single butt-strap and single-riveted; while Figs. 4, 5, and 6 are "butt-joints" with double butt-straps, Fig. 4 being single-riveted, while the others are double-riveted. Fig. 5 shows zigzag riveting, and Fig. 6 chain riveting. All these joints are designed to resist tension and to convey stress from one single thickness of plate to another. Two or three other joints peculiar to bridge and roof work will hereafter be shown.

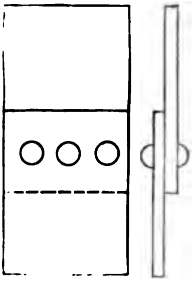


Fig. 1.

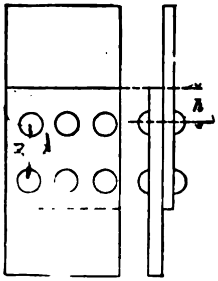


Fig. 2.

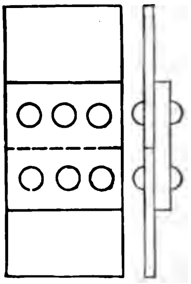


Fig. 3.

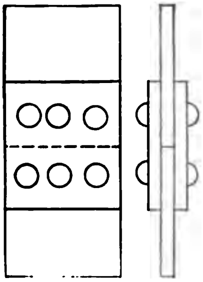


Fig. 4.

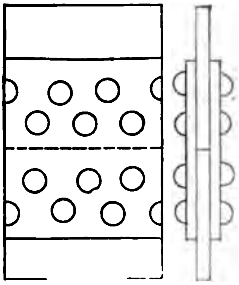


Fig. 5.

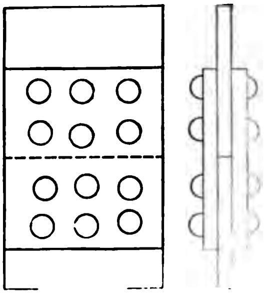


Fig. 6.

In the cases of bridges and roofs these "butt-straps" are usually called "cover-plates."

Art. 89.—Distribution of Stress in Riveted Joints.

The Bending of the Plates.

A very little consideration of the question will show that only an approximate determination of the distribution of stress in a riveted joint can be reached.

In order that rivets, butt-straps, or cover-plates, and different portions of the same main plate may take their proper portions of stress, an absolutely accurate adjustment of these different parts must be attained; but all shop work must necessarily be more or less imperfect, and the requisite conditions can never be maintained during and after construction. Hence the precise amount of stress carried by each rivet, or each cover-plate, or each portion of the main plate at the joint, cannot be found.

In the cases of lap-joints with three or more rows of rivets (frequently found in truss work), or in similar work when two rows of rivets join a small plate to a much larger one, the outside rows, or row, in consequence of the stretching of the material at the joint, must take far more than their portion of stress, if indeed they do not carry nearly all. The same condition of things will exist in butt-joints if two or more rows are found, under similar circumstances, on the same side of the joint.

If a strip of plate in which the ratio of width over thickness is very considerable be so gripped in a testing-machine that the applied stress be approximately uniformly distributed over its ends and if it be tested to breaking, it will be found, if the broken pieces be joined at the place of breaking, that the central portions of the fracture are widely separated, while the edges are in contact. This is due to the cause explained in Art. 41, "Coefficient of Elasticity." If a hole or holes be made in or near the centre of the specimen, a portion of the material in the front and rear of these holes will be relieved from stress, and the total stress in the central section of the specimen will be more nearly uniformly distributed in the remaining material. And again, these holes will "neck" the specimen down to a short one. The influences noticed in Art. 41, "Ultimate Resistance and Elastic Limit," will thus be called into action. For both these reasons the existence of the hole, or holes, *in itself*, will increase the intensity of the ultimate resistance of the plate.

On the other hand, the effect of the punch, if the hole is punched, as will presently be shown, is to decrease the resistance of the metal about the hole. If the hole is in a joint, also, the bearing pressure between the rivet and plate is very great, and as this pressure must be carried as tension to the material adjacent to the rivet-hole, and through that in its immediate vicinity, the latter (i.e., the material at the extremities of diameters parallel to the joint) will receive much greater tension than that in the central portion between the holes.

These last two influences tend to reduce the mean intensity of ultimate resistance of the material of the joint, and sometimes more than counterbalance the increase caused by the existence of the holes *simply as such.* In other cases the resultant effect can only be determined by experiment.

In Figs. 1 and 2 it will be observed that the stresses in the plates of a lap-joint act eccentrically, and, let it first be assumed, with a lever-arm equal to half the sum of the thickness of the two plates. If, however, a specimen joint is put in a testing-machine, the resultant stress may be made to pass through the centre of the joint, thus making the lever-arm for each plate about half its thickness.

If, therefore, t is the thickness of one plate and t' that of the other while T and T' are the mean intensities of tension in the plates, p the pitch of the rivets, and d the diameter, in the first case a width of each plate equal to the pitch will be subjected to the bending moment,

$$M = Tt(p-d)\left(\frac{t+t'}{2}\right) = T't'(p-d)\left(\frac{t+t'}{2}\right), \quad . \quad . \quad (1)$$

and in the second,

$$M = Tt^2\left(\frac{p-d}{2}\right), \quad \text{or} \quad T't'^2\left(\frac{p-d}{2}\right). \quad . \quad . \quad (2)$$

If K is the greatest intensity of tensile bending stress, then

$$K=\frac{tM}{2I}, \quad \text{or} \quad \frac{t'M}{2I}. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The greatest intensity of tension in the plate will therefore be

$$T+K, \quad \text{or} \quad T'+K. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The moment of inertia I will have the value

$$\frac{(p-d)t^3}{12}, \quad \text{or} \quad \frac{(p-d)t'^3}{12}.$$

If each plate has the same thickness, $t=t'$ and $T=T'$; hence

$$\text{By eq. (1)} \quad K=6T. \quad . \quad . \quad . \quad . \quad . \quad (5)$$
$$\text{By eq. (2)} \quad K=3T. \quad . \quad . \quad . \quad . \quad . \quad (6)$$

These values of K are very large and appear excessive. It is to be remembered, however, that the formula used, eq. (3), is strictly applicable only within the elastic limit.

There is no reason to doubt, therefore, that within that limit the greatest intensity of tension in the plates of the joint may reach from $4T$ to $7T$.

From these considerations it is to be expected that the true elastic limit of the joint, as a whole, would be very low.

The preceding investigations in the flexure of the joint are based upon the virtual assumption that the plates remain straight after the application of external stress. In reality such a condition of things does not obtain. Even below the elastic limit the plates begin to take positions which are shown in an exaggerated manner in Fig. 7. On account of the bending, the material at the points AA stretches much more than that at the points BB (with low values of T that at the latter points may be in compression), so that the centre lines of the plates P and P' are brought more nearly into coincidence, thus lessening the bending moment to which the joint is subjected. After the elastic limit of the material at AA is passed, a considerable increase of strain or stretch takes place at those points for the same increment of stress. Two important results follow this increase of strain between the elastic limit and failure: the joint becomes very markedly distorted, so that the plates P and P' become much more nearly in line, and the stress becomes much more nearly uniformly distributed in the sections AB, AB. This is equivalent to saying that the joint is subject to a greatly decreased bending moment.

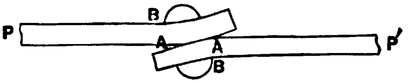


Fig. 7.

If the plates are thin, the excess of strain at AA over that at BB, requisite to bring the plates PP' essentially into line, may easily be within the stretching capacity of the material. If, however, the plates are thick, that condition will not hold, and the material at AA will begin to fail before P and P' are nearly in line. Hence the mean intensity of stress in a thick plate, other things being equal at the instant of rupture, will be considerably less than that in a thin one. It might thus happen that a lap-joint with thin plates would be found stronger, even, than one with thicker plates.

Reference will hereafter be made to experiments which verify these conclusions.

It will now be well to turn back a moment to the consideration of eqs. (5) and (6). Those equations show the effect of bending to be dependent on T *only*, and entirely surface of contact of the joined plates, while the unavoidably varying "fit" of the rivet in its hole, even in the best of work, throws the pressure towards the front portion of the surface of the rivet shaft. The intensity thus varies both along the axis and around the circumference of the rivet.

If any arbitrary law is assumed, the greatest intensity of pressure is easily determined. Such laws, however, are mere hypotheses and possess no real value. All that can be done is to determine, by experiment, the mean safe working intensity on the diametral plane of the rivet which is equivalent to a fluid pressure of the same intensity against its shaft.

Thus, if f is this mean (empirically determined) intensity, d the diameter of the rivet, and t the thickness of the plate, the total pressure carried by one rivet pressing against one plate is

$$R = fdt. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (10)$$

The Bending of Plate Metal Immediately in Front of Rivets.

There yet remains to be considered the condition of that portion of the plate on which the pressure $R = fdt$ is applied, and which is situated immediately in front of the rivet.

This portion of the plate is really in the condition of a beam fixed at each end, with a span equal to the diameter of the rivet. The beam, however, is not a straight one. At each end of the diameter the direct bending stress will be tension; and, on account of the position of the material, its direction will be approximately, at least, that of the proper tension of the plate. At those points, therefore, the proper and bending tension will act to some extent together, and the metal will usually be more highly stressed than anywhere else. This accounts for the usual manner of tensile fracture of a joint, in which the metal begins to tear on each side of the rivets, the metal between (generally in a diagonal direction in zigzag riveting) being the last to give way.

In the interior of the joint it is quite impossible to determine the value of this tensile bending stress on each side of the rivet. On the exterior of the joint, however, an approximate result may be reached, and hence the depth h, Fig. 2, from the centre of the outside row of rivets to the edge of the plate. The depth of the beam will be taken as $\left(h-\frac{d}{2}\right)$, and the pressure or load will be considered concentrated at the middle of the diameter or span. If t is the thickness of the plate, p the pitch of the rivets, and T the mean intensity of tension between the rivets, the load on the beam will be $(p-d)Tt$, and the moment of inertia of the cross-section will be

$$I=\frac{t\left(h-\frac{d}{2}\right)^3}{12}.$$

From what has been shown in the chapter on bending the modulus of rupture in the present case may be safely taken at $\frac{3}{2}T$.

In Art. 22 the moment at the centre and end of a span fixed at each end and loaded in the centre was shown to be equal to one eighth the load into the span.

Hence, by the usual formulæ,

$$M=\frac{d}{8}(p-d)Tt=\frac{2KI}{\left(h-\frac{d}{2}\right)}=\frac{3}{2}T\frac{t\left(h-\frac{d}{2}\right)^2}{6};$$

$$\therefore\ h=0.71\sqrt{(p-d)d}+0.5d. \quad . \quad . \quad . \quad (11)$$

Reviewing the results of this section, it may be concluded that the bending of the plates about axes parallel to them, or normal to them in the interior of the joint, and the bending of the rivets, as well as the law of the distribution of pressure against them, cannot be expressed by formula with any useful degree of accuracy; but that such influences must be recognized in the empirical determination of the shearing and tensile resistances of the joint and the mean intensity of pressure against the diametral plane of the rivet.

Effect of Punching.

The injurious effect of punching steel plates has been fully considered on pages 315 to 325, and the results there set forth should be carefully remembered in connection with riveted joints.

Art. 90.—Diameter and Pitch of Rivets and Overlap of Plate. Distance between Rows of Riveting.

Diameter of Rivets.

The diameter of rivet may at least approximately be expressed in terms of the thickness of the plate which it pierces. There are various arbitrary or conventional rules based upon this method of determining the rivet diameter. If the unit is the inch, the diameter d may be expressed as ranging between the two following values for ordinary thicknesses of plate:

$$\left.\begin{aligned} d &= .75t + .375, \\ d &= .875t + .375, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which t is the thickness of the plate. Unwin gives the following expression for the diameter of somewhat different form from that which precedes:

$$d = 1.2\sqrt{t}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Neither of the preceding expressions can be applied for all thicknesses of plates. If the thickness is great, those expressions make the diameter of the rivet too large, the diameter rarely exceeding 1 inch even for the heaviest plates. The application of eq. (1) to different thicknesses of plates will give the following diameters of rivets expressed by the nearest $\frac{1}{16}$ in.:

t	d
$\frac{1}{4}$ in.	$\frac{9}{16}$ in.
$\frac{3}{8}$	$\frac{3}{4}$
$\frac{1}{2}$	$\frac{7}{8}$
$\frac{5}{8}$	$\frac{15}{16}$
$\frac{3}{4}$	$1\frac{1}{16}$
$\frac{7}{8}$	$1\frac{1}{8}$
1	$1\frac{3}{16}$

In structural work for ordinary thicknesses of metal the prevailing diameters of rivets are $\frac{3}{4}$ in. and $\frac{7}{8}$ in. For light work, such as sidewalk railings or light highway construction, rivets as small as $\frac{1}{2}$ in. or $\frac{5}{8}$ in. in diameter are used. On the other hand, 1-in. rivets are occasionally employed for specially heavy sections.

Pitch of Rivets.

It is possible to determine the pitch of rivets approximately by an equation expressing equality between the tensile resistance of the net section between two adjacent rivets and the shearing or bearing capacity of a single rivet, but it is scarcely practicable to proceed in that manner as a rule. Again, the pitch will vary to some extent with the number of lines of riveting on either side of the joint. In single-riveting the pitch must be less than in double- or other multiple-riveting. In boiler or other similar riveting, also, the pitch must be usually less than in structural work, as questions of steam- and water-tightness or other similar considerations are involved in the former class of joints. Finally, the pitch will also obviously depend largely upon the thickness of plates. In single-riveting for comparatively thin plates the following relation may be taken, p being the pitch in inches:

$$p = 1.75 \text{ in. to } 2.25 \text{ in.} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

For comparatively thick plates in single-riveting the following relation may hold:

$$p = 2.375 \text{ in. to } 3 \text{ in.} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

In double-riveting, p and t still being the pitch and thickness respectively, the following relation may be taken for comparatively thin plates:

$$p = 2.6875 \text{ in. to } 3.25 \text{ in.} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Again, for comparatively thick plates in double-riveting,

$$p = 3.375 \text{ in. to } 3.75 \text{ in.} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The values given by eqs. (3) to (6) are for boiler or other similar work.

In structural work the pitch of rivets is seldom less than about three times the diameter of the rivet, and it is frequently specified not to exceed sixteen times the thickness of the thinnest plate pierced by the rivet.

Overlap of Plate.

The overlap of a plate, h in Fig. 2, Art. 88, in a riveted joint is the distance from the edge of the plate to the centre line of the nearest row of rivets. This distance, like other elements of riveted joints, will depend somewhat upon the thickness of the plate as well as the diameter of rivet and other similar considerations. It is a common practice to make the overlap not less than about $1.5d$, d being the diameter of the rivet. Occasionally in riveted joints it is made a little less, but 1.5 times the diameter of the rivet is about as small as the overlap should be made. Sometimes $\frac{1}{8}$ in. is added to the preceding value of the overlap.

The width of overlap (h) may also be determined in terms of d by the aid of eq. (11) of Art. 89. Since the load on the rivet is represented by $(p-d)Tt$, p must be taken in terms of d for a single-riveted joint, in which $p = 2\frac{1}{4}d$ to $2\frac{3}{4}d$. As a margin of safety, and as it will at the same time simplify the resulting expression, let $p = 3d$.

Eq. (11) of Art. 89 then gives, in confirmation of the preceding rule,

$$h = 1.5d. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Experience has shown that this rule gives ample strength, and is about right for calking, in boiler joints.

It is to be remembered that the preceding conventional rules for the diameter of rivet, pitch, and overlap of plate are necessarily to a large extent conventional or approximate, and in special cases they cannot be applied with mathematical exactness. As practical rules, however, they are sufficiently close to give good general ideas of those features of riveted joints.

Distance between Rows of Riveting.

The distance between the rows of riveting is not susceptible of accurate expression by formulæ, although the considerations involved in the establishment of eq. (11) of Art. 89 would lead to an approximate value. It is evident, however, that this distance should never be as small as *h*. Apparently, in more than double-riveted joints, this distance should increase as the centre line of the joint is receded from, in consequence of the bending action of the rivet. There are other reasons, however, besides that of inconvenience, why such a practice is not advisable.

In chain riveting the distance between the centre lines of the rows of rivets may be taken equal to the pitch in a single-riveted joint, or, as a mean, at 2.5 *the diameter of a rivet.*

In zigzag riveting (Fig. 5) *this distance may be taken at three quarters its value for chain riveting.*

Art. 91.—Lap-joints, and Butt-joints with Single Butt-strap for Steel Plates.

A butt-joint with single butt-strap, similar to that shown in Fig. 3, Art. 88, is really composed of two lap-joints in contact, since each half of the butt-strap or cover-plate with its underlying main plate forms a lap-joint. It is unnecessary, therefore, to give it separate treatment.

From these considerations it is clear that the thickness of the butt-strap or cover-plate should be at least equal to that of the main plate; it is usually a little greater.

Let t = thickness of plates;
d = diameter of rivets;
p = pitch of rivets (i.e., distance between centres in the same row);
T = mean intensity of tension in net section of plates between rivets;
T' = mean intensity of tension in main plates;
f = mean intensity of pressure on diametral plane of rivet;
S = mean intensity of shear in rivets;
n = number of rivets in one main plate;

q = number of rows in one main plate;

h = amount of extreme lap as shown in Fig. 2, Art. 77.

If all the dimensions are in inches, then T, T', f, and S are in pounds per square inch.

The starting-point in the design of a joint is the thickness t of the plate. The rivet diameter may then be expressed in terms of t, and the pitch in terms of the diameter. Such rules, like those given in Art. 89, may be useful within a certain range of application, but they cannot be depended upon in all cases.

The thickness t of boiler-plate depends upon the internal pressure, and is to be determined in accordance with the principles laid down in Art. 6, after having made allowance for the metal punched out at the holes to find the net section.

In truss work the thickness depends upon the amount of stress to be carried, and the same allowance is to be made for rivet-holes in finding the net section.

The relation existing between T and T' is shown by the following equations:

$$t(p-d)T = tpT'; \quad \therefore \quad \frac{T}{T'} = \frac{p}{p-d},$$

or

$$\frac{T'}{T} = \frac{p-d}{p} = 1 - \frac{d}{p}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

In order that the joint may be equally strong in reference to all methods of failure, the following series of equalities must hold:

$$\frac{n}{q}tpT' = \frac{n}{q}t(p-d)T = nfdt = 0.7854nd^2S.$$

$$\therefore \quad tpT' = t(p-d)T = qfdt = 0.7854qd^2S. \quad . \quad . \quad (2)$$

It is probably impossible to cause these equalities to exist in any actual joint, but none of the intensities T', T, f, or S should exceed a safe working value.

The method of failure by tearing through the gross section of the main plate is practically impossible under ordinary circumstances, and it is neglected in designing riveted joints. This neglect is expressed by dropping the first member of eq. (2) and thus reaching eq. (3):

$$t(p-d)T=qfdt=0.7854qd^2S. \quad . \quad . \quad . \quad . \quad (3)$$

This equation shows that the usual design of a riveted joint must provide against failure in three principal ways:

1. *Tearing through the net section of the plate.*
2. *Compression of the metal where the rivets bear against the plate.*
3. *Shearing of the rivets.*

Although these are the three principal methods of failure of riveted joints, whatever may be their type or form, the proper design of such joints should be so performed as to afford provision also against the secondary stresses caused by rivet bending, bending of the plates, and other indirect influences discusssd in preceding articles. This latter end is attained by determining the empirical intensities T, f, and S of eq. (3) by testing to failure actual riveted joints in which those secondary stresses exist. In that manner the design against the three principal methods of failure, described above, will also afford provision against the secondary or indirect stresses of rivet and plate bending or other similar conditions. The determination of the intensities T, f, and S by tests of actual riveted joints will be fully shown in the following articles.

Among the older tests of steel-riveted joints are those of Sharp, Martell, Kirkaldy, and Greig and Eyth, some of the results of which are shown in Table I. Values of the shearing stress S are shown for a considerable variety of joints and thicknesses of plate. A considerable reduction is noticed with the increase in plate thickness, due probably to increased bending and size of rivet.

TABLE I.

SHEARING OF STEEL RIVETS.

Joint.	**Mean of**	**Thickness of Plate.**	**S in Pounds per Square Inch.**
Single-riveted......................	2	——	57,570
"	6	——	53,690
Double-riveted (chain)...............	8	——	53,310
" "	1	——	50,650
Treble-riveted "	7	——	60,930
" "	1	$\frac{1}{4}$ in.	56,220
" "	1	$\frac{1}{2}$ "	57,120
" "	1	$\frac{3}{4}$ "	53,540
" "	1	1 "	53,980
" (zigzag)...............	1	$\frac{7}{8}$ "	43,560
" "	1	$\frac{7}{8}$ "	46,140
Quadruple-riveted (zigzag)............	1	$\frac{7}{8}$ "	43,010

Four experiments by Mr. Kirkaldy on single-riveted lap-joints, during 1881 gave S varying from 52,106 to 54,042 lbs. per square inch.

Those and other tests have shown that

$$S = .7T \text{ to } .8T; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

the larger value belonging to comparatively thin plates. As a mean there may be taken for steel joints

$$S = .75T. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

It will be found hereafter that f may be taken at least $1.25T$. If these values be substituted in the third and fourth members of eq. (3), in which $q = 2$, there will result

$$d = 2t \text{ (nearly)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

This value of d is too large for thick plates.

The rivet diameter, therefore, for steel plates may be said to vary from $2t$ for thin plates to $1.6t$ for thick ones, with a maximum diameter of $1\frac{1}{8}$ to $1\frac{3}{16}$ inches. The distance between the centre lines of the rows of rivets may be taken at $2.5d$ to $3d$ for chain riveting and three fourths of that distance for zigzag riveting.

The best designed single-riveted lap-joints give from 55 to 64 per cent. the strength of the solid plates.

Well-designed double-riveted lap-joints should give from 65 to 75 per cent. the resistance of the solid plate.

Equally well-constructed treble- and quadruple-riveted joints should have an efficiency of 70 to 80 per cent. of the solid plate.

It is therefore seen that there is little economy in more than double-riveting ordinary joints.

Art. 92.—Steel Butt-joints with Double Cover-plates.

Butt-joints with double butt-straps or covers differ in two respects, and advantageously, from lap-joints and butt-joints with a single cover; i.e., in the former the rivets are in double shear and the main plates are subjected to no bending. The cover-plates, however, are subjected to greater flexure than the plates of a lap-joint, for there is no opportunity to decrease the leverage by stretching. As the covers form only a small portion of the total material, these, with economy, may be made sufficiently thick to resist this tendency to failure.

Let t' = thickness of each cover-plate; and let the remaining notation be the same as in Art. 91. The intensity of compression between the walls of the holes in the cover-plates and the rivets, and the tension in the former, will be ignored on account of the excess in thickness of the two cover plates combined over that of the main plate. This excess in thickness is required on account of the bending in the covers noticed above.

The thickness of each cover should be from $\frac{5}{8}$ to $\frac{7}{8}$ the thickness of the main plates, or $t' = .625$ to $.875t$.

The combined thickness of the covers will thus be from 1.25 to 1.75 that of the main plates.

The four principal methods of rupture in the main plate will then lead to the following equations, corresponding to eq. (2), Art. 91:

$$\frac{n}{q}tpT' = \frac{n}{q}t(p-d)T = nfdt = 1.5708nd^2S.$$

$$\therefore \ tpT' = t(p-d)T = qfdt = 1.5708qd^2S. \quad . \quad . \quad (1)$$

As in Art. 91, and for the reasons there given, the first member of eq. (1) may be omitted, thus giving

$$t(p-d)T = qfdt = 1.5708qd^2S. \quad . \quad . \quad . \quad . \quad (2)$$

Table I exhibits the results of some of the older tests of steel-riveted joints. The intensities of tension and compression, T and f, existed at failure. The first of the last set of results in the table, by Mr. Kirkaldy, was found with zigzag riveting in which the distance beween the centre lines of the rows of rivets was too small.

These results are quite irregular, but it would seem to be a safe deduction to take $f = 1.25T$, with T equal to 70,000 to 75,000 pounds per square inch for thin plates, and 55,000 to 60,000 for thick ones.

With this value of f, and $q = 2$, the first and second members of eq. (2) give for *double-riveted butt-joints with two covers*

$$p = 3.5d. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

TABLE I.

DOUBLE-RIVETED BUTT-JOINTS.

Experimenter or Authority.	Holes by	Pounds per Square Inch for	
		T.	*f.*
Henry Sharp	Drill	96,160	83.330
" "	Punch	87,600	75.170
Martell	Drill	55.100	76.205
"	"	51,740	71,680
"	"	64,290	88,890
"	"	58,690	89.130
Boyd	"	55,200	76,160
"	"	51,230	70,800
"	"	64,320	88,930
Kirkaldy, annealed plates	Punch	68,990	93.160
" " "	"	75,490	101,900
" " "	"	82,450	99,660
" " "	"	83,180	100.510
" " "	"	76,590	90.460
" " "	"	78,220	92.380
" " "	"	74,030	92,850
" " "	"	70,540	88,500
" " "	"	73,920	84,630
" " "	"	72,560	83,080
" " "	"	72,390	107,110
" " "	"	76,520	112,780
Greig and Eyth	Drill	67,670	92,270
" " "	"	57.360	71,440
Parker	"	49,370	49.100
"	"	50,920	50,760
"	"	62,140	107,410
Kirkaldy 7/8-inch Siemens steel plates. Mean of two	"	61,800	74.520
" 9/16-inch Landore " "13/16" rivets	"	66,200	112,000
" " " " " " "	"	63,260	107,700
" " " " " " "	Bored	63,560	104,100
" " " " " " "	"	69,590	117,300
" " " " " " "	Punch	67,540	98.630
" " " " " " "	"	66,750	108,000
" " " " " " "	Bored	67,260	121,060

If the same value of f be preserved, there will result for *single-riveted butt-joints with two covers*

$$p = 2.5d. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

If, as in the preceding article, there be taken $S = .75T$ and $f = 1.25T$, the second and third members of eq. (2) give

$$d = 1.06t. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

This value of the rivet diameter is too small for thin plates, but about right for thick plates.

Double-riveted butt-joints designed in accordance with the foregoing deductions should give a resistance ranging from 65 to 75 per cent. of that of the solid plate.

Single-riveted joints will give an efficiency somewhat less; perhaps from 60 to 65 per cent.

It is to be supposed, in applying the rules just established, that all steel plates are drilled or punched and reamed.

As in the preceding cases, the distance between the centre lines of the rows of rivets may be taken at 2.5 to $3d$ for chain riveting, and three quarters that distance for zigzag.

Art. 93.—Tests of Full-size Riveted Joints.

There have not been many tests of full-size riveted joints of either iron or steel, and those which have been made seldom include such heavy steel plates as are now frequently employed both in boiler work and for structural purposes. The most valuable tests available and with the greatest range in size of rivet and thickness of plate are those which have been made at the U. S. Arsenal, Watertown, Mass. The results shown in Table I were taken from "Senate Ex. Doc. No. 1, 47th Congress, 2d Session," while those in Table II are taken from "Senate Ex. Doc. No. 5, 48th Congress, 1st Session." The results shown in Table III are from the same source and are given in the "U. S. Report of Tests of Metals and Other Materials" for 1896. The character of plates, rivets, and holes is shown in the tables, and the intensities of tension in the net sections of plates, compression or bearing on diametral surface, and shearing on rivets are those which existed at the instant of failure. The bold-face figures show the kind of failure, and when such figures are found, for the same test, in two or three columns, they show that the same two or three kinds of failure took place simultaneously.

TABLE I.

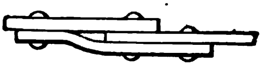
RIVETED JOINTS—IRON AND STEEL.

No.	Size of Rivet and Kind.	Pitch of Rivet.	Maximum Stresses, Pounds per Square Inch.			Efficiency of Joint, Per Cent.	Remarks.
			Tension on Net Area of Plate (T.	Compression on Diametral Surface (f).	Shearing on Rivets (S).		
			Single-riveted lap-joints; ¼-inch iron plates.				
35	⅝″ iron	2 ins.	43,230	76,140	34,900	57.7	11/16″ punched holes.
36	⅝″ "	2 "	45,520	82,910	38,640	61.4	" " "
37	⅝″ "	2 "	38,580	73,260	34,870	52.8	" drilled "
38	⅝″ "	2 "	41,790	79,360	38,660	57.1	" " "
39	⅝″ "	1⅝ "	52,160	65,420	33,420	60.6	" punched "
40	⅝″ "	1⅝ "	54,930	68,390	35,200	64.0	" " "
41	⅝″ steel	2 "	49,420	87,670	39,640	65.9	" " "
42	⅝″ "	2 "	47,260	83,940	40,610	63.1	" " "
43	½″ "	1⅝ "	45,890	78,220	45,300	60.3	9/16″ " "
44	½″ "	1⅝ "	49,720	84,660	48,420	65.5	" " "
45	7/16″ iron	1 5/16 "	41,095	66,778	44,204	53.1	½″ drilled "
46	7/16″ "	1 5/16 "	37,500	60,886	42,038	48.3	" " "
			Single-riveted lap-joints; ¼-inch steel plates.				
426	⅝″ iron	2 ins.	46,340	82,480	37,890	53.2	11/16″ punched holes.
427	⅝″ "	2 "	46,010	81,780	37,860	52.8	" " "
436	⅝″ steel	2 "	60,250	107,260	49,270	69.2	" " "
437	⅝″ "	2 "	59,240	105,290	48,750	68.0	" " "
428	⅝″ iron	2 "	40,950	77,870	36,350	48.2	" drilled "
429	⅝″ "	2 "	42,370	80,200	36,710	49.6	" " "
438	⅝″ steel	2 "	63,190	120,160	56,100	74.3	" " "
439	⅝″ "	2 "	61,310	116,090	52,460	71.8	" " "
430	⅝″ "	1⅝ "	66,860	90,000	41,790	68.8	" " "
31	⅝″ "	1⅝ "	70,000	94,230	43,750	72.0	" " "
47	7/16″ "	1 5/16 "	62,496	101,180	65,220	69.0	½″ " "
48	7/16″ "	1 5/16 "	58,338	94,800	60,382	64.8	" " "
49	⅝″ "	2 "	60,184	114,603	52,742	70.6	11/16″ " "
50	⅝″ "	2 "	57,439	109,650	50,645	67.6	" " "
			Double-riveted lap-joints; ¼-inch plates.				
85	7/16″ iron	2 ins.	38,535	64,120	43,110	60.3	½″ drilled holes.
86	7/16″ "	2 "	41,750	69,710	41,750	65.3	" " "
617	½″ "	1⅝ "	50,592	42,118	28,691	65.8	9/16″ punched "
618	½″ "	1⅝ "	49,950	41,660	28,660	65.3	" " "
			Double-riveted lap-joints; ¼-inch steel plates.				
432	⅝″ iron	2 ins.	61,510	54,640	25,400	70.4	11/16″ punched holes.
433	⅝″ "	2 "	60,300	53,715	25,530	69.4	" " "
434	⅝″ "	2 "	65,400	64,600	30,430	74.9	" " "
435	⅝″ "	2 "	64,600	63,430	30,430	74.3	" " "
87	7/16″ steel	2 "	56,944	94,910	57,910	76.3	½″ " "
88	7/16″ "	2 "	59,130	98,360	61,130	79.5	" " "
			Double-welt butt-joints; ¼-inch iron plates.				
615	⅝″ iron	1⅝ ins.	53,475	67,321	16,944	62.2	11/16″ punched holes.
616	⅝″ "	1⅝ "	50,959	64,138	16,719	59.3	" " "

TABLE I.—*Continued.*

No.	Size of Rivet and Kind.	Pitch of Rivet.	Maximum Stresses, Pounds per Square Inch.			Efficiency of Joint, Per Cent.	Remarks.
			Tension on Net Area of Plate (*T*)	Compression on Diametral Surface (*f*).	Shearing on Rivets (*S*).		
			Single-riveted lap-joints; ⅝-inch iron plates.				
62	1 1/16"iron	2 ins.	37,460	60,840	38,280	49.0	¾" punched holes.
63	" "	2 "	36,180	58,150	35,520	47.2	" " "
64	" "	2 "	38,190	60,730	37,580	49.7	" drilled "
65	" "	2 "	36,210	57,530	36,050	47.1	" " "
66	" "	1¾ "	41,750	54,130	34,230	50.0	" punched holes.
67	" "	1¾ "	41,290	53,400	34,150	49.3	" " "
720	1" "	2 1/16 "	61,700	52,970	26,180	60.4	1 1/16" " "
721	1" "	2 1/16 "	58,510	50,220	24,830	57.1	" " "
			Single-riveted lap-joints; ⅝-inch steel plates.				
51	1 1/16"iron	2 ins.	39,220	63,210	39,740	45.4	¾" punched holes.
52	" "	2 "	37,700	60,760	38,190	43.6	" " "
53	" steel	2 "	55,215	89,580	56,430	64.1	" " "
54	" "	2 "	54,740	88,660	55,460	63.5	" " "
55	" "	1¾ "	63,650	80,930	50,650	66.7	" drilled "
56	" "	1¾ "	63,976	81,600	50,900	67.2	" " "
238	¾" "	2 "	65,460	89,490	58,560	70.9	1 13/16" punched "
239	¾" "	2 "	65,210	88,990	58,600	70.6	" " "
718	1" iron	2 5/16 "	73,394	79,510	36,614	71.4	1 1/16" " "
719	1" "	2 5/16 "	73,970	80,200	36,590	72.0	" " "
			Double-riveted lap-joints; ⅝-inch iron plates.				
68	1 1/16"iron	2 ins.	48,450	39,160	24,760	63.5	¾" punched holes.
69	" "	2 "	50,780	41,070	26,150	66.4	" " "
58	" "	2 "	50,220	40,640	25,330	65.7	" " "
70	" "	2 "	46,255	41,480	27,550	60.5	" " "
71	" "	2 "	46,110	41,270	27,010	60.4	" " "
81	" "	3¼ "	30,920	58,700	39,130	50.4	" drilled "
82	" "	3¼ "	30,130	57,340	38,410	49.1	" " "
			Double-riveted lap-joints; ⅝-inch steel plates.				
57	1 1/16"iron	2 ins.	62,800	50,760	32,310	73.2	¾" punched holes.
59	" "	2 "	64,720	52,450	32,930	75.2	" " "
60	" "	2 "	63,210	56,860	34,710	73.2	" " "
61	" "	2 "	54,980	49,530	30,830	63.8	" " "
83	" steel	3¼ "	44,660	84,460	52,750	64.4	" drilled "
84	" "	3¼ "	43,650	83,000	51,845	63.0	" " "
			Reinforced riveted lap-joints; ⅝-inch iron plates. (See figure next page.)				
244	⅝" iron	2 ins. joint 4 " welt	38,870	59,080	40,360	67.6	1 1/16" drilled hole, ⅝" welt.
245	¾" "	" " " " " "	43,770	56,640	34,460	74.0	1 13/16" " " " "
296	¾" "	" " " " " "	44,840	57,910	33,890	75.7	" " " ¼" "
297	¾" "	" " " " " "	42,680	55,350	31,810	71.9	" " " " "

TABLE I.—*Continued.*



No.	Size of Rivet and Kind.	Pitch of Rivet.	Maximum Stresses, Pounds per Square Inch.			Efficiency of Joint, Per Cent.	**Remarks.**
			Tension on Net Area of Plate (*T*)	Compression on Diametral Surface (*f*).	Shearing on Rivets (*S*).		
Reinforced riveted joints; 5/8-inch steel plates. (See above figure.)							
246	7/8" steel	2 ins. joint 4 " welt	**62,050**	67,320	32,960	89.0	15/16" drilled holes.
247	7/8" "	" " " " " "	**62,880**	68,135	33,900	90.1	" " "
298	7/8" iron	" " " " " "	61,020	**67,300**	**34,250**	87.8	" " "
299	7/8" "	" " " " " "	61,710	**68,040**	**34,750**	88.9	" " "
Single-riveted lap-joints; 1/2-inch iron plates.							
240	3/4" iron	2 ins.	**31,100**	41,500	34,280	39.8	13/16" punched holes.
241	" "	2 "	**31,895**	41,955	34,060	39.7	" " "
292	" "	2 "	32,376	47,850	**38,020**	42.9	" drilled "
293	" "	2 "	33,180	48,890	**39,220**	44.3	" " "
327	" steel	2 "	39,000	**58,880**	**47,020**	52.2	" " "
328	" "	2 "	**40,500**	**59,900**	47,830	54.2	" " "
Single-riveted lap-joints; 1/2-inch steel plates.							
242	3/4" iron	2 ins.	38,204	50,940	**41,100**	38.2	13/16" punched holes.
243	3/4" "	2 "	35,915	47,800	**38,636**	35.9	" " "
294	15/16" "	2 "	**60,210**	56,980	**36,770**	51.2	1" " "
295	15/16" "	2 "	**49,590**	47,060	30,540	42.2	" " "
Double-riveted lap-joints; 1/2-inch iron plates.							
329	3/4" iron	2 ins.	**44,320**	50,640	25,280	57.0	13/16" punched holes.
635	3/4" "	2 "	**42,920**	57,950	24,560	55.2	" " "
Double-riveted lap-joints; 1/2-inch steel plates.							
619	15/16" iron	2 ins.	**64,602**	29,354	19,670	53.8	1" punched holes.
620	15/16" "	2 "	**64,519**	29,371	19,644	53.8	" " "
Single-riveted lap-joints; 5/8-inch iron plates.							
730	1" iron	2 5/8 ins.	**34,680**	47,510	35,460	44.9	1 1/16" punched holes.
731	1" "	2 5/8 "	**34,230**	46,790	34,930	42.0	" " "
Double-riveted lap-joints; 5/8-inch iron plates.							
732	1" iron	2 5/8 ins.	**43,580**	29,740	22,960	56.3	1 1/16" punched holes.
733	1" "	2 5/8 "	**45,850**	31,310	23,670	59.3	" " "
Single-riveted lap-joints; 5/8-inch steel plates.							
734	1" steel	2 5/8 ins.	49,650	56,760	**43,490**	50.5	1 1/16" punched holes.
735	1" "	2 5/8 "	52,770	60,150	**46,080**	53.6	" " "
Double-riveted lap-joints; 5/8-inch steel plates.							
736	1" steel	2 5/8 ins.	**69,680**	30,780	30,470	70.9	1 1/16" punched holes.
737	1" "	2 5/8 "	**67,100**	38,300	29,340	68.3	" " "

TABLE II.

RIVETED JOINTS—IRON AND STEEL.

No.	Thickness of Plate and Kind.	Diameter and Kind of Rivet.	Pitch of Rivet.	Maximum Stresses, Pounds per Square Inch.			Efficiency of Joint, Per Cent.	Remarks.
				Tension on Net Area of Plate (*T*)	Compression on Diametral Surface (*f*).	Shearing on Rivets (*S*).		
				Single-riveted iron lap-joints.				
1	⅜″ iron	1¹⁄₁₆″ iron	1¾ ins.	39,300	50,850	33,710	47.0	¾″ punched holes.
2	" "	" "	" "	41,000	53,050	35,170	49.0	1¹⁄₁₆″ " "
3	½″ "	¼″ "	2 "	35,650	47,350	37,300	45.6	1³⁄₁₆″ " "
4	⅜″ "	" "	" "	35,150	46,690	36,780	44.9	" " "
				Single-riveted iron butt-joints.				
5	⅜″ iron	1¹⁄₁₆″ iron	2 ins.	46,360	72,390	25,380	59.9	¾″ punched holes.
6	" "	" "	" "	46,875	73,050	25,450	60.5	" " "
7	½″ "	¾″ "	" "	46,400	61,940	24,630	59.4	¹³⁄₁₆″ " "
8	" "	" "	" "	46,140	61,740	24,310	59.2	" " "
9	⅝″ "	1″ "	2⅝ "	44,260	60,330	23,010	57.2	1¹⁄₁₆″ " "
10	" "	" "	" "	42,350	58,080	22,310	54.9	" " "
11	¾″ "	1⅛″ "	2.9 "	42,310	57,000	21,870	52.1	1³⁄₁₆″ " "
12	" "	" "	" "	41,920	56,540	22,140	51.7	" " "
				Single-riveted steel lap-joints.				
13	⅜″ steel	¾″ iron	1¾ ins.	61,270	65,760	40,390	59.5	¹³⁄₁₆″ punched holes.
14	" "	" "	" "	60,830	65,320	39,900	59.1	" " "
15	½″ "	¹⁵⁄₁₆″ iron	2 "	47,530	44,590	29,390	40.2	1″ " "
16	⅝″ "	" "	" "	49,340	46,960	31,070	42.3	" " "
				Single-riveted steel butt-joints.				
17	⅜″ steel	1¹⁄₁₆″ iron	2 ins.	62,770	97,940	31,240	71.7	¾″ punched holes.
18	" "	¹⁵⁄₁₆″ "	" "	61,210	95,210	31,020	69.8	1″ " "
19	½″ "	" steel	" "	68,920	62,220	20,370	57.1	" " "
20	2″ "	" "	" "	66,710	59,580	19,890	55.0	" " "
21	⅝″ "	1″ "	2⅜ "	62,180	71,450	27,750	63.4	1¹⁄₁₆″ " "
22	" "	" "	" "	62,590	71,980	27,940	63.8	" " "
23	¾″ "	1⅛″ "	2½ "	54,650	55,610	23,190	54.0	1³⁄₁₆″ " "
24	" "	" "	" "	54,200	55,840	22,810	53.4	" " "

It is important to notice that in general the highest ultimate resistances of tension and compression or bearing are found with the thin plates, and that those quantities diminish appreciably as the thickness of plate increases, both for iron and steel. This law is not so well defined in reference to the diameter of rivet, if indeed these tests show it at all, except for steel.

The length of these test joints varied from 9.75 to 13 inches for Tables I and II, and from 10 to 27 inches for Table III.

Although the results of these tables are somewhat irregular, they confirm the general accuracy of the relations established between the values of T, f, and S in the preceding articles, as well as other general rules and conclusions for boiler work.

Some efficiencies are lower than those given for similar joints in Art. 94, but such instances can, by the aid of the tables, be traced either to indifferent design or a phenomenally low value of some one of the three resistances. In general the results compare well with those given in that article.

The pitches of rivets are seen to be adapted to boiler work, being much less than are ordinarily used in bridge work; yet the corresponding resistances show what may legitimately be done and expected when unusual conditions demand a departure from ordinary rules.

Before deducing working intensities for bridge construction from the preceding results it is to be first explained that those results are as given in the government reports, and that the net section used is the gross section of the plate, less the actual metal removed by the punch or drill, with no allowance for deterioration by the former in the immediate vicinity of the hole. Again, in Tables I and II the diametral bearing surface and the shearing area of the rivet are taken to be those of the drill, or a mean between the punch and die in case of punched holes. In bridge work, in determining the net section, metal is deducted for a diameter equal to that of the cold rivet before driving plus one eighth of an inch; and the shearing and bearing are computed for the section and diameter of the cold rivet before driving.

TABLE III.

TESTS OF STEEL-RIVETED JOINTS; ½-INCH PLATES.

Joint.	Rivet.	Maximum Stresses: Pounds per Square Inch. Tension on Net Area of Plate (T).	Compression on Diametral Surface (t).	Shearing on Rivets (S).	Efficiency of Joint, Per Cent.	Remarks.
A	¾" steel	38,940	57,960	41,760	47.1	13/16" drilled holes.
B	" "	39,450	81,530	35,560	57	" " "
C	" "	62,200	59,950	22,480	83.5	" " "
"	" "	56,410	77,900	29,640	80.3	" " "
"	" "	63,000	88,510	20,930	85.5	" " "
"	" "	59,330	78,900	29,410	85.3	" " "
" *	" "	55,050	71,890	29,850	79.4	" " "
H	" "	51,340	76,550	36,030	78	" " "
"	" "	52,150	50,170	20,790	78.6	" " "
"	" "	62,390	54,660	21,530	90.1	" " "
"	" "	58,550	51,350	20,620	84.7	" " "
"	" "	55,030	67,490	27,030	82.5	" " "

* Joint not fractured.

A. Double-riveted lap-joint; ½-inch plate.
B. Double-riveted butt-joint, two splice-plates; ½-in. plate.
C. Treble-riveted " " " " "
H. Quadruple-riveted butt-joint, two splice-plates; ½-in. plate.

The pitch of the outside rows of rivets in joints B, C, and H was double that of the adjoining rows. In the same joints one splice-plate was narrower than the other, so that it took one less row of rivets on either side of the joint than the other.

With these explanations in view, the preceding tests justify the following working stresses for the plate-girder floor-beams and stringers of railway bridges with machine-driven rivets.

Rivet shearing........... { 7,500 lbs. per sq. in. for iron.
10,000 " " " " " steel.

Rivet bearing............ { 14,000 lbs. per sq. in. for iron.
18,000 " " " " " steel.

Tension in net section of plate { 8,000 lbs. per sq. in. for iron.
15,000 " " " " " steel.

The bearing resistances are taken rather low, especially for steel, for the reason that thick plates are frequently used in bridge construction, and the ultimate bearing resistance for them is appreciably less than for the thin plates used in most of the preceding tests.

The preceding working stresses are based on steel for rivets giving from 56,000 to 64,000 pounds per square inch tensile resistance, while the steel for plates, in test specimens, should offer from 58,000 to 66,000 pounds per square inch ultimate tensile resistance.

In the government report from which Table I is abstracted, can be found a large number of tests made for the purpose of determining the proper minimum distance from the centres of rivet-holes to the edge of plates. As a result of those tests and other experience on the same subject, it may be stated that the least distance from the centre of a rivet-hole to the edge of a plate may be taken at one and one half the diameter of the *hole* for steel and one and five eighths the diameter of the *hole* for iron, in cases where it is important to secure the maximum resistance of the joint.

Art. 94.—Efficiencies.

The values of the quantity which has been termed the "efficiency" of the joint, i.e., the ratio of the resistance of a given width of joint over that of an equal width of solid plate, in the preceding investigations, are those actually determined by experiments with the joints themselves. They may, therefore, be relied upon. Some values which have for many years been considered as standard, but which in reality are of a somewhat arbitrary nature, and at best belonging to a limited class of joints, have been disregarded.

Table I gives the results of Mr. Kirkaldy's experiments in reference to the comparative resistance of chain and zigzag riveting. The difference is not great, but what there is is in favor of the chain riveting.

TABLE I.

BUTT-JOINTS WITH TWO COVERS—1877.

No. of Tests.	Plate Thickness.	Rivet Diameter.	Pitch of Rivets.	Holes.	Riveting.	Efficiency.
2	$\frac{7}{16}$ in.	$\frac{5}{8}$ in.	$2\frac{1}{8}$ in.	Punched	Chain	0.672
2	$\frac{7}{16}$ in.	$\frac{5}{8}$ in.	3 in.	Punched	Zigzag	0.669
2	$\frac{1}{2}$ in.	$\frac{3}{4}$ in.	$2\frac{1}{4}$ in.	Drilled	Chain	0.662
2	$\frac{1}{2}$ in.	$\frac{3}{4}$ in.	3 in.	Drilled	Zigzag	0.633

TABLE II.

KIRKALDY'S TESTS—1872.

Joint.	Riveting.	Holes.	Rivet Diameter in Terms of t.	Pitch in Terms of d.	Efficiency.
Lap............	Single	Punched	$d=2t$	$p=3d$	0.55
Lap............	Single	Drilled	$d=2t$	$p=2\frac{5}{8}d$	0.62
Lap............	Double	Punched	$d=2t$	$p=4\frac{1}{2}d$	0.69
Lap............	Double	Drilled	$d=2t$	$p=4d$	0.75
Butt, 1 cover....	Single	Punched	$d=2t$	$p=3d$	0.55
Butt, 1 cover....	Single	Drilled	$d=2t$	$p=2\frac{5}{8}d$	0.62
Butt, 1 cover....	Double	Punched	$d=2t$	$p=4\frac{1}{2}d$	0.69
Butt, 1 cover....	Double	Drilled	$d=2t$	$p=4d$	0.75
Butt, 2 covers...	Single	Punched	$d=1\frac{1}{4}t$	$p=3\frac{1}{4}d$	0.57
Butt, 2 covers...	Single	Drilled	$d=1\frac{1}{4}t$	$p=3d$	0.67
Butt, 2 covers...	Double	Punched	$d=1\frac{1}{4}t$	$p=5\frac{1}{4}d$	0.72
Butt, 2 covers...	Double	Drilled	$d=1\frac{1}{4}t$	$p=4\frac{3}{4}d$	0.79

Table II gives the results of the same experimenter on the relative value of punched and drilled work.

The drilled work is seen to give decidedly the greatest efficiency in every case.

The joints to which Tables I and II belong were of wrought iron.

Experiments by Mr. Kirkaldy during 1881 show that well-designed double-riveted steel butt-joints with two covers may be expected to give efficiencies varying from 0.65 to 0.75.

The tests of full-size wrought-iron and steel-riveted joints exhibited in Art. 93 show, as a rule, that thin plates give materially higher efficiencies than thick plates. Although there are irregularities, single-riveted lap-joints may yield efficiencies running from 50 to 74 per cent. for ¼-inch plates, but dropping to 50 to 54 per cent. for ⅝-inch plates and materially lower for ½-inch plates. On the whole, the double-riveted lap-joints show somewhat higher efficiencies than the single-riveted, but not quite the same relative differences between ¼-inch and ⅝-inch plates, the values being found more generally between about 60 and 80 per cent.

The single-riveted butt-joints of Table II, Art. 93, give efficiencies ranging from about 52 to 72 per cent.

Some unusually high efficiencies are found in Table III of the same article for butt-joints, i.e., about 78 to 90 per cent. Those high values are due to the special design of the joints, and they cannot ordinarily be attained in practice, but they show that well-considered designs will yield greatly increased efficiencies.

In general, efficiencies running from 65 to 70 per cent. may be considered excellent for the usual conditions of practice.

Art. 95.—Riveted-truss Joints.

The circumstances in which riveted joints are used in truss work render permissible many special forms which can find no place in boiler-riveting. If joints are found under the same circumstances, as far as the transference of stress is concerned, precisely the same forms would be used, except that calking is, of course, only required in boiler work.

Fig. 1 shows a common form of chord construction in riveted-truss work, with the relative proportions exaggerated.

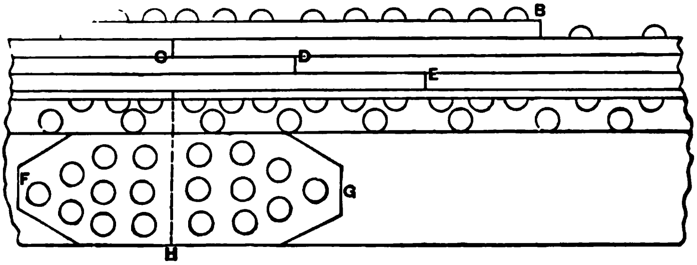
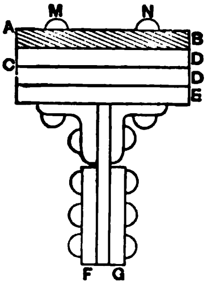


FIG. 1.



The lower portion of the figure shows a section of the chord in which the cover- or splice-plate is shaded. The joint is supposed to be in tension.

In this form of joint the splice-plate material is reduced to a minimum. These are, in reality, two lap-joints *CD* and *DE* with the two plates *C* and *E* to be spliced. In each lap-joint there should be sufficient rivets determined by the methods of Art. 91. The splice-plate *AB* should be long enough to give the requisite plate *AC* to the left of *C*, with the same length from *B* to a point vertically over *E*.

In most cases one or two plates only should be spliced at the same point.

The joint in the vertical plate should be formed as at *FG*; i.e., it should be a double-cover butt-joint. The principles already established in a preceding section, in regard to the thickness of covers and diameter of rivets, should be observed here.

The two or more full rows of rivets on either side of the joint may as well be chain-riveted with a pitch of 3½ to 4 diameters. Other rivets should then be staggered in until the group of rivet centres on each side is brought to a point, as shown in the upper part of Fig. 1. In this manner the available section of a width of plate equal to that of the cover becomes approximately equal to the total, less the material from one rivet-hole. Hence the efficiency of the joint becomes correspondingly increased.

If the joint is in compression the preceding observations hold without change, except that all covers should have the same thickness as the plates covered.

Even if the joints *C*, *D*, *E*, and *H* are of planed edges, little or no reliance should be placed upon their bearing on each other, since the operation of riveting will draw them apart more or less, however well the work may be done. Melted zinc or, in the case of very wide chords, four longitudinal rows of rivets should be used in such joints as are exemplified by Fig. 1.

Unless great caution is observed and excellence of design secured, there will frequently be excessive bending in the riveted joints of truss work, on account of the great variety of connections required.

Diagonal Joints.

It has been proposed to form riveted joints the edges of whose plates are neither perpendicular nor parallel to the stress transferred. In this manner a greater number of rivets and a greater section of metal will resist the stress exerted in the body of the plate.

Mr. Kirkaldy made some tests on such lap-joints, single-riveted, with $\frac{3}{8}$-inch plates, the joints of which lay at 45° with the applied force, with the following results:

Entire plate.	100
Square joint.	59.4
Diagonal joint.	87.2

The diagonal joints are thus seen to give by far the best results. They are, however, much the most expensive also.

Riveted Joints in Angles.

It has been found by tests of full-size angles that if a riveted joint be formed by riveting one leg only, the ultimate tensile resistance per square inch of the net angle section may be but 75 per cent. of the ultimate tensile resistance of test specimens cut from the same angle. On the other hand, if both legs are riveted the ultimate tensile resistance per square inch of the net section may easily be 90 per cent. of the ultimate resistance of test specimens cut from the same angle. These results show that both legs of angles should always be riveted at joints.

Friction of Riveted Joints.

There are not lacking experiments to show that the friction between the plates of a riveted joint is very great. This, however, cannot be relied upon to give additional resistance to the joint, since a sensible relative movement of the plates takes place in advance of its greatest resistance and essentially destroys the friction.

The experiments of Edwin Clarke, Harkort and Lavelley show that this friction may range from 8330 to 22,400 lbs. per square inch of rivet section.

The specimens were prepared with one slotted plate, so that friction was the only resistance to the parting of the plates.

Hand and Machine Riveting.

Pneumatic, steam, and hydraulic riveting-machines have been brought to such a degree of perfection that machine work is now always preferred to hand riveting.

The resistances of joints will vary to some extent with the method of riveting. Usually, however, the variation will not be greater than may be found for the same kind of riveting in different places and under different circumstances.

As a rule, machine riveting is much more reliable than hand, in that the hole is better filled and the rivet more quickly headed, in consequence of the great excess of pressure exerted. There is thus much less liability of loose rivets.

Most of the preceding experimental results were obtained from machine work.

Art. 96.—Welded Joints.

At the present time the process of welding can, with proper care and material, be made to give excellent results.

Scarf-welds give much better results than lap-welds, on account of the bending to which the latter are subjected.

Mr. Kirtley (Institute of Mechanical Engineers of Great Britain) made some experiments with small strips, 7.5 inches long and $\frac{7}{16}$ inch thick, cut across welded joints. These strips were taken out of boilers whose longitudinal joints had been welded. Twenty-three experiments with strips varying from one to one and a half inches wide gave the following results per square inch of plate section:

	Welded.	Solid Plate.
Greatest	53,310 lbs.	57,790 lbs.
Mean	46,140 "	52,860 "
Least	36,960 "	46,370 "

The preceding results are for wrought iron. The ordinary welding processes cannot be applied satisfactorily to structural steel. Welding may, however, be successfully done in steel by electric processes, but under such special conditions as usually not to be available for structural purposes.

Art. 97.—Pin Connections.

A pin connection consists of two sets of eye-bars or links, through the heads at one end of each of which a single pin passes. Fig. 1 shows a pin connection; *A*, *A*, *B*, *B*, are eye-bars or links, and *P* is the pin.

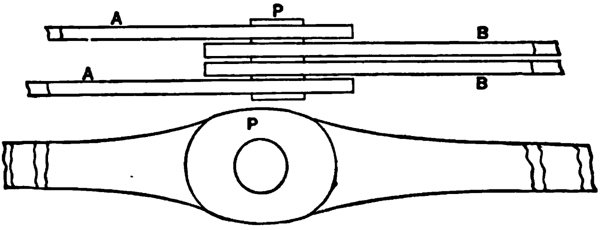


FIG. 1.

The head of the eye-bar (one is shown in elevation in Fig. 2) requires the greatest care in its formation. It is imperfect unless it be so proportioned that when the eye-bar is tested to failure, fracture will be as likely to take place in the body of the bar as in the head; in other words, unless its efficiency is unity.

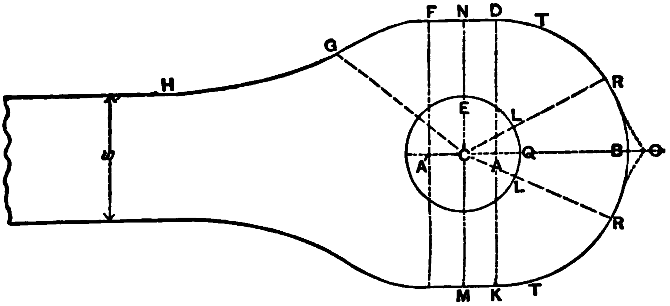


FIG. 2.

In Fig. 2 the head of the eye-bar, or link, is supposed to be of the same thickness as that of the body of the bar whose width is *w*.

If t is the thickness of the bar, so that wt is the area of its normal section, then t is almost invariably included between the limits of $\frac{1}{3}w$ and $\frac{1}{8}w$. In fact these extreme values are each too extreme for the full resistance of the bar, although they are sometimes used. These ratios, as well as the diameter of the pin in terms of w, can only be determined by experiments on full-size bars. A large number of such experiments have been made both in this country and in Great Britain, and while the resistance of the bar as a whole depends to a considerable extent on the mode of manufacture or formation of the head, it has been found that for the best proportioned head t should range from $\frac{1}{4}w$ to $\frac{1}{6}w$, and the diameter, d, of the pin from $0.75w$ to w.

It is extremely difficult to reach more than a general idea of the condition of stress in an eye-bar head, although an approximate mathematical treatment of the question may be found in the "Trans. Am. Soc. of Civ. Engrs.," Vol. VI, 1877, in which the results agree essentially with those of experiment.

Before taking a general view of the stresses which may arise in an eye-bar head, it must be premised that a difference of $\frac{1}{64}''$ to $\frac{1}{100}''$ between the diameter of the pin and that of the pin-hole is exceptionally good practice. Before the eye-bar is strained, therefore, there is a line of contact only between the pin and eye-bar head, but, on account of the elasticity of the material, this line changes to a surface when the bar is under stress, and increases with the degree of stress to which the bar is subjected. The line and surface of contact is, of course, in the vicinity of Q, Fig. 2, i.e., on that side of the pin toward the nearest end of the bar. The consequence of this is that, when the bar is strained, the portion about QB, Fig. 2, is subject to direct compression and extension; that about RL, NE, and GS to direct tension and bending, while in the vicinity of T there is a point of contraflexure, and the stress in the direction of the circumference changes from compression to tension as E is approached from Q.

As a result of many of the exper ments which have been made, the following mode of proportioning the head has heretofore been very extensively used: Let r represent the radius of the pin, while reference is made to Fig. 2. Then take $EN = 0.66w$. The curve $DRBK$ is a semicircle with a radius equal to $r + 0.66w$, with a centre, A, so taken on the centre line of the bar that $QB = 0.87w$. GF is a portion of the same curve, with A' as the centre ($AC' = AC$); GH is any curve with a long radius joining GF gradually with the body of the bar. HG should be very gradual in order that there may be a large amount of metal in the vicinity of CG, for there the metal is subjected to flexure as well as direct tension. FD is a straight line parallel to the centre line of the bar.

As the preceding rule gives a head whose outline causes a more expensive die than a simple circle, at the present day eye-bar heads are usually formed as shown in Fig. 3.

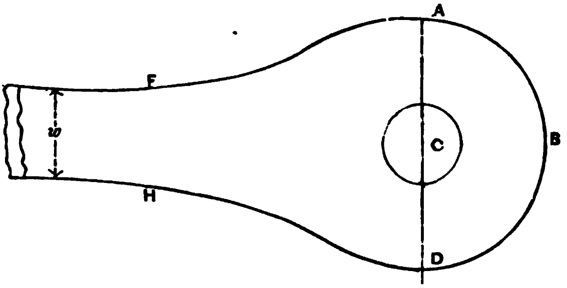


Fig. 3.

ABD is a semicircle with a radius equal to $r + 0.8w$ to $r + 0.9w$, and whose centre, C, is the centre of the pin-hole. The portions FA and HD are formed as before.

There should be no weld across the bar in the vicinity of *FH*. Consequently heads are usually formed by placing proper sized pieces upon the upset ends of the plain bars, and then, after insertion in a heating furnace, forcing the head to the desired shape in a die under hydraulic or steam pressure.

The intensity of this pressure will affect, to a considerable extent, the permissible dimensions of the head. The greater the pressure, the better will be the results.

The unfinished head is sometimes rolled on the bar, as by the Kloman process.

The thickness of the head is sometimes made greater than that of the body of the bar. If the head is circular, as in Fig. 3, the section of metal on each side of the pin (through *AC* or *CD*) should be not far from eight tenths that in the body of the bar.

This thickening of the eye-bar head is an excellent thing for the bar, but subjects the pin to a great increase of bending, and hence requires increased pin diameter.

In pin connections the pin is subjected to very heavy bending.*

If M is the bending moment to which the pin is subjected, K the greatest intensity of bending stress developed, and A the area of the normal section of the pin, eq. (4) of Art. 69 gives

$$M = K\frac{Ad}{8} = 0.1Kd^3 \text{ (nearly)}, \quad . \quad . \quad . \quad . \quad (1)$$

or

$$d = 2.16\sqrt[3]{\frac{M}{K}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Values of K, for circular sections, may be found in Art. 69.

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* For a detailed treatment of this subject the author's "Bridge and Roof Trusses" may be consulted.

CHAPTER XIV.

PLATE GIRDERS.

Art. 98.—The Design of a Plate Girder.

A PLATE girder is a flanged girder or beam built usually of plates and angles, the flanges being secured to the web by the proper number of rivets suitably distributed. The flanges, unlike those of rolled beams, are usually of varying sectional area, although occasionally either flange may be of uniform section throughout when formed of two angles, or two angles and a cover-plate. Fig. 1 is a general view of a plate girder, while Figs. 2, 3, 4, 5, and 6 show some of the general features of design.

The total length of a plate girder is materially more than the length of clear span over which the girder is designed to carry load. Blocks or pedestals of masonry or metal, as the case may be, support the ends of the girders and rest on the masonry or other supporting masses or members carrying the girder and its load. The distance between the centres of these blocks or pedestals is called the effective span of the girder, as it is the span length which must be used in computing bending moments, shears, or reactions. Plate girders must evidently be somewhat longer than the effective span. In Fig. 2 the relations of the various parts at the end of the plate girder are shown in detail. The girder illustrated in Fig. 1 has an effective span of 80 ft. with the centre of the pedestal block 15 inches from the face of the masonry abutment and 6.5 inches from the extreme end of the girder. The effective depth of the girder is the vertical distance or depth between the centres of gravity of the two flanges. When the girder has cover-plates this effective depth may be greater than the depth of web plate at the centre of span and less than that at the ends, even when the web plate is of uniform depth. It is always customary, however, to take the effective depth of a plate girder with uniform depth of web as constant. Frequently that depth is taken equal to the depth of the web plate; or, again, it may be taken equal to the depth between the centres of gravity of the flanges at mid-span without sensible error. In case the web plate is not of uniform depth the effective depth might still be taken as the depth of web plate at the various sections of the girder, or it may be taken as the depth between centres of gravity of the flanges at the same sections.

The plate girder shown in Fig. 1 and to be assumed for the purposes of design has a clear span of 77 ft. 6 ins., an effective span of 80 ft., and a length over all of 81 ft. 1 in. The differences between the effective span and the clear span and total length are obviously dependent upon the length of span. For short spans those differences are relatively small, and relatively large for long spans. The depth of web plate will be taken as 7 ft., and it will be found later that at and in the vicinity of the centre of span three cover-plates will be needed. The girder will be assumed to be of mild structural steel and will be supposed to carry a single-track railroad moving load with the concentrations and spacings shown in Table I, p. 69.

The dead load or own weight of the girder and track will depend somewhat upon whether the girder is of the

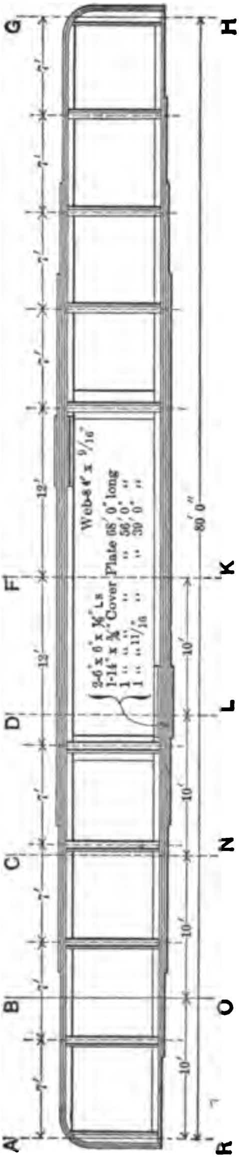


FIG. 1.

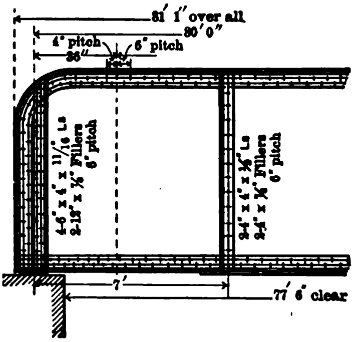


FIG. 2.

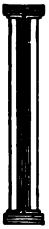


FIG. 3.

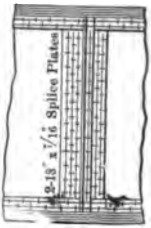


FIG. 4.

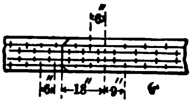


FIG. 5.

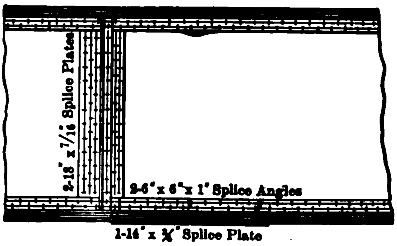
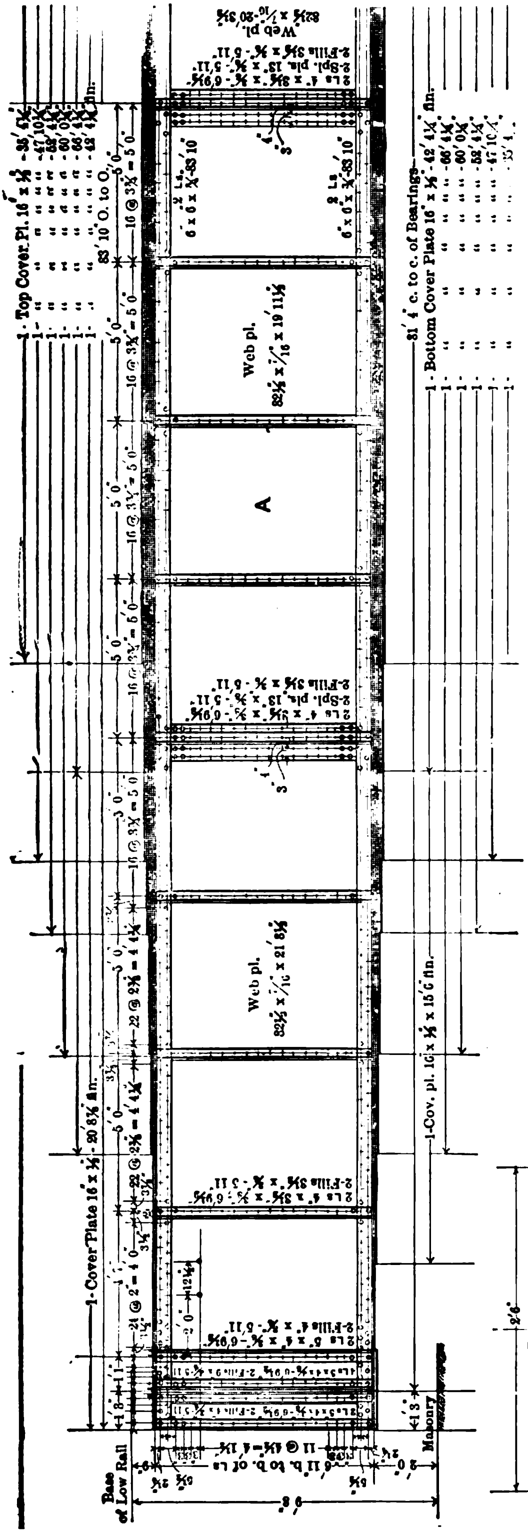
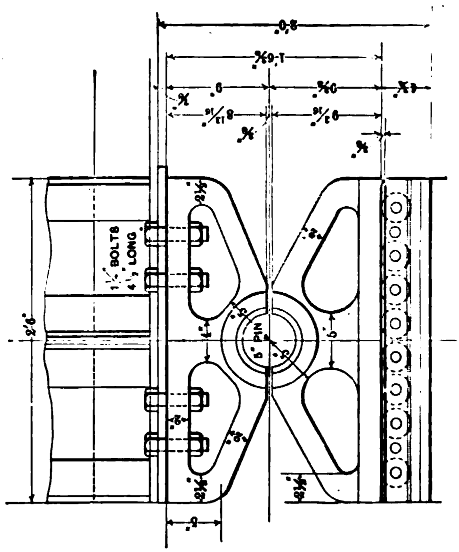


FIG. 6.


1-Top Cover Pl. 16" x ⅝ - 35' 5¼"
83' 10" O. to O.
Web pl. 82½ x 7/16 x 19' 11½
A
Web pl. 82½ x 7/16 x 21' 8½
81' 4" c. to c. of Bearings
1-Bottom Cover Plate 16" x ⅝ - 42' 4¼" fin.
1-Cov. pl. 16 x ⅝ x 15' 6" fin.
1-Cover Plate 16 x ⅝ - 20' 8¾" fin.
Base of Low Rail
Masonry
8' 6"

81-foot Railroad Plate-girder Span with End Pin and Roller Bearing.



End Pin and Roller Bearing.

through or deck type. The only difference in computation arising in those two types is due to the fact that if the girders are of the deck class (i.e., carrying the moving load directly on their upper flanges) the rivets connecting the upper flanges with the webs must be assumed to carry the wheel concentrations in addition to their other duties, as will be shown in the following computations. The total dead load or own weight will be taken as 1400 lbs. per linear foot. Inasmuch as there are two girders, each will carry one half of the moving load and one half of the dead load or own weight. It should be observed that the effective length of span being 80 ft., the two locomotives at the head of the train load will more than cover the span, so that the uniform train load will not appear in the computations.

Bending Moments.

The first computations necessary are those required to determine the bending moments, and from them the flange stresses at different points of the span. Those points may be taken at 6, 8, or 10 ft. apart as may be desired for the purposes of design; the closer together the sections are taken the greater will be the degree of accuracy attained. In the present instance it will suffice to take those sections 10 ft. apart as shown at *BO*, *CW*, *DL*, and *FK* in Fig. 1, the latter section being at the centre of the span. After the bending moments are obtained, the flange stresses at once result by dividing the former by the effective depth.

As each girder will carry 700 lbs. of dead load per linear foot, and as the effective span is 80 ft., the expression for the dead-load bending moment in foot-pounds at any point will be as follows:

$$M = \frac{700}{2}(80x - x^2). \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The application of eq. (1) to the sections of the girder 10, 20, 30, and 40 ft. from the ends will give the following expressions for the bending moments in foot-pounds:

x.	Moment.
10 ft..................	245,000 ft.-lbs.
20 "	420,000 "
30 "	525,000 "
40 "	560,000 "

The moving-load bending moments are next to be found by using the concentrations shown on p. 69. For this purpose the criterion for the maximum bending moment, eq. (5), Art. 16, must be applied at the assumed sections in which l' (equal to x in the above dead-load computations) has the values 10, 20, 30, and 40 ft. The application of that criterion to the section BO, Fig. 1, 10 ft. from the end of the span shows that W_2, or the first driving-wheel, must rest at the section in question for the maximum bending moment, the loads W_1 to W_{13} inclusive resting on the span. By the aid of Table I, p. 69, the maximum bending moment takes the form

$$M = \tfrac{1}{8}(7,668,000 + 212,000 \times 4) - 80,000 = 984,500 \text{ ft.-lbs.}$$

The application of the same criterion to the section CW of Fig. 1 shows that maximum bending moments will be found by placing either W_3 or W_4 at the section in question. The greatest of those two moments is to be found by placing W_3 on the second driving-wheel at the section. By the aid of the same table as before, that bending moment takes the value

$$M = \tfrac{1}{4}(6,708,000 + 192,000 \times 4) - 230,000 = 1,639,000 \text{ ft.-lbs.}$$

Similarly the criterion equation is satisfied by placing either W_4 or W_5 at the section DL, Fig. 1, 30 ft. from the end of the effective span, but the greatest moment results from placing W_4 at the section in question with eleven wheel loads on the span. That moment has the value

$$M = \tfrac{3}{8}(5,848,000 + 172,000 \times 4) - 480,000 = 1,971,000 \text{ ft.-lbs.}$$

By proceeding in the same manner it will be found that there are several positions of the loading satisfying the criterion for the centre section FK of the span, but that the greatest moment of all will be found by placing W_{12} at the centre of the span, which requires that W_6 to W_{18} inclusive rest on the girder. The resulting value of M, found in the same manner as before, will be $M = 2,160,500$ ft.-lbs.

In designing plate girders it is customary in the best practice to recognize the effect of shock due to inevitable inequalities of track, lack of vertical balance of driving-wheels and other agencies acting under high speeds of fast trains, by adding to the stresses produced by the moving load certain percentages depending usually upon the length of span. In the present instance 60 per cent. for shock will be added to the bending moments found for the moving load in the preceding computations.

The total greatest bending moments therefore due to dead load, moving load, and shock are found by taking the sums of the various computed moments for the different sections as follows:

$l' =$	10 Feet.	20 Feet.	30 Feet.	40 Feet.
	Ft.-lbs.	Ft.-lbs.	Ft.-lbs.	Ft.-lbs.
Dead.	245,000	420,000	525,000	560,000
Moving.	984,500	1,639,000	1,971,000	2,160,500
Shock.	590,700	983,400	1,182,600	1,296,300
Total.	1,820,200	3,042,400	3,678,600	4,016,800

The greatest permitted or working tensile stress to be used in the tension flanges of these girders is 14,000 lbs. per square inch of net section. The effective depth at the centre of the girder will be somewhat greater than the actual depth of the web, and it will be assumed at 7.1 ft. It will further be assumed in this case, as is frequently done in practice, that the web of the girder takes shear only, its resistance to bending being neglected. The total bending resistance of the girder therefore will be due to the flanges only. This assumption involves a small error on the side of safety.

It should be stated at this point that the bending resistance of a plate girder is sometimes computed by the general formula involving the moment of inertia of section, $M = \frac{KI}{d}$, d being the distance from the centre of gravity of section, i.e., usually the mid-depth, to the extreme fibre, while K is the greatest permitted intensity of stress in that extreme fibre.

Flange Stresses and Sectional Areas of Flanges.

If each of the preceding bending moments be divided by the effective depth 7.1 ft., the results will be the maximum flange stresses at the different sections considered. Again, if those flange stresses each be divided by 14,000 the quotients will be the net sectional areas in square inches at the sections considered. Those results will be as follows:

$l' =$	10 Feet.	20 Feet.	30 Feet.	40 Feet.
Flange stresses......	256,370 lbs.	428,500 lbs.	518,110 lbs.	565,800 lbs.
Net sections.......	18.3 sq. ins.	30.6 sq. ins.	37 sq. ins.	40.8 sq. ins.

These net sections must be made up of cover-plates and angles in the manner shown in section in Fig. 3. Before making the actual design, the arrangement of rivets connecting the flange angles with the web plate is not precisely determinable. In the present instances a large steel angle with legs 6 ins. long must be used, and it is practically certain that two lines of rivets in each leg should be employed. It does not follow that two rivet-holes should be taken out of each leg to find the net section of these angles, since by staggering the rivet-holes a less amount of material may be wasted. In the case of staggered rivet-holes there are not sufficient experiments on record to determine the degree of staggering of two lines of rivets to make it proper to deduct one hole only for the net section. With the riveting scheme used in this plate girder it is entirely feasible so to arrange the four rows that the net section will properly be obtained by deducting three rivet-holes from each angle, and that procedure will be followed.

The sizes of rivets will obviously vary to some extent, but ¾- and ⅞-inch rivets are commonly used in practice, those diameters belonging to the cold rivet as delivered at the shop. The increased diameter caused by driving the rivet is not recognized in rivet computations. In this instance the rivets will be assumed to be ⅞ inch in diameter, i.e., before being driven. The reamed hole for the rivet in the angle must, however, be taken ⅛ inch larger than the diameter of the rivet. These holes must therefore be taken as 1 inch in diameter, since ⅞-inch rivets are to be used.

The selection of the angles to be used is to some extent a matter of judgment. As this girder is a large one, two 6×6×⅞-in. angles, weighing 33.1 lbs. per linear foot, will be selected for each flange. The gross area of section of each of these angles is 9.73 sq. ins., as will be found by referring to the Cambria Steel Company's tables at the end of this book. In view of what has been stated in the preceding paragraphs, the metal taken out of each 6×6-in. angle is $3 \times .875 = 2.63$ sq. ins. Hence the total net section of the two angles will be

$$\textit{Net section} = 2 \times (9.73 - 2.63) = 14.2 \text{ sq. ins.}$$

This net sectional area of the two angles runs throughout the entire length of the tension flange. By referring back to the total net sectional areas required it will be seen that there must be made up 4.1 sq. ins 10 ft. from the end; 16.4 sq. ins. 20 ft. from the end; 22.8 sq. ins. 30 ft. from the end; and 26.6 sq. ins. at the centre of the span, in excess of the net areas of section of the two angles.

These additional areas can be supplied by resorting to cover-plates. The selection of these cover-plates so as best to make up the areas required is largely a matter of judgment. There are, however, some general considerations always to be observed. No cover-plate should be used thicker than the one underneath it, and as a rule the width of cover-plates should be but little if any less than the width of the two flange angles and the web plate between them, but the cover-plate may have a somewhat greater width than that of the two angles and web plates between. In the present instance the cover-plates certainly should not be less than 12 ins. in width and they will be taken 14 inches wide. As there will be four lines of staggered rivets the net width of plate will be taken as 12 ins.

The first cover-plate, i.e., that lying immediately on the angles, must evidently begin where the net section of the two angles is just enough to afford the total net flange section needed at that point. The second cover-plate must similarly begin where the net section of the two angles added to that of the first cover-plate is just equal to the total flange section required; and so on with all the cover-plates. It is to be remembered, however, that in the end of a cover-plate begun just at the section where its area is required, the rivets might not transfer to it precisely at that point the stress which it ought to carry. Hence it is customary to make a cover-plate from a foot to 18 inches longer at each end than the length required by computation. Bearing all these considerations in mind it will be found that the following series of cover-plates will fulfil all the requirements of computation and of all the practical considerations which have been set forth:

One top cover-plate 14 ins. × $\frac{11}{16}$ ins., 39 ft. long.

One 14-in. × $\frac{3}{4}$-in. plate under the above, 56 ft. long.

One 14-in. × $\frac{3}{4}$-in. thick cover-plate immediately on the flange angles, 68 ft. long.

By referring to Fig. 1 the disposition of these cover-plates will be entirely clear. It will be found that the net section is ample to meet the demands of the flange stresses at all points without wastefulness of metal. Obviously at the ends of the flanges there will be some excess of net section in the angles, just as there will also be found some excess of section at and near the end of each cover-plate, but those conditions are inevitable. In designing and arranging the cover-plates of a plate girder, care should be taken to use plates neither too thick nor too thin. It is seldom necessary to use a plate as thick as one inch and no plate less than $\frac{3}{8}$ inch thick should be used for a railroad plate girder.

The length of cover-plate has been determined by direct reference to computations for the flange stresses, but there are other methods by which those lengths can be computed and two such methods will be given in the next article. There is little if any gain in using such methods, but it may be desirable occasionally to employ them.

Rivets between Web and Flanges and End Stiffeners.

Stresses are given to the flanges of a plate girder through the rivets which bind the flange angles to the web plate. It is necessary therefore to have a sufficient number of rivets through the legs of the flange angles lying against the web plate to carry to the flange metal those flange stresses which have been computed. Each of these rivets will obviously be in double shear, but the bearing capacity of the rivet can be determined only after the thickness of the web plate is known, and it will next be necessary to ascertain that thickness. It is frequently assumed that the web plate is to carry transverse shear only, and that assumption is made in this case. The working intensity of shear in a steel web plate where proper allowance has been made for shock may safely be taken at 7000 lbs. per sq. in., and it is sometimes taken as high as 10 000 lbs. per sq. in. The maximum transverse shear will be at the end of the span, i.e., it will be the maximum reaction. In the present instance it will exist when the first driving-wheel (wheel 2) is at one end or, strictly speaking, at an indefinitely short distance from one end. Assuming that position of the moving load, therefore, the reaction will have the value,

$$\textit{Reaction} = 20 + \frac{70}{80} \times 3 \times 20 + \frac{48}{80} \times 4 \times 13 + \frac{32}{80} \times 10 + \frac{16.5}{80} \times 4 \times 20 = 124{,}200 \text{ lbs.}$$

The reaction due to dead load will be $700 \times 40 = 28{,}000$ lbs. After adding 60 per cent. to the moving-load shear to provide for shock the total end shear will be

$$\textit{End shear} = 124{,}700 + 74{,}820 + 28{,}000 = 227{,}520 \text{ lbs.}$$

Hence

$$\textit{Required sectional area} = \frac{227{,}520}{7000} = 32.5 \text{ sq. ins.}$$

The rivets through the end stiffeners will take away some of the metal of the web plate, possibly as much as 25 per cent. of the section. The gross sectional area of the web plate must therefore be $32.5+12.2=44.7$ sq. ins. The sectional area of a web plate $84\times\frac{9}{16}$ ins. is 47.25 sq. ins. Hence a $\frac{9}{16}$-in. web plate will be assumed.

In determining the number of rivets required to give the flange stresses to the flanges it must be observed that each rivet which passes through the vertical legs of the flange angles and the web plate is subjected to double shear at the two surfaces of the latter, while it bears against the $\frac{9}{16}$-in. web plate. The double shearing value of one $\frac{7}{8}$-in. rivet at 9000 lbs. per sq. in. is 10,820 lbs., whereas the bearing value of each $\frac{7}{8}$-in. rivet against a $\frac{9}{16}$-in. plate, at 18,000 lbs. per sq. in., is 8860 pounds. Hence the bearing value of the rivet against the web plate, being but little more than three quarters of its double shearing value, will necessarily govern the number of rivets needed to secure each flange to the web-plate. If the flange stress at each 10-foot point be divided by 8860, the number of rivets required in each 10-foot section will at once be shown. Hence

$$\text{Rivets in first 10 ft. of flange} = \frac{256{,}370}{8{,}860} = 30.$$

$$\text{" " second " " " "} = \frac{428{,}500-256{,}370}{8{,}860} = 20.$$

$$\text{" " third " " " "} = \frac{518{,}110-428{,}500}{8{,}860} = 10.$$

$$\text{" " fourth " " " "} = \frac{565{,}800-518{,}110}{8{,}860} = 6.$$

The numbers of rivets found above are the least which must be employed in securing the vertical legs of the flange angles to the web plate in each 10-foot section. As a matter of fact, the actual number needed is considerably larger for the reason that the pitch of rivets must not exceed

6 ins. in a single row, or about 9 ins. in either one of double lines, in order that the different parts of the girder may be secured to each other with sufficient rigidity or stiffness. At and near the ends of the flanges the rivets must be pitched closer together than in the intermediate portions so as to give the ends of the girder sufficient solidity, and for the further reason that the flange stresses increase much more rapidly in that portion of the flange than in other portions.

In the present instance, as shown in Fig. 2, there will be two lines of staggered rivets in the vertical legs of the flange angles. For a distance of 3 ft. 7 ins. from the centre of the end stiffener the pitch in each line will be 4 ins., and 6 ins. throughout the remainder of the flanges. It will be found that this arrangement will nearly double the number of rivets required in the first 10 ft. of the flange angle, and more than double the number required in the other 10-foot sections, but the arrangement is no more abundant in rivets than is required for the best results. Obviously, the shorter the sections are taken in making the preceding computations the closer will the variation in the number of rivets correspond to the variation of flange stresses.

When the moving load rests directly upon the upper flanges of the plate girder, as in the case of most deck-girders, the rivets through the vertical legs of the flange angles and the web plate carry the driving-wheel concentrations as well as the flange stresses, the latter acting in a horizontal direction, while the former are vertical. It is uncertain in such a case over what length of flange a single driving-wheel concentration is distributed, but it is reasonable to assume that the distribution is essentially uniform over a distance not less than 2 feet. If that distance be chosen the amount of flange stress near the ends of the girder given to the flange within that same distance may be taken as one fifth of 256,370, or about 51,000 pounds.

The greatest driving-wheel concentration in this case is 20,000 pounds. The rivets in a length of 2 ft. therefore would carry the resultant force, $\sqrt{(51,000)^2+(20,000)^2}$ $=55,000$ pounds. Hence the number of rivets required would be $55,000 \div 8860 = 6$ (nearly). In the double row of rivets in each of which the pitch is 6 ins. there would be 8 rivets within a distance of 2 ft.; hence that number is more than enough for the worst cast of driving-wheel concentration combined with the direct chord stress. In some cases it is essential that this effect of the wheel concentration be recognized.*

Rivets in Cover-plates.

The number of rivets required in the cover-plates must next be determined. Obviously the rivets at and near the end of a cover-plate must be sufficient to give to the latter the total stress which it is designed to carry. The general arrangement of these rivets will be as shown in Fig. 5. There are four rows of staggered rivets, the pitch in each being 6 ins. for the first 18 ins. of the cover-plate and 9 ins. adjacent to that end portion of 18 ins. With these rivet pitches it is reasonable to assume that the net sections of the cover-plate will be found by deducting the metal taken out of 2 rivet-holes only. The net width of the 14-in. cover-plate will therefore be 12 ins. Consequently the total stress carried by each ¾-in. cover-plate will be $12 \times \frac{3}{4} \times 14,000 = 126,000$ lbs. In this detail each rivet will be subjected to single shear and that single shear value is 5410 lbs., much less than the bearing value against the ¾-in. plate. The total number of rivets required for the ¾-in. cover-plates will then be $126,000 \div 5410 = 23$ rivets. By referring to Fig. 5 it will be seen that this number is much more than afforded by the arrangement shown. It is not feasible, however, for the requisite stiffness and solidity of the flanges of the girder to make rivet pitches greater than those shown. Each ¾-in. cover-plate will obviously have the same arrangement of rivets at and near the ends as well as throughout other portions of their lengths. Precisely the same arrangement will also be used for the $\frac{11}{16}$-in. top cover-plate. The single shear value of each rivet will remain as before. As the bearing value at 18,000 lbs. per sq. in. of a ⅞-in. rivet against a $\frac{11}{16}$-in. plate is about 10,800 lbs., it will be at once seen that the single shear value will give the number of rivets required. Hence the arrangement of rivets shown in Fig. 5 will be taken for each of the three cover-plates.

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* Another method for finding pitch of rivets will be found in Appendix II.

Compression Flanges or Chords.

The design of the tension flange being completed, including all its details except such splice-joints as may be needed, the compression flange might be designed in precisely the same manner, except that gross sections would be considered instead of net sections, since all rivet-holes are filled with the driven rivets. The unit working intensity of compressive stress would be somewhat less than the tensile intensity already used. Hence it is usually customary, although not the invariable practice, first to design the tension flange completely and then make the gross section of the compression flange precisely the same. Indeed not only the gross section of the compression flange is frequently made identical with the tension flange, but the dimensions also of its various component parts. The only exception from this procedure arises from the fact that it is advisable to carry the cover-plate immediately adjacent to the flange angles in the compression flange throughout its entire length. In this manner the angles near the ends of the girder are bound more solidly together and the flange as a whole in that vicinity is in better condition to resist compression. These observations acquire added significance in those cases where the ties of the railroad track rest directly upon the upper flange of the girder. The inevitable deflection of the ties tends to bend downward the horizontal leg of one flange angle in each girder, and this tendency is effectively resisted if those angles carry a cover-plate throughout their entire length. It is better practice, therefore, to make the first cover-plate extend the whole length of the compression chord, and that is done in this case, as shown in Fig. 1. A little metal might be saved by reproportioning the angles and cover-plates of this compression chord retaining the gross sections as already found, but the gain would be slight and the cross-sections of both angles and cover-plates will be made the same in both flanges.

For the purpose of a neater finish the lowest cover-plate on the top flange is reduced to a half-inch thickness 18 ins. from the end stiffeners, and curved so as to run down over the latter to the bottom of the girder as shown in Fig. 1 and Fig. 2.

End Stiffeners.

The end stiffeners shown in Fig. 2 immediately over the pedestal at the end of the effective span must be considered as taking the reaction of the girder and its load. The lower ends of the end stiffeners resting closely against the lower flange angles sustain the entire reaction. That reaction, however, is gradually given to the web of the girder in passing to the upper ends of the stiffeners through the rivets securing the latter to the web plate. The number of rivets required, therefore, in an end stiffener must be sufficient to convey the entire reaction to the web plate in the manner indicated. It has already been shown that the maximum reaction at the end of the girder, including 60 per cent. for shock, is 255,520 lbs. It is clear from what has been stated that the cross-section of the end stiffener must be sufficient to sustain the compression produced by the reaction somewhat as in a short column, but obviously with little tendency to flexure. It is reasonable, therefore, to assume under the circumstances outlined that compression in the end stiffeners may reach 10,000 lbs. per sq. in. as the greatest permissible working stress. The sectional area required for the end stiffeners will therefore be

$$255{,}520 \div 10{,}000 = 25.6 \text{ sq. in.}$$

Four $6 \times 4 \times \frac{11}{16}$ steel angles will give an aggregate area of 25.6 sq. ins., which happens to be precisely equal to the area required; hence each end stiffener will be taken as composed of a pair of these steel angles, the 6-inch legs of which must be placed against a $12 \times \frac{7}{8}$-inch filler, as shown in Fig. 2. The $12 \times \frac{7}{8}$-inch filler on each side of the web plate is required in order that the end stiffeners may be used without bending, since the thickness of the lower flange angles is also $\frac{7}{8}$ inch. The upper ends of these angles will be bent or cut to form the curved end of the girder as shown in Figs. 1 and 2, the upper flange angle closing against the $12 \times \frac{7}{8}$-inch filler. The half-inch end cover makes a finish for the end of the girder and turns over the upper ends of the stiffeners so as to run horizontally about 18 inches before abutting against the $\frac{7}{8}$-inch cover-plate.

The rivets piercing the end stiffeners will evidently be in double shear on the two sides of the web plate in which each rivet has the value of 10,800 lbs. The bearing value, however, of each $\frac{7}{8}$-inch rivet on the $\frac{9}{16}$-inch web is 8860 lbs. The number of rivets required, therefore, must be determined by dividing 255,520 by 8860, and that division gives 29 rivets as the number required. This would give less than 15 rivets to each pair of stiffening angles, a far less number than is required to make a satisfactory joint between the end stiffeners and the web plate. The end stiffeners take the immediate effect of all the hammering at the end of the span produced by heavy and rapidly moving loads, and they are required to be held to the plate by a riveted joint especially rigid and strong. Each 6-in. leg, therefore, of the stiffening angles will have two lines of staggered rivets, the pitch in each line being 6 in. As the depth of the girder is 7 ft. this arrangement will call for a total of about 60 rivets passing through the end stiffeners, and it is none too many for the purpose. Fig. 2 shows this arrangement of rivets. The ends of the end stiffeners, and indeed of all the stiffeners, should be machine finished or faced to fit exactly the fillets of the flange angles in the chords or flanges.

Intermediate Stiffeners.

The dimensions and the location of the intermediate stiffeners, one of which is shown in Fig. 2, are chiefly matters of judgment. It has been shown in considering the internal stresses of a beam or girder that in directions at right angles to each other, and at 45 degrees with the neutral axis, two stresses of tension and compression exist, and hence that those compressive stresses tend to buckle the web plate of a plate girder in directions at 45 degrees with the neutral axis. In order to prevent most effectively this tendency to buckle the web plate, the intermediate stiffeners should be placed on that plate also at 45 degrees with the neutral axis. That procedure, however, involves some inconvenience in construction and ample stiffening can be secured by placing the intermediate stiffeners at right angles to the neutral axis or, in the ordinary position of girders, in a vertical direction; hence that arrangement is always followed in plate-girder construction. It is usual to place these intermediate stiffeners at distances apart of once to once and a half the depth of the girder. In large girders like that under discussion, it is probably best to make these intermediate stiffeners, especially when near the ends, not farther apart than the depth of the girder. Inasmuch as the transverse shear increases towards the ends of girder and is relatively small at the centre of the span, the stiffeners might be placed farther apart as the centre of span is approached, and that is sometimes done. Indeed they are frequently omitted altogether at the centre of span. In the present instance there will be four intermediate stiffeners in each half of the girder placed 7 feet apart, as shown in Fig. 1, and there will be no intermediate stiffener in the central 24 feet of girder. These intermediate stiffeners are frequently bent to fit the flange angles and web plates and that procedure is a good one for small girders. For large girders it is preferable to use a filling plate, as shown in Fig. 2. The cross-section of the intermediate stiffener angles is a matter of judgment, and in this case they will be taken as 4 in. × 4 in. × $\frac{1}{2}$ in. steel angles, each with one line of rivets pitched 6 in. apart; also as shown in Fig. 2.

There will be a $4 \times \frac{5}{8}$ inch filler, making it unnecessary to bend the stiffening angles to fit the flanges and web. As has already been stated the ends of the intermediate stiffeners should be machine faced to fit the fillets and horizontal legs of the flange angles. It is obvious that the filling plates under the stiffening angles add materially to the effectiveness of the stiffeners.

Web-plate Joint.

It is impossible to secure single plates large enough for the webs of large plate girders, and it therefore becomes necessary to make joints or splices in the web-plate, and such a joint is shown in Fig. 4. It is feasible to secure steel web plates 84 in. wide and $\frac{9}{16}$ in. thick more than 30 ft. long, but that will not be necessary in this case. The joint in the web-plate will be taken at the fourth intermediate stiffener from the end of the span, which will make a plate about 29 ft. long for each end portion of the girder and about 24 ft. long for the central portion. The chief function of the splicing of the web plate is to resist shear, although it will clearly act to some material extent in resisting flexure. In no case should the splice have less than two rows of rivets on each side of the plate-joint, and in a large girder like that under consideration there should be three. The thickness of the splice-plate should also be substantially more than one half the thickness of the web plate. In the present instance the web plate is $\frac{9}{16}$ inch thick and the splice-plates will be taken $\frac{7}{16}$ inch thick. There will be three rows of staggered rivets on each side of the plate-joint, with rivets pitched 6 in. apart. It is frequently advisable to place the joint at one of the intermediate stiffeners, as has been done in this case. One row of the splice-rivets will therefore be identical with the row of rivets through the intermediate stiffener angle and filling plate. In this case the latter, i.e., the filling plate, will be 4 in. $\times \frac{7}{16}$ in. The entire arrangement of the web-joint with the rivets distributed as indicated is shown in Fig. 4.

Joints in Flanges or Chords.

Although it is usually practicable to secure flange angles 80 ft. long, as an illustration a splice will be made at some intermediate point. This point might be taken at the centre of span and it would have the advantage of making the angle-bars of the same length, but that is not important. The maximum sectional area is required at the centre of span, and if the splice-joint be placed 9 feet from the centre there will be the material advantage of extra sectional area where it is needed for the splice. The greatest length of angle will then be but 50 ft., the length of the other pieces being 32 ft. Similarly, the longest cover-plates run throughout the entire length of the girder, and although they can be secured in one length the method of splicing them will be shown. It will usually be advisable to place the angle and cover-plate splices at different points, but as there is excess of sectional area needed at the point taken in this case, and as there is but one cover-plate to be spliced, one joint will be made to answer for both flange angles and cover. Both chords have the same splice-joint, but those joints will be placed on opposite sides of the centre of span. The net section of one of the flange angles has been seen to be 7.1 sq. ins. or 14.2 sq. ins. for the two. At 14,000 lbs. per sq. in. the total stress carried by the tension flange angles will be 198,800 lbs. The rivets used at the splice will be in double shear, each having a value of 10,800 lbs. As this double-shear value is much less than the bearing value of one rivet upon the two thicknesses of ⅞-in. angle leg, the double-shear value will control the number of rivets required. The latter number is equal to 198,800 ÷ 10,800 = 19. The splicing of each ⅞-in. angle will be done by a single piece of 6×6 angle 1 inch thick with its corner fitted to the fillet of the flange angle, thus giving a net section a little larger than that of the flange angle itself. The arrangement of the rivets will be as shown in Fig. 6, in which it is seen that there are 5 rivets in each staggered row, making 20 on either side of the joint, a little more than fulfilling the requirements of the case.

The splicing of the longest ¾-in. cover-plate is done by the joint-plate on the outside of the flange. The net section of the ¾-in. cover-plate is $12 \times \frac{3}{4} = 9$ sq. ins. Hence $9 \times 14{,}000 = 126{,}000$ lbs. is the amount of stress which it carries. The single-shear value of one rivet is 5410 lbs., and as that is less than its bearing value against the ¾-in. plate, the number of rivets required will be $126{,}000 \div 5410 = 23$ rivets. In order to give a little excess of section the splice-plate will be taken ⅞ in. thick and there will be placed six rivets in each staggered row on either side of the cover-joint. As the rivet pitch is 6 ins. the total length of splice-plate will be about 48 ins. The position and arrangement of this splice-plate and its rivets is fully shown in Fig. 6.

Other tension joints at other points in the girder or in any girder may be formed in precisely the same manner, whether it is designed to splice the flange angles or the cover-plates. Inasmuch as there is some extra section at the point taken for this joint it would be sufficient to make the splice-plate but ¾ in. thick. A little greater thickness has been taken, however, in order to illustrate the general principle of procedure that a joint should have a little greater section of metal and show a little calculated excess of resistance over that of the main section of the flange of the member spliced.

By placing a flange-joint at the end of a cover-plate, the extension of that plate may be used for the splice, thus avoiding a separate splice- or joint-plate.

It will be supposed that the corresponding splice in the compression flange is made precisely the same as in the tension flange, although the ends of the flange angles and spliced cover-plates, being machine-finished, as they should be, may abut. The process of riveting may draw the abutting ends apart to a slight degree and thus render it advisable to make the resisting capacity of the rivets of the joint the same as in the tension flange.

General Considerations.

The girder proper with its flanges, web, and stiffeners has been designed in this article without indicating the manner of connecting such lateral or cross bracing as would be required in the complete design of a railroad plate-girder span. The design of such bracing would be supplementary to the actual design of the girder as made, and it is the purpose here to illustrate only those principles belonging to the design of the girder proper. The design of the bracing and the details of its connection with the girder belong rather to bridge construction than to the subject treated.

It is to be remembered that large plate girders are not always built complete in the shop, although girders 80 ft. in length are frequently and perhaps usually so completed at the present time. When it is necessary to build them in portions and rivet the portions together in the field, the general principles governing the construction of the necessary field-joints are precisely the same as those illustrated in this article. They are simply adjusted or adapted to the exigencies of each particular case.

It is sometimes required in specifications that there shall be sufficient rivets in the web plate along a length of flange adjacent to the end of the span and equal to the depth of girder to take the entire end shear as flange stress. That end shear has been seen to be 255,520 pounds. The bearing value of a $\frac{7}{8}$-in. rivet on a $\frac{9}{16}$-in. web plate is 8860 pounds. Hence the number of rivets required is $255,520 \div 8860 = 29$. As actually designed, the first 7 feet of flange has about 33 rivets in the vertical legs of the flange angles. This requirement therefore is more than fulfilled.

The bill of material and estimated weight of a single girder as designed is as follows, the bill being written as if the ends of the girder were square:

Two $6\times6\times\frac{7}{8}$ in. angles, 33.1 lbs. per lin. ft. (upper flange), 79 ft. long	$158\times$	33.1	=	5,230 lbs.
Two $6\times6\times\frac{7}{8}$ in. angles, 33.1 lbs. per lin. ft. (lower flange), 81 ft. long	$162\times$	33.1	=	5,362 "
One $14\times\frac{3}{4}$ in. cover-plate, 76 ft. long	$76\times$	35.7	=	2,713 "
One $14\times\frac{3}{4}$ in. cover-plate, 68 ft. long	$68\times$	35.7	=	2,428 "
Two $14\times\frac{3}{4}$ in. cover plates, 56 ft. long	$112\times$	35.7	=	3,998 "
Two $14\times\frac{11}{16}$ in. cover-plates, 39 ft. long	$78\times$	32.72	=	2,552 "
One $84\times\frac{9}{16}$ in. web plate, 81 ft. long	$81\times$	160.65	=	13,013 "
Eight $6\times4\times\frac{11}{16}$ in. angles (end stiffeners), 7 ft. long	$56\times$	21.8	=	1,221 "
Four $12\times\frac{7}{8}$ in. end fillers, 7 ft. long	$28\times$	35.7	=	1,000 "
Sixteen $4\times4\times\frac{1}{2}$ in. angles (intermediate stiffeners), 7 ft. long	$112\times$	12.8	=	1,444 "
Twelve $4\times\frac{7}{8}$ in. fillers, 6 ft. long	$72\times$	11.9	=	857 "
Four $4\times\frac{7}{16}$ in. fillers, 6 ft. long	$24\times$	5.95	=	144 "
Four $18\times\frac{7}{16}$ in. splice-plates, 6 ft. long	$24\times$	26.78	=	643 "
Four $6\times6\times1$ in. angles (splices), 5.5 ft. long	$22\times$	37	=	814 "
Two $14\times\frac{7}{8}$ in. cover splice-plates, 6 ft. long	$12\times$	41.65	=	500 "
Two $14\times\frac{1}{2}$ in. end covers, 9.5 ft. long	$19\times$	23.8	=	452 "
2800 rivets			=	1,680 "
Total			=	44,051 "

The weight of girder per linear foot of span will therefore be $44,051 \div 80 = 551$ lbs. The assumed weight of 700 lbs. per linear foot is a little scant for a half-track weight taken at 200 lbs. per linear foot, as the latter would have to be added to the weight of the steel. The revision, however, would not add materially to the weight of the girder.

Art. 99.—Length of Cover-plates.

There are various methods of determining the lengths of cover-plates of plate girders involving simple computations only, which are well illustrated by the following procedures:

The first of these procedures is based on the assumption that the depth of the girder is uniform and that the bending moment throughout the length of girder varies as the ordinate of a parabola as in the case of uniform loading. The following notation is required:

l = effective length of span either in feet or inches;
L = length of cover-plate required in the same unit as l;
A = total net flange area;
a = net cover-plate area required.

Since the flange and cover-plate areas vary directly as the flange stresses, and as the latter vary as the ordinates of a parabola when the depth of girder is constant, the following equation will result:

$$\frac{L^2}{l^2}=\frac{a}{A},$$

or

$$L=l\sqrt{\frac{a}{A}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Eq. (1) will give the length of the cover-plate whose area of section is a. Any convenient unit may be taken for a and A, but the square inch is ordinarily employed.

If there are several cover-plates, a is to be taken successively the area of the first, second, third, etc., cover-plates in summation, i.e., it will first be taken as the net sectional area of the top cover, then as the net sectional area of the top cover added to that of the cover-plate below it, and so on.

The second method is the following, and is applicable to the case of a girder with varying depth, the notation being as follows:

Let w = uniform load per linear foot, or "equivalent uniform load" per linear foot;

d and d' represent the effective depths of girder in feet at the centre of span and at the end of the cover-plate respectively;

$A-a=a'$ = area of flange section at the end of cover-plate;

T = permissible flange stress per square inch;

the bending moment at the end of the cover-plate will then be

$$M=w\frac{l^2}{8}-\frac{w}{2}\left(\frac{L}{2}\right)^2=AdT-w\frac{L^2}{8}=d'a'T. \quad . \quad . \quad (2)$$

By solving the second and third members of the preceding equation there will result

$$L=2\sqrt{2}\sqrt{\frac{(Ad-a'd')T}{w}}=2.83\sqrt{\frac{(Ad-a'd')T}{w}}. \quad . \quad . \quad (3)$$

It must be remembered that the application of either of the two preceding methods will give the net length of the cover-plate. There must be added 12 to 18 ins. at each end with rivets closely pitched so that the cover-plate may certainly take its stress at the points where its effectiveness should begin.

CHAPTER XV.

ROPES AND CHAIN CABLES.

Art. 100.—Iron, Steel, and Hemp Cables or Ropes—Wrought-iron Chain Cables.

THE following tables of resistance and other properties of cables are those published by John A. Roebling's Sons Co.

It will be observed that the figures for hemp ropes are given in comparison with either iron or steel in each of the tables.

In considering the resistance of iron and steel cables composed of wire twisted into strands, it is of the highest importance to keep clearly in view the circumstances or conditions produced by the manner of fabrication, as they are peculiar to all classes of ropes, whether of hemp or wire.

In this class of material the fibres or strands no longer lie parallel to the direction of the stress which they carry, but the process of twisting causes each fibre or wire to take a helical form, the pitch of which is not constant for the different portions of the rope. The consequence is that if the process of fabrication were absolutely perfect, so that each wire or fibre could take its proper portion of load, the stress in that wire or fibre would be its portion of load multiplied by the secant of its inclination to the axis of the rope. As a matter of fact, however, each wire does not take its proper portion of load; the imperfections unavoidably incident to the processes of manufacture render such a result impossible. Hence the increased necessity of experimental determination of the ultimate resistances of metallic and hemp ropes.

The same composite character of these productions renders anything like an approximately elastic character, even, an essential impossibility. It is true that any rope will yield to a considerable extent while under stress, and then return nearly to its original condition; but this behavior is only apparently elastic; it is almost entirely due to the increase of helical pitch of the strands caused by the external loading. During this operation the strands endeavor to place themselves more nearly parallel to the direction of stress, and give rise to a corresponding decrease in diameter. Since these influences preclude the existence of either coefficient of elasticity or elastic limit ultimate resistances only will be given in this section.

The preceding observations evidently do not apply to suspension-bridge cables which are built up of parallel wires. The operations leading to the production of such a cable are of such a refined and exact character that the total resistance of the cable may be assumed without essential error to be the sum of the resistances of all the wires taken separately; the coefficient of elasticity and elastic limit may, and usually do, exist with perfect definition.

The ultimate resistance of the Swedish iron wire used in the standard hoisting rope of Table I may run from 50,000 to 100,000 lbs. per sq. in., while the cast (crucible) steel wire of the same table may be 150,000 to 190,000 lbs. per sq. in. The latter values also apply to the crucible-cast steel wire of Table II. The plough-steel wire of Table III may have an ultimate resistance of 190,000 to 350,000 lbs. per sq. in. The plough-steel rope, therefore, is seen to have much the highest carrying capacity.

TABLE I.

STANDARD HOISTING ROPE.

(Composed of 6 strands and a hemp centre, 19 wires to the strand.)

SWEDISH IRON.

Trade Number.	Diameter in Inches.	Approximate Circumference in Inches.	Weight per Foot in Pounds.	Approximate Breaking Strain in Tons of 2000 Lbs.	Allowable Working Strain in Tons of 2000 Lbs.	Minimum Size of Drum or Sheave in Feet.
—	2¾	8⅝	11.95	114	22.8	16
—	2½	7⅞	9.85	95	18.9	15
1	2¼	7⅛	8.00	78	15.60	13
2	2	6¼	6.30	62	12.40	12
3	1¾	5½	4.85	48	9.60	10
4	1⅝	5	4.15	42	8.40	8½
5	1½	4¾	3.55	36	7.20	7½
5½	1⅜	4¼	3.00	31	6.20	7
6	1¼	4	2.45	25	5.00	6½
7	1⅛	3½	2.00	21	4.20	6
8	1	3	1.58	17	3.40	5¼
9	⅞	2¾	1.20	13	2.60	4½
10	¾	2¼	0.89	9.7	1.94	4
10¼	⅝	2	0.62	6.8	1.36	3½
10½	9/16	1¾	0.50	5.5	1.10	2¾
10¾	½	1½	0.39	4.4	0.88	2¼
10*a*	7/16	1¼	0.30	3.4	0.68	2
10*b*	⅜	1⅛	0.22	2.5	0.50	1½
10*c*	5/16	1	0.15	1.7	0.34	1
10*d*	¼	¾	0.10	1.2	0.24	¾

CAST STEEL.

Trade Number.	Diameter in Inches.	Approximate Circumference in Inches.	Weight per Foot in Pounds.	Approximate Breaking Strain in Tons of 2000 Lbs.	Allowable Working Strain in Tons of 2000 Lbs.	Minimum Size of Drum or Sheave in Feet.
—	2¾	8⅝	11.95	228	45.6	10
—	2½	7⅞	9.85	190	37.9	9½
1	2¼	7⅛	8.00	156	31.2	8½
2	2	6¼	6.30	124	24.8	8
3	1¾	5½	4.85	96	19.2	7¼
4	1⅝	5	4.15	84	16.8	6¼
5	1½	4¾	3.55	72	14.4	5¾
5½	1⅜	4¼	3.00	62	12.4	5½
6	1¼	4	2.45	50	10.0	5
7	1⅛	3½	2.00	42	8.40	4½
8	1	3	1.58	34	6.80	4
9	⅞	2¾	1.20	26	5.20	3½
10	¾	2¼	0.89	19.4	3.88	3
10¼	⅝	2	0.62	13.6	2.72	2¼
10½	9/16	1¾	0.50	11.0	2.20	1¾
10¾	½	1½	0.39	8.8	1.76	1½
10*a*	7/16	1¼	0.30	6.8	1.36	1¼
10*b*	⅜	1⅛	0.22	5.0	1.00	1
10*c*	5/16	1	0.15	3.4	0.68	¾
10*d*	¼	¾	0.10	2.4	0.48	½

TABLE II.

EXTRA-STRONG CRUCIBLE CAST-STEEL ROPE.

(Composed of 6 strands and a hemp centre, 19 wires to the strand.)

Trade Number.	Diameter in Inches.	Approximate Circumference in Inches.	Weight per Foot in Pounds.	Approximate Breaking Strain in Tons of 2000 Lbs.	Allowable Working Strain in Tons of 2000 Lbs.	Minimum Size of Drum or Sheave in Feet.
—	2¾	8⅝	11.95	266	53	10
—	2½	7⅞	9.85	222	45	9½
1	2¼	7⅛	8.00	182	36.4	8½
2	2	6¼	6.30	144	28.8	8
3	1¾	5½	4.85	112	22.4	7¼
4	1⅝	5	4.15	97	19.4	6¼
5	1½	4¾	3.55	84	16.8	5¾
5½	1⅜	4¼	3.00	72	14.4	5½
6	1¼	4	2.45	58	11.6	5
7	1⅛	3½	2.00	49	9.80	4½
8	1	3	1.58	39	7.80	4
9	⅞	2¾	1.20	30	6.00	3½
10	¾	2¼	0.89	22	4.40	3
10¼	⅝	2	0.62	15.8	3.16	2¼
10½	9/16	1¾	0.50	12.7	2.54	1¾
10¾	½	1½	0.39	10.1	2.02	1½
10*a*	7/16	1¼	0.30	7.8	1.56	1¼
10*b*	⅜	1⅛	0.22	5.78	1.15	1
10*c*	5/16	1	0.15	4.05	0.81	⅔
10*d*	¼	¾	0.10	2.70	0.54	½

(7 wires to the strand.)

Trade Number.	Diameter in Inches.	Approximate Circumference in Inches.	Weight per Foot in Pounds.	Approximate Breaking Strain in Tons of 2000 Lbs.	Allowable Working Strain in Tons of 2000 Lbs.	Minimum Size of Drum or Sheave in Feet.
11	1½	4¾	3.55	79	15.8	8½
12	1⅜	4¼	3.00	68	13.6	8
13	1¼	4	2.45	56	11.2	7¼
14	1⅛	3½	2.00	46	9.20	6¼
15	1	3	1.58	37	7.40	5¾
16	⅞	2¾	1.20	28	5.60	5
17	¾	2¼	0.89	21	4.20	4½
18	11/16	2⅛	0.75	18.4	3.68	4
19	⅝	2	0.62	15.1	3.02	3½
20	9/16	1¾	0.50	12.3	2.46	3
21	½	1½	0.39	9.70	1.94	2½.
22	7/16	1¼	0.30	7.50	1.50	2¼
23	⅜	1⅛	0.22	5.58	1.11	2
24	5/16	1	0.15	3.88	0.77	1¾
25	9/32	⅞	0.125	3.22	0.64	1½

TABLE III.

PLOUGH-STEEL ROPE.

(Composed of 6 strands and a hemp centre, 19 wires to the strand.)

Trade Number.	Diameter in Inches.	Approximate Circumference in Inches.	Weight per Foot in Pounds.	Approximate Breaking Strain in Tons of 2000 Lbs	Allowable Working Strain in Tons of 2000 Lbs.	Minimum Size of Drum or Sheave in Feet.
—	2¾	8⅝	11.95	305	61.0	11
—	2½	7⅞	9.85	254	50.8	10
1	2¼	7⅛	8.00	208	41.6	9
2	2	6¼	6.30	165	33.0	8
3	1¾	5½	4.85	128	25.6	7½
4	1⅝	5	4.15	111	22.2	6
5	1½	4¾	3.55	96	19.2	5½
5½	1⅜	4¼	3.00	82	16.4	5¼
6	1¼	4	2.45	67	13.4	5
7	1⅛	3½	2.00	56	11.2	4½
8	1	3	1.58	44	8.80	4½
9	⅞	2¾	1.20	34	6.80	3¾
10	¾	2¼	0.89	25	5.00	3½
10¼	⅝	2	0.62	18	3.60	3
10½	9/16	1¾	0.50	14.5	2.90	2½
10¾	½	1½	0.39	11.4	2.28	2
10*a*	7/16	1¼	0.30	8.85	1.77	1½
10*b*	⅜	1⅛	0.22	6.55	1.31	1
10*c*	5/16	1	0.15	4.50	0.90	⅞
10*d*	¼	¾	0.10	3.00	0.60	¾
(7 wires to the strand.)						
11	1½	4¾	3.55	91	18.2	8½
12	1⅜	4¼	3.00	78	15.6	8
13	1¼	4	2.45	64	12.8	7¼
14	1⅛	3½	2.00	53	10.6	6¼
15	1	3	1.58	42	8.40	5½
16	⅞	2¾	1.20	32	6.40	5
17	¾	2¼	0.89	24	4.80	4
18	11/16	2⅛	0.75	21	4.20	3½
19	⅝	2	0.62	17	3.40	3
20	9/16	1¾	0.50	14	2.80	2¾
21	½	1½	0.39	11	2.20	2½
22	7/16	1¼	0.30	8.55	1.71	2
23	⅜	1⅛	0.22	6.35	1.27	1½
24	5/16	1	0.15	4.35	0.87	1¼
25	9/32	⅞	0.125	3.65	0.73	1

All the ropes of the Tables I, II, and III have tarred hemp centres, the latter affording a cushioning form for the surrounding steel strands; at the same time it also lubricates the strands. If the hemp be replaced by a steel strand centre the resistance of the rope will be increased 7 to 10 per cent., but it will be less flexible.

All the allowable working resistances in these tables are one fifth of the ultimate resistances or breaking loads of the ropes. For smaller drums than those prescribed in the tables or for relatively high speeds, one eighth or one tenth of the ultimate resistance is used for the working loads. The high resisting plough-steel rope is used where it is necessary to reduce the weight of the cable, as when it is so long as to make its own weight a material element of its loading.

The average ultimate resistance of the best grades of steel ropes is scarcely more than eight tenths of the aggregate ultimate resistance of the wires of which it is composed if tested singly.

Wrought-iron Chain Cables.

It might at first sight be supposed that the pull which the link of a chain cable could resist would be twice that offered by a bar of round iron equal in cross-section to that of one side of the link. But a weld exists at one end of the link and a bend at the other, each requiring at least one heat for the portion of the link in which it is located. These manipulations produce a considerable decrease in the resistance of the link.

The United States Committee on "Tests of Chain Cables," of which Commander L. A. Beardsley was chairman, made many experiments on the iron of which chain cables are made, as well as on the finished cables.

The following conclusions and table are taken from the report of that committee: ". . . that beyond doubt, when made of American bar iron, with cast-iron studs, the studded link is inferior in strength to the unstudded one.

ULTIMATE RESISTANCE AND PROOF TESTS OF CHAIN CABLES.

Diameter of Bar.	Average Resistance = 163 Per Cent. of Bar.	Proof Test.	Diameter of Bar.	Average Resistance = 163 Per Cent. of Bar.	Proof Test.
Inches.	Pounds.	Pounds.	Inches.	Pounds.	Pounds.
1	71,172	33,840	$1\frac{9}{16}$	162,283	77,159
$1\frac{1}{16}$	79,544	37,820	$1\frac{5}{8}$	174,475	82,956
$1\frac{1}{8}$	88,445	42,053	$1\frac{11}{16}$	187,075	88,947
$1\frac{3}{16}$	97,731	46,468	$\frac{3}{4}$	200,074	95,128
$1\frac{1}{4}$	107,440	51,084	$1\frac{13}{16}$	213,475	101,499
$1\frac{5}{16}$	117,577	55,903	$1\frac{7}{8}$	227,271	108,058
$1\frac{3}{8}$	128,129	60,920	$1\frac{15}{16}$	241,463	114,806
$1\frac{7}{16}$	139,103	66,138	2	256,040	121,737
$1\frac{1}{2}$	150,485	71,550			

"That, when proper care is exercised in the selection of material, a variation of five to seventeen per cent. of the strongest may be expected in the resistance of cables. Without this care the variation may rise to twenty-five per cent.

"That with proper material and construction the ultimate resistance of the chain may be expected to vary from 155 to 170 per cent. of that of the bar used in making the links, and show an average of about 163 per cent.

"That the proof test of a chain cable should be about 50 per cent. of the ultimate resistance of the weakest link."

The decrease of the resistance of the studded below the unstudded cable is probably due to the fact that in the former the sides of the link do not remain parallel to each other up to failure, as they do in the latter. The result is an increase of stress in the studded link over the unstudded in the proportion of unity to the secant of half the inclination of the sides of the former to each other.

From a great number of tests of bars and finished cables, the committee considered that the average ultimate resistance and proof tests of chain cables made of the bars, whose diameters are given, should be such as are shown in the accompanying table.

Manila and Hemp Rope.

The results given below were obtained in the Govt. machine at Watertown, Mass., and are found in "Ex. Doc. No. 35, 49th Congress, 1st Session." "The rope was secured in the testing machine either with a hitch made by taking a round turn over a 4½-inch pin held from turning and then making fast the end to a smaller pin or by passing the ends over 3½-inch thimbles and securing to the standing part by means of seizing."

Kind.	Circumference, Inches.	Fastening.	Elongation just before Failure, Per Cent.	Breaking Load in Pounds.		
				Greatest.	Mean.	Least.
Manila.......	3¼	*Hitch*	———	9,600	8,620	8,250
"	3¼	*Seizing*	14.6 to 18.1	10,820	10,015	9,050
"	3½	*Hitch*	———	11,400	10,650	10,050
"	3½	*Seizing*	14.1 to 17.2	12,150	10,810	9,500
"	3¾	*Hitch*	———	12,200	11,350	10,550
"	3¾	*Seizing*	12.2 to 17.0	11,900	11,020	9,400
Hemp........	3¼	*Hitch*	———	7,950	6,990	6,000
"	3¼	*Seizing*	8.3 to 10.9	12,900	11,350	9,850
"	3½	*Hitch*	———	10,100	8,315	6,500
"	3½	*Seizing*	8.1 to 13.0	12,480	10,700	8,050
"	3¾	*Hitch*	———	10,500	8,535	7,300
"	3¾	*Seizing*	8.1 to 11.4	14,200	12,900	9,350

The rope used in these tests had never been in service, but was ten years old, having been made at the Boston navy-yard and kept in store that length of time at the Washington navy-yard. All the manila rope had 3 strands, but of the hemp about half had 3 strands and the other half 4.

CHAPTER XVI.

MISCELLANEOUS PROBLEMS.

Art. 101.—Resistance of Flues to Collapse.

If a circular tube or flue be subjected to external normal pressure, such as that of steam or water, the material of which it is made will be subjected to compression around the tube, in a plane normal to its axis. If the following notation be adopted,

l = length of tube;
d = diameter of tube;
t = thickness of wall of the tube;
p = intensity of excess of external pressure over internal;

then will any longitudinal section lt, of one side of the tube, be subjected to the pressure $\frac{pld}{2}$. But let a unit only of length of tube be considered. This portion of the tube is approximately in the condition of a column whose length and cross-section, respectively, are πd and t.

The ultimate resistance of such a column is (Art. 24)

$$P = \frac{\pi^2 EI}{\pi^2 d^2}.$$

As this ideal column is of rectangular section,

$$I = \frac{t^3}{12},$$

and

$$P = \frac{Et^3}{12d^2}.$$

But $P = pd$, hence

$$p = \frac{Et^3}{12d^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

is the greatest intensity of external pressure which the tube can carry. But the formulæ of Art. 24 are not strictly applicable to this ideal column. The curvature on the one hand and the pressure on the other tend to keep it in position long after it would fail as a column without lateral support. *Hence p will vary inversely as some power of d much less than the third.*

Again, it is clear that a very long tube will be much more apt to collapse at its middle portion than a short one, as the latter will derive more support from the end attachments; and this result has been established by many experiments. Hence p must be considered as some inverse function of the length l.

Eq. (1), therefore, can only be taken as typical in form, and as showing in a general way, only, how the variable elements enter the value of p. If x, y, and z, therefore, are variable exponents to be determined by experiment, there may be written

$$p = c\frac{t^x}{l^y d^z}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

in which c is an empirical coefficient.

Sir Wm. Fairbairn ("Useful Information for Engineers, Second Series") made many experiments on wrought-iron tubes with lap- and butt-joints single riveted. He inferred from his tests that $y=z=1$. Two different experiments would then give

$$pld=ct^x, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$p'l'd'=ct'^x. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Hence

$$\log (pld)=\log c+x \log t,$$
$$\log (p'l'd')=\log c+x \log t';$$

in which "log" means "logarithm." Subtracting one of these last equations from the other, the value of x becomes

$$x=\frac{\log (pld)-\log (p'l'd')}{\log t-\log t'}=\frac{\log \left(\frac{pld}{p'l'd'}\right)}{\log \left(\frac{t}{t'}\right)}. \quad . \quad . \quad . \quad (5)$$

As p, l, d, t, p', l', d', and t' are known numerical quantities in every pair of tests, x can at once be computed by eq. (5); c then immediately results from either eq. (3) or eq. (4). By the application of these equations to his experimental data, Fairbairn found for wrought-iron tubes:

$$p=9,675,600\frac{t^{2.19}}{ld}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

in which p is in pounds per square inch, while t, l, and d are in inches. *Eq.* (6) *is only to be applied to lengths between* 18 *and* 120 *inches.*

He also found that the following formula gave results agreeing more nearly with those of experiment, though it is less simple:

$$p=9,675,600\frac{t^{2.19}}{ld}-0.002\frac{d}{t}. \quad . \quad . \quad . \quad . \quad (7)$$

Fairbairn found that by encircling the tubes with stiff rings he increased their resistance to collapse. *In cases where such rings exist, it is only necessary to take for l the distance between two adjacent ones.*

In 1875 Prof. Unwin, who was Fairbairn's assistant in his experimental work, established formulæ with other exponents and coefficients ("Proc. Inst. of Civ. Engrs.," Vol. XLVI). He considered x, y, and z variable, and found *for tubes with a longitudinal lap-joint:*

$$p = 7{,}363{,}000\frac{t^{2.1}}{l^{0.9}d^{1.16}}. \quad . \quad . \quad . \quad . \quad . \quad (8)$$

From *one tube with a longitudinal butt-joint*, he deduced:

$$p = 9{,}614{,}000\frac{t^{2.21}}{l^{0.9}d^{1.16}}. \quad . \quad . \quad . \quad . \quad . \quad (9)$$

For *five tubes with longitudinal and circumferential joints*, he found:

$$p = 15{,}547{,}000\frac{t^{2.35}}{l^{0.9}d^{1.16}}. \quad . \quad . \quad . \quad . \quad (10)$$

By using these same experiments of Fairbairn, other writers have deduced other formulæ, which, however, are of the same general form as those given above. It is probable that the following, which was deduced by J. W. Nystrom, will give more satisfactory results than any other:

$$p = 692{,}800\frac{t^2}{d\sqrt{l}}. \quad . \quad . \quad . \quad . \quad . \quad (11)$$

At the same time, it has the great merit of more simple application.

From one experiment on an elliptical tube, by Fairbairn, it would appear that the formulæ just given can be approximately applied to such tubes by substituting for *d* twice the radius of curvature of the elliptical section at either extremity of the smaller axis. If the greater diameter or axis of the ellipse is *a* and the less *b*, then, for *d*, there is to be substituted $\frac{a^2}{b}$.

Art. 102.—Approximate Treatment of Solid Metallic Rollers.

An approximate expression for the resistance of a roller may easily be written. The approximation may be considered a loose one, but it furnishes a basis for an accurate empirical formula.

The following investigation contains the improvements by Prof. J. B. Johnson and Prof. H. T. Eddy on the method originally given by the author.

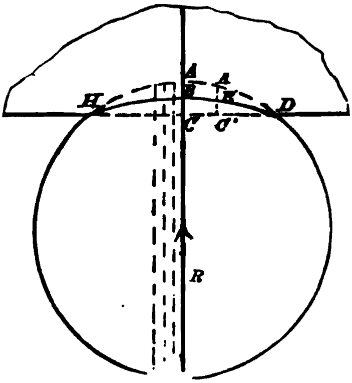

FIG 1

The roller will be assumed to be composed of indefinitely thin vertical slices parallel to its axis. It will also be assumed that the layers or slices act independently of each other.

Let E' be the coefficient of elasticity of the metal over the roller.

Let E be the coefficient of elasticity of the metal of the roller.

Let R be the radius of the roller and R' the thickness of the metal above it.

Let w = intensity of pressure at A;

p = " " " " any other point;

Let P = total weight which the roller sustains per unit of length.

x be measured horizontally from A as the origin;

$d = AC$;

$e = DC$.

From Fig. 1:

$$AB = \frac{wR}{E}; \quad A'B' = \frac{pR}{E}.$$

$$BC = \frac{wR'}{E'}; \quad C'B' = \frac{pR'}{E'}.$$

$$\therefore\ d = AC = AB + BC = w\left(\frac{R}{E} + \frac{R'}{E'}\right) \quad . \quad . \quad . \quad (1)$$

and

$$A'C' = A'B' + B'C' = p\left(\frac{R}{E} + \frac{R'}{E'}\right). \quad . \quad . \quad . \quad (2)$$

Dividing eq. (2) by eq. (1),

$$p = A'C'\frac{w}{d}.$$

But

$$P = \int_{-e}^{+e} p dx = \frac{w}{d}\int_{-e}^{+e} A'C'dx.$$

If the curve DAH be assumed to be a parabola, as may be done without essential error, there will result:

$$\int_{-e}^{+e} A'C'dx = \frac{4}{3}ed.$$

Hence

$$P = \frac{4}{3}we. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

But

$$e = \sqrt{2Rd - d^2} = \sqrt{2Rd}, \text{ nearly.}$$

By inserting the value of d from eq. (1) in the value of e, just determined, then placing the result in eq. (3),

$$P = \frac{4}{3}\sqrt{2w^3 R\left(\frac{R}{E} + \frac{R'}{E'}\right)}. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

If $R = R'$,

$$P = \frac{4}{3}R\sqrt{2w^3 \frac{E+E'}{EE'}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The preceding expressions are for one unit of length. If the length of the roller is l, its total resistance is

$$P' = Pl = \frac{4}{3}l\sqrt{2w^3 R\left(\frac{R}{E} + \frac{R'}{E'}\right)}. \quad . \quad . \quad . \quad . \quad (6)$$

Or if $R = R'$,

$$P' = \frac{4}{3}Rl\sqrt{2w^3 \frac{E+E'}{EE'}}. \quad . \quad . \quad . \quad . \quad . \quad (7)$$

In ordinary bridge practice eq. (7) is sufficiently near for all cases.

A simple expression for conical rollers may be obtained by using eqs. (4) or (5).

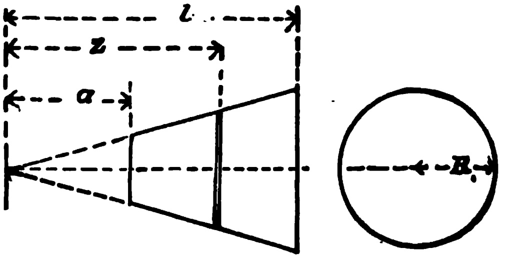

FIG 2

As shown in Fig. 2, let z be the distance, parallel to the axis, of any section from the apex of the cone; then consider a portion of the conical roller whose length is dz. Let R_1 be the radius of the base. The radius of the section under consideration will then be

$$R = \frac{z}{l}R_1,$$

and the weight it will sustain, if $R_1 = R'$,

$$dP' = \frac{R_1}{l}\sqrt{2w^3\frac{E+E'}{EE'}}\,.\,zdz.$$

Hence

$$P' = \int_a^l dP' = \frac{l^2-a^2}{2l}R_1\sqrt{2w^3\frac{E+E'}{EE'}}. \quad . \quad . \quad . \qquad (8)$$

Eqs. (6), (7), and (8) give ultimate resistances if w is the ultimate intensity of resistance for the roller.

It is to be observed that the main assumptions on which the investigation is based lead to an error on the side of safety.

If for wrought iron, $w = 12,000$ pounds per square inch, and $E = E' = 28,000,000$ pounds, eq. (5) gives

$$P = \frac{8}{3}R\sqrt{\frac{w^3}{E}} = 664R.$$

Art. 103.—Resistance to Driving and Drawing Spikes.

Some very interesting experiments on driving and drawing rail spikes were made by Mr. A. M. Wellington, C.E., and reported by him in the "R. R. Gazette," Dec. 17, 1880. He experimented with wood both in the natural state and after it had been treated by the Thilmeny (sulphate of baryta) preserving process.

"The test-blocks were reduced to a uniform thickness of 4.5 inches, this thickness being just sufficient to give a full bearing surface to the parallel sides of the spikes when driven to the usual depth, and to allow the point of the spike to project outwards. It was considered that the bevelled point could add very little to the holding power of the spike, and it was desired to press the spike out again by direct pressure after turning the block over. . . ."

TABLE. I.

SPIKES WERE STANDARD: 5.5 INCHES × $\frac{9}{16}$ INCH.

Kind of Wood.	Natural Wood.				Prepared Wood.			
	To Driving Spike, Pounds.		To Pulling Spike, Pounds.		To Driving Spike, Pounds.		To Pulling Spike, Pounds.	
		Mean.		Mean.		Mean.		Mean.
Beech	5,216 6,743	5,980	5,673 6,282	5,978	7,288 7,656	7,472	8,873 8,267	8,420
White oak, green	5,970 5,670	5,820	7,179 5,869	6,523	—		—	
Pin oak	5,216 5,521	5,368	6,638 6,469	6,553	6,117 4,589	5,353	6,135 6,267	6,201
White ash		5,953		4,560	6,588 5,978	6,283	(Split)	
White oak, well seasoned	6,433 6,433	6,433	5,128 3,435	4,281	—		—	
Black ash	3,996 4,202	4,090	4,408 4,868	4,638	4,453 4,453 4,301 3,380	4,147	3,340 3,028 3,300 3,493	3,290
Elm	4,453 4,758	4,606	3,536 3,843	3,690	4,453 4,148	4,300	4,148 4,202	4,175
Chestnut, green	3,996 3,386	3,691	2,730 3,790	3,260	—		—	
Soft maple	4,148 3,538	3,843	2,578 3,645	3,111	3,843 3,448	3,645	2,725 3,030	2,877
Sycamore	4,103 3,493	3,798	3,188 3,188	3,188	3,691 3,976	3,833		1,968
Hemlock		2,910		1,996	—		—	

The forces exerted in pulling and driving the spikes were produced by a lever. A few tests with a hydraulic press showed that the friction of the plunger varied from about 6 to 18 per cent. The experimental results are given in Table I.

Some very excellent tests of the holding power of railroad spikes and lag-screws were made by Mr. A. J. Cox, of the University of Iowa, during 1891, in the engineering laboratory of that institution, the results of which were

TABLE II.

RESISTANCE OF RAILROAD SPIKES TO PULLING OUT AND PRESSING IN.

Kind of Tie and Spike.	No. of Tests.	Greatest Resistance in Pounds.			Average Resistance in Pounds per Square Inch Surface of Spike†	Average Resistance per Ounce of Spike.
		Maximum.	Average.	Minimum.		
Seasoned White-oak Tie.						
Common spike................	20	7,700	5,514	3,500	643	664
Common spike, ½-in. bored hole.	9	6,660	4,936	3,950	575	595
Common spike, redrawn.......	1	——	5,120* 4,460	——	520	537
Common spike, ½-in bored hole, redrawn..................	2	——	4,040* 3,240	——	378	390
Hill curved spike.............	4	6,580	5,843	5,290	——	632
Bayonet spike.................	3	6,850	6,350	5,600	——	934
Unseasoned White Oak.						
Common spike................	7	6,130	4,706	4,050	548	567
Common spike, ½-in. bored hole.	3	5,950	5,807	5,720	716	740
Hill curved spike.............	5	5,680	5,130	4,400	——	555
Bayonet spike.................	3	6,930	5,334	4,030	——	784
Unseasoned White Cedar.						
Common spike................	2	1,240	1,140	1,040	133	137
Common spike, ½-in. bored hole.	2	1,460	1,400	1,340	162	169
Hill curved spike.............	2	1,830	1,775	1,720	——	192
Bayonet spike.................	2	980	955	930	——	140

PRESSING SPIKES INTO TIES UNDER STEADY PRESSURE OF TESTING MACHINE.

White-oak Ties.						
Curved spike, pressing in......	2	7,430	7,375	7,320	——	——
Curved spike, pulling out......	2	6,830	6,615	6,400	——	——
Bayonet spike, pressing in.....	2	6,660	6,530	6,400	——	——
Bayonet spike, pulling out.....	2	4,400	3,845	3,290	——	——

* These values are the first resistance to drawing out. The spikes were then redriven in the same holes and redrawn, with the results shown.

† Wedge surface not considered.

published in the technical journal ("The Transit") of the university for September, 1891; they will be found somewhat rearranged in Tables II and III. Three kinds of spikes were used, viz., the common spike (length 5.5 ins., 0.5625 in. square, weight 8.3 oz.), Hill's curved spike (length 5.875 ins., weight 9.25 oz.), and the bayonet or grooved spike (length 5.5 ins., weight 6.8 oz.). The timber of the ties is shown in the two tables. The spikes were forced into the wood by the pressure exerted by the 100,000-pound testing machine used in the tests, and by which they were pulled out of the ties.

TABLE III.

RESISTANCE OF LAG-SCREWS TO PULLING OUT.*

Kind of Wood.	Diameter of Screw, Inches.	Diameter of Bored Hole, Inches.	Length Screw in Hole, Ins.	Maximum Average Resistance in Pounds.	Resistance pounds per Square Inch.	No. of Tests.
Seasoned white oak. . . .	⅝	½	4½	8,037	1,024	3
Seasoned white oak. . . .	9/16	7/16	3	6,480	1,223	1
Seasoned white oak. . . .	½	⅜	4½	8,780	1,239	2
Yellow-pine stick.	⅝	½	4	3,800	484	2
White cedar unseasoned	⅝	½	4	3,405	434	2

* The area of surface for these lag-screws used in finding the resistance per square inch was computed as that of a cylinder whose diameter was equal to the diameter of the screw considered. In pulling the first lag-screw of Table III, the resistance of 8037 pounds at the end of a ⅛-inch movement decreased to 4550, 2476, 1475, and 410 pounds at the ends of movements of 0.5, 1, 2, and 2.75 inches respectively.

The greatest pulling resistance of any spike is offered at the very beginning of motion, and it then rapidly decreases. A common spike which resisted 5120 pounds at the beginning of motion offered but 3050 pounds after having moved a half-inch, 2,440 pounds after 1 inch of motion, 1,300 pounds after 1.75 inches, 940 pounds after 2 inches, and 440 pounds after moving 3 inches; the original penetration of the spike was 4.375 inches in a seasoned white-oak tie. Similar results were reached with other timbers.

When spikes were pressed into the ties the timber offered an increasing resistance to penetration, but at a rate less rapid than that of the decrease in pulling out. A ½-inch penetration in a seasoned white-oak tie gave a resistance to a common spike of 2,320 pounds which increased to 3,340 pounds for 1-inch penetration, to 4550 pounds for

2 inches, to 5580 pounds for 3.5 inches, and to 6555 pounds for 4.5 inches.

Art. 104.—Shearing Resistance of Timber behind Bolt or Mortise Holes.

Col. T. T. S. Laidley, U.S.A., made some tests during 1881 at the United States Arsenal, Watertown, Mass., on the resistance offered by timber to the shearing out of bolts or keys when the force is exerted parallel to the fibres.

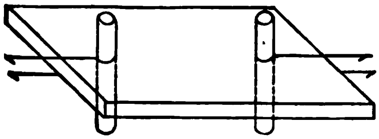

Fig. 1.

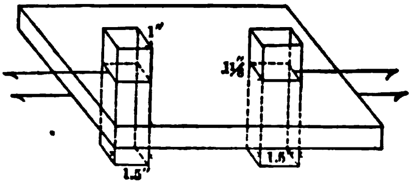

Fig. 2.

The test specimens are shown in Figs. 1 and 2. Wrought-iron bolts and square wrought-iron keys were used. All the timber specimens were six inches wide and two inches thick. The diameter of the bolts used (Fig. 1) was one inch for all the specimens. The keys were 1" × 1."5 and 1".125 × 1".5 as shown in Fig. 2. In all the latter specimens, failure took place in front of the smaller key where the pressure was greatest.

In many cases the specimen sheared and split simultaneously in front of the hole. By putting bolts through the pieces in a direction normal to the force exerted, so as to prevent splitting, the resistance was found (in most cases) to be considerably, though irregularly, increased.

Unless otherwise stated, the wood was thoroughly seasoned.

The accompanying table gives the results of Col. Laidley's tests.

Kind of Wood.	[illegible]	Total Area of Shearing.	Ultimate Shearing Resistance per Square Inch, in Pounds.
	[illegible]	Sq. Ins.	
Spruce blocks,	2	8	369
	4	16	359
	6	24	275
	8	32	262
White pine blocks,	2	8	457
	4	16	611
	6	24	450
	8	32	327
Yellow pine (blocks)	2	8	605
	4	16	720
	6	24	456
	8	32	337
Yellow pine (square keys)	2	8	599
	4	16	369
	6	24	572
	7	28	438
White pine (square keys)	2	8	550
	4	16	412
	6	24	332
	7	28	236
Spruce (square keys)	2	8	410 (Not thoroughly seasoned.)
	4	16	329 " " "
	6	24	242 (Wet timber.)
	7	28	279

Art. 105.—Bulging of Plates.

A plate offers resistance to "bulging" when it is simply supported or firmly fixed around its entire edge and carries a single or uniformly distributed load acting normal to its surface. The very complicated nature of the stresses and strains existing in a plate thus acted upon, together with the fact that its conditions just before rupture are entirely different from those accompanying the initial loading, give to the problem a character of unusual intricacy, and, indeed, preclude a solution possessing a degree of approximation commonly obtained in questions relating to the elasticity and resistance of materials.

An elegant analysis of the problem, considered as one of pure elasticity, may be found in "Die Theorie der Elasticität Fester Körper," by Clebsch. It is, however, of little value in connection with questions of ultimate resistance.

The following roughly approximate but simple analysis may be used to suggest the form of an empirical formula which can be completed by the aid of experiments.

Let the length, breadth, and thickness of a rectangular plate simply supported around its edges be represented by a, b, and t, respectively, and let it first be loaded by a uniformly distributed pressure whose intensity (per unit of ab) is w.

If the plate is supposed to consist of two sets of small strips or beams parallel to a and b, those crossing each other at the centre must have the same deflection at middle. If, further, the uniform load w be supposed to be so divided into two parts, w_1 and w', that they would cause two rectangular beams whose spans are a and b to have the same centre deflection, the following equation (see eq. (26) of Art. 22) must obtain:

$$\frac{5w_1a^4}{384EI}=\frac{5w'b^4}{384EI}.$$

Then, since $w'+w_1=w$, there must result:

$$w'=\frac{a^4w}{a^4+b^4} \quad \text{and} \quad w_1=\frac{b^4w}{a^4+b^4}.$$

The bending moments at the centres of such beams would be (eq. (27), Art. 22, and eq. (10), Art. 9):

$$\frac{w_1a^2}{8}=\frac{2K_1I}{t} \quad \text{and} \quad \frac{w'b^2}{8}=\frac{2K'I}{t}.$$

Since the beams are rectangular in section, $I = \frac{t^3}{12}$.

Hence

$$K_1 = \frac{3w_1 a^2}{4t^2} \quad \text{and} \quad K' = \frac{3w' b^2}{4t^2}.$$

According to these hypothetical conditions the greatest intensity of stress at the centre of the plate will have the value

$$K = \left(\frac{K_1^2 + K'^2}{2}\right)^{\frac{1}{2}} = \frac{3a^2b^2w}{4t^2\sqrt{2(a^4+b^4)}}. \quad . \quad . \quad . \quad (1)$$

Hence

$$t = \frac{ab}{2}\left(\frac{3w}{K\sqrt{2(a^4+b^4)}}\right)^{\frac{1}{2}}. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

For square plates, $a = b$.
Hence

$$K = \frac{3a^2w}{8t^2} \quad \text{and} \quad t = 0.615a\sqrt{\frac{w}{K}}. \quad . \quad . \quad . \quad (3)$$

If the edges are fixed, the greatest bending will occur along those lines; and for K_1 and K' then are to be put $\frac{2}{3}K_1$ and $\frac{2}{3}K'$.
Hence

$$K_1 = 2 \cdot \frac{a^2b^4w}{4t^2(a^4+b^4)} \quad \text{and} \quad K' = 2 \cdot \frac{b^2a^4w}{4t^2(a^4+b^4)}. \quad . \quad (4)$$

Since the greatest bending occurs along the edges, these are the expressions for the greatest intensities of stress. If ab^2 is greater than a^2b, then is K_1 greater than K'; and *vice versa*.

In the first case the expression for t is

$$t = 0.707ab^2\sqrt{\frac{w}{(a^4+b^4)K_1}}. \quad . \quad . \quad . \quad . \quad (5)$$

But if $K' > K_1$, or $a^2b > ab^2$,

$$t = 0.707a^2b\sqrt{\frac{w}{(a^4+b^4)K'}}. \quad . \quad . \quad . \quad . \quad (6)$$

If the plate is square,

$$K = \frac{a^2w}{4t^2} \quad \text{and} \quad t = \frac{a}{2}\sqrt{\frac{w}{K}}. \quad . \quad . \quad . \quad . \quad (7)$$

If a plate is loaded with a single weight P, it may be supposed to be divided in the same manner as w; so that

$$P_1 + P' = P.$$

The equation of middle deflections for ends simply supported then becomes

$$\frac{P_1a^3}{48EI} = \frac{P'b^3}{48EI}.$$

Hence

$$P' = \frac{a^3P}{a^3+b^3} \quad \text{and} \quad P_1 = \frac{b^3P}{a^3+b^3}.$$

Proceeding in precisely the same manner as before,

$$K = 1.06\frac{abP}{t^2(a^3+b^3)}\sqrt{a^4+b^4} \quad . \quad . \quad . \quad . \quad (8)$$

and

$$t = 1.03\left(\frac{abP}{K(a^3+b^3)}\sqrt{a^4+b^4}\right)^{\frac{1}{2}}. \quad . \quad . \quad . \quad (9)$$

If the plate is square,

$$K = 0.75\frac{aP}{t^2} \quad \text{and} \quad t = 0.87\sqrt{\frac{aP}{K}}. \quad . \quad . \quad . \quad (10)$$

If the edges are fixed in position, the hypothetical beams are fixed at each end and loaded at the centre, and the greatest bending moments (at centre and ends alike) are thereby reduced to one half their preceding values, or, what is the same thing, $2t^2$ is to take the place of t^2 in eqs. (8), (9), and (10).

Hence

$$K = 0.53\frac{abP}{t^2(a^3+b^3)}\sqrt{a^4+b^4}, \quad . \quad . \quad . \quad . \quad (11)$$

$$t = 0.73\left(\frac{abP}{K(a^3+b^3)}\sqrt{a^4+b^4}\right)^{\frac{1}{2}}. \quad . \quad . \quad (12)$$

If the plate is square,

$$K = 0.375\frac{aP}{t^2} \quad \text{and} \quad t = 0.613\sqrt{\frac{aP}{K}}. \quad . \quad . \quad (13)$$

These equations are of little value as they stand, except as indicating a form of formula to which empirical coefficients are to be fitted. The hypothetical division of the plate into small beams is very far indeed from being correct. In the empirical determinations which follow, therefore, K will not be the greatest intensity of stress in the plate, but a coefficient or quantity partly analytical and partly experimental.

Circular plates have not been considered, because square ones furnish the requisite type of formula.

Experiments have thus far been made on square and circular plates only; hence oblong rectangular plates will not again be noticed.

Kirkaldy's experiments on Fagersta steel plates and Fairbairn's on wrought-iron ones would seem to indicate that the thickness t varies about as $(w)^{0.8}$ or $(P)^{0.8}$; but the variation in diameter or side of square was not sufficient to establish any relation between t and a, while other elements remain the same. Regarding, therefore, K as an empirical quantity which may have different values for square and circular plates, eqs. (3), (7), (10), and (13) may be written as follows:

$$K=\frac{3a^2}{8t^2}w^{1.6} \quad \text{and} \quad t=0.615a\frac{w^{0.8}}{\sqrt{K}}. \quad . \quad . \quad (14)$$

$$K=\frac{a^2}{4t^2}w^{1.6} \quad \text{and} \quad t=0.5a\frac{w^{0.8}}{\sqrt{K}}. \quad . \quad . \quad . \quad (15)$$

$$K=\frac{3a}{4t^2}P^{1.6} \quad \text{and} \quad t=0.87\frac{\sqrt{a}P^{0.8}}{\sqrt{K}}. \quad . \quad . \quad (16)$$

$$K=\frac{3a}{8t^2}P^{1.6} \quad \text{and} \quad t=0.613\frac{\sqrt{a}P^{0.8}}{\sqrt{K}}. \quad . \quad . \quad (17)$$

Kirkaldy made twenty experiments with mild Fagersta steel circular plates, 12 inches in diameter. He forced these through an aperture 10 inches in diameter by the pressure of a very blunt point. The edge of the aperture on which the plate rested was rounded; hence the initial diameter of aperture was somewhat more than 10 inches. Eqs. (16) are the ones to be used in connection with these experiments.

From the first member of that equation, K was computed for a number of different experiments by substituting the numerical values of P, t, and a. In this manner the following values were found to give good results:

For unannealed mild Fagersta steel circular plates:

$$K=6,760,000,000.$$

Hence

$$t=0.000,010,6\sqrt{a}P^{0.8}. \quad . \quad . \quad . \quad . \quad (18)$$

For annealed mild Fagersta steel circular plates:

$$K=5,710,000,000.$$

Hence

$$t=0.000,011,52\sqrt{a}P^{0.8} \quad . \quad . \quad . \quad . \quad (19)$$

Eq. (16) gives

$$P=\left(\frac{t\sqrt{K}}{0.87\sqrt{a}}\right)^{1\frac{1}{4}}.$$

Table I contains the results of computation by this formula and those obtained in the tests. On account of the rounded edge of the supporting ring, K was so taken that P, as computed, is a little larger than its experimental value. None of these plates was cracked, but they were bulged at the centre from 3.00 to 3.45 inches.

In "Engineering" for Sept. 28, 1877, Robert Wilson describes four experiments on unstayed flat boiler heads subjected to hydraulic pressure. These flat circular plates were riveted to angles encircling the body of the boiler. The edges of the plates were thus fixed, and eqs. (15) are therefore to be used. Proceeding in precisely the same manner as before, the following values were established:

For wrought-iron flat boiler heads with fixed edges:

$$K=11,000,000.$$

Hence

$$t=0.000,015aw^{0.8}. \quad . \quad . \quad . \quad . \quad (20)$$

w was taken in pounds per square inch; it has the value

$$w=\left(\frac{4t^2K}{a^2}\right)^{0.625}.$$

TABLE I.

CIRCULAR PLATES SIMPLY SUPPORTED.

Unannealed.			Annealed.		
t, in Inches.	P, in Pounds.		t, in Inches.	P in Pounds.	
	Experimental.	By Formula.		Experimental.	By Formula.
$\frac{5}{8}$	215,690	219,420	$\frac{5}{8}$	198,000	196,530
$\frac{1}{2}$	162,740	166,000	$\frac{1}{2}$	154,330	148,690
$\frac{3}{8}$	104,850	115,860	$\frac{3}{8}$	95,600	103,780
$\frac{1}{4}$	71,800	69,800	$\frac{1}{4}$	59,430	62,520
$\frac{1}{8}$	35,400	29,350	$\frac{1}{8}$	25,430	26,290

Each "experimental" result is a mean of two.

The results of the experiments, and of this formula, are:

Diameter. Inches.	t, Inch.	q, in Pounds per Square Inch.	
		Experimental.	By Formula.
34.5	$\frac{9}{16}$	280	349
34.5	$\frac{3}{8}$	200	211
26.25	$\frac{3}{8}$	371	296
28.25	$\frac{3}{8}$	300	270

The agreement, in this case, is not satisfactory. It is probably due to the lack of a proper exponent of a. These plates were fractured along the lines of rivet-holes in the edges.

Two means of four experiments by Fairbairn remain to be considered. His plates were square ones of wrought iron firmly fixed to a square frame 12 inches by 12 inches in the aperture. The force was applied by a blunt point at the centre, consequently eqs. (17) are to be used.

By precisely the same method already used the following results were established:

For wrought-iron 12-inch square plates, with edges firmly fixed:

$$K = 390{,}000{,}000;$$

$$t = 0.000{,}031\sqrt{a}P^{0.8}. \qquad (21)$$

The expression for the indenting force is

$$P = \left(t\sqrt{\frac{8K}{3a}}\right)^{1\frac{1}{4}}.$$

The experiments and computations are:

Diameter, Inches.	t, Inch.	P, in Pounds. Experimental.	By Formula.
12	$\frac{1}{4}$	16,780	16,350
12	$\frac{1}{2}$	37,720	38,890

The plates gave way at the centre, under the blunt point.

Some experiments by Kirkaldy, in 1875, on *wrought-iron circular plates simply supported around the edge,* show that for 12-inch plates forced through a 10-inch aperture with rounded edge, there may be safely taken:

$$t = 0.000{,}013\sqrt{a}P^{0.8}. \qquad (22)$$

In all the preceding formulæ, a and t are to be taken in inches, w in pounds per square inch, and P in pounds.

The investigations can only be considered provisional. Although they give, as a whole, tolerably satisfactory results, the range of the experiments is far too small for the establishment of thoroughly reliable formulæ. Experiments on which a proper exponent of a can be based are yet wholly lacking, and as the only resort, that found in the rough analysis has been retained.

Art. 106.—Special Cases of Flexure.

There are a few cases of flexure which, while not frequently found in engineering experience, are of some practical importance, and are occasionally required. The two or three which follow involve the integration of some linear differential equations that are treated in all the advanced works on the integral calculus; consequently the operations of integration will not be given here, but the general integrals will be assumed.

Flexure by Oblique Forces.

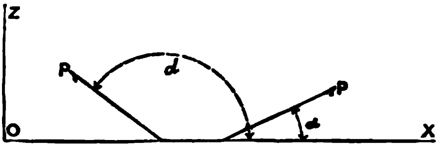

Fig. 1.

In Fig. 1 let OX represent a beam acted upon by the oblique forces P, which make angles α with the axis of X. The origin O is supposed to be taken anywhere on the axis of the beam. If right-hand moments are positive, and left-hand negative, the component $P \sin \alpha$ will have the negative moment $-P \sin \alpha x$ about O. The lever-arm of $P \cos \alpha$, if the deflection w is positive, is $+w$, and its moment $P \cos \alpha . w$ is positive. Hence the resultant moment of any force, P, in reference to the origin O is

$$EI \frac{d^2w}{dx^2} = -P \sin \alpha . x + P \cos \alpha . w. \quad . \quad . \quad . \quad (1)$$

If α is greater than 90°, cos α is negative, so that if

$$A = \pm \frac{P \cos \alpha}{EI} \quad \text{and} \quad V = -\frac{P \sin \alpha . x}{EI},$$

the two cases may be expressed by the equation

$$\frac{d^2w}{dx^2} + Aw = V. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If $a = +\sqrt{-A}$, and $b = -\sqrt{-A}$, the general integral of eq. (2) is

$$w = Ce^{ax} + C'e^{bx} + \frac{e^{ax}}{a-b}\int Ve^{-ax}dx - \frac{e^{bx}}{a-b}\int Ve^{-bx}dx, \quad (3)$$

in which C and C' are arbitrary constants to be determined by the special conditions of any given problem, and $e = 2.71828$.

When α is less than 90° and

$$a = \sqrt{\frac{P\cos\alpha}{EI}}, \quad b = -\sqrt{\frac{P\cos\alpha}{EI}};$$

eq. (3) becomes

$$w = Ce^{ax} + C'e^{bx} + x\tan\alpha; \quad . \quad . \quad . \quad . \quad (4)$$

C and C' yet remain to be determined by the particular circumstances of a given case.

The conditions on which the determination of these constants rests are expressed by giving known values to w and $\frac{dw}{dx}$ for values of x, also known.

If α is greater than 90°,

$$a = \sqrt{\frac{P\cos\alpha}{EI}}\,.\sqrt{-1} \quad \text{and} \quad b = -\sqrt{\frac{P\cos\alpha}{EI}}\,.\sqrt{-1},$$

and eq. (3)* becomes

$$w = C\cos\left(\frac{P\cos\alpha}{EI}\right)^{\frac{1}{2}}x + C'\sin\left(\frac{P\cos\alpha}{EI}\right)x - \tan\alpha\,.\,x. \quad (5)$$

---

* The symbolical method by which eq. (3) was established shows that if $a = -\frac{1}{2}B + \sqrt{A - \frac{1}{4}B^2}$ and $b = -\frac{1}{2}B - \sqrt{A - \frac{1}{4}B^2}$, the complete integral of the equation $\frac{d^2w}{dx^2} + B\frac{dw}{dx} + Aw = V$ is given by eq. (3).

As before, C and C' are to be determined by the circumstances of each case to which the equation is applied; and the value of $\cos\alpha$, it is to be remembered, is to be substituted with the positive sign.

Let a column with one fixed and one free end, and with the force P acting parallel to its original axis, be considered. Since $\alpha = 180°$,

$$w = C\cos\left(\sqrt{\frac{P}{EI}}\cdot x\right) + C'\sin\left(\sqrt{\frac{P}{EI}}\cdot x\right). \quad . \quad . \quad (6)$$

With the origin of coordinates at the free end w must equal zero, for $x = 0$; hence $C = 0$.

The value of w then becomes

$$w = C'\sin\left(\sqrt{\frac{P}{EI}}\cdot x\right). \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$\therefore \frac{dw}{dx} = C'\sqrt{\frac{P}{EI}}\cos\left(\sqrt{\frac{P}{EI}}\cdot x\right). \quad . \quad . \quad . \quad (8)$$

But if l is the length of the column, $\frac{dw}{dx} = 0$ for $x = l$.

Hence

$$\cos\left(\sqrt{\frac{P}{EI}}\cdot l\right) = 0;$$

or if n is any whole number from 0 to infinity,

$$\sqrt{\frac{P}{EI}}\cdot l = \tfrac{1}{2}(2n+1)\pi. \quad . \quad . \quad . \quad . \quad . \quad (9)$$

If the value of $\sqrt{\frac{P}{EI}}$ be taken from eq. (9) and inserted in eq. (8), there will result

$$\frac{dw}{dx} = C'\sqrt{\frac{P}{EI}}\cos\left(\frac{x(2n+1)}{2l}\right)\pi. \quad . \quad . \quad . \quad (10)$$

Eq. (10) shows that for values of x equal to 1, 3, 5, 7 . . . times $\frac{l}{2n+1}$, $\frac{dw}{dx}=0$. The most dangerous supposition, i.e., that which requires the greatest value of P, is $n=0$.

This value of n in eq. (9) gives

$$P=\frac{\pi^2EI}{4l^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

The ultimate resistance of the column is thus seen to be independent of the deflection, as was found for a different case in Art. 24. The end of the column, in this case, which carries the load is free to deflect laterally, but in Art. 24 both ends were supposed to be fixed in a lateral direction in reference to each other. In the latter case the resistance is seen to be nine times as great as in the present.

Since

$$\cos\left(\sqrt{\frac{P}{EI}}.l\right)=0, \quad \sin\left(\sqrt{\frac{P}{EI}}.l\right)=1.$$

Hence if w_1 is the deflection of the free end from a vertical tangent to the fixed, eq. (7) becomes, for $x=l$,

$$w_1=C'.$$

In general, therefore,

$$w=w_1\sin\left(\sqrt{\frac{P}{EI}}.x\right). \quad . \quad . \quad . \quad . \quad (12)$$

For the same value of x, therefore, w varies directly as w_1, and the relative deflections may be computed by the equation

$$\frac{w}{w_1}=\sin\left(\frac{(2n+1)\pi x}{2l}\right); \quad . \quad . \quad . \quad . \quad . \quad (13)$$

or in the ordinary case

$$\frac{w}{w_1}=\sin\frac{\pi x}{2l}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (14)$$

Eq. (1) was written for one force only. If any number of forces act,

$$EI\frac{d^2w}{dx^2}=\Sigma(-P\sin\alpha.x+P\cos\alpha.w),$$

and in place of w there is to be put Σw.

General Flexure by Continuous Normal Load.

The most general case of flexure by a continuous normal load is that in which the intensity (load per unit of length of beam) is a variable quantity. Let x be an abscissa measured along the original axis of the beam, and let w represent the deflection. Then the intensity of the load may be represented by $f(x, w)$. It was shown in Art. 20 that

$$\frac{d^2M}{dx^2}=EI\frac{d^4w}{dx^4}=p=f(x, w) \quad . \quad . \quad . \quad . \qquad (15)$$

The integration of the equation

$$\frac{d^4w}{dx^4}=\frac{f(x, w)}{EI}$$

will depend upon the form of the function $f(x, w)$.

Let it be supposed that $f(x, w)=cx$, c being a constant. Then if A, A_1, A_2, and A_3 are constants of integration, there will result

$$w=\frac{cx^5}{120EI}+\frac{Ax^3}{6}+\frac{A_1x^2}{2}+A_2x+A_3. \quad . \quad . \qquad (16)$$

Again, if $f(x, w) = cw$, c, as before, being a constant,

$$\frac{d^4w}{dx^4} = \frac{cw}{EI}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

For simplicity of notation let

$$a^4 = \frac{c}{EI};$$

then the general integral of eq. (17) becomes

$$w = Ae^{ax} + A_1e^{-ax} + A_2 \cos ax + A_3 \sin ax. \quad . \quad (18)$$

In eq. (18) $e = 2.71828$ is the base of the Naperian logarithms; while in both eqs. (16) and (18) A, A_1, A_2, and A_3 are arbitrary constants to be determined by the circumstances of each individual case.

CHAPTER XVII.

WORKING STRESSES AND SAFETY FACTORS.

Art. 107.—Definitions.

IN all metallic and timber constructions the greatest (supposed) possible loads are determined from the attendant circumstances of the different cases, and then the stresses induced by these greatest loads are computed. These stresses are called the "*working stresses.*"

The ultimate resistance of any piece in a structure divided by the working stress gives a number called the "*safety factor.*" Occasionally the reciprocal of this number is called the safety factor, though but seldom.

The intensity of the ultimate resistance of any piece in a structure divided by the intensity of the working stress will also give the safety factor. This is the more usual and convenient form, since it does not involve the cross-section of the piece.

The values of safety factors depend upon many circumstances, such as kind and character of material, kind of stress, conditions under which material is used and the amount of variation of stress in the piece or the fatigue of the material. The safety factor is intended also to cover both computed stresses and others which are recognized, but are not within the reach of exact analysis. The latest practice among American engineers will be illustrated in the following articles by extracts from specifications drawn from some first-class sources.

Art. 108.—General Specifications for Iron Railroad Bridges and Viaducts, by Theodore Cooper, C.E., 1901.

The excerpts given in this article are from the general specifications of Mr. Theodore Cooper, C.E., consulting engineer, which have secured a wide adoption in American railway practice.

2. *Kind of Girders.*—The following kinds of girders shall preferably be employed:

Spans	up to	20	feet....	Rolled beams or longitudinal trough floors.
"	20 to	75	"	Riveted plate girders.
"	75 to	120	"	Riveted plate or lattice girders.
"	120 to	150	"	Lattice or pin-connected trusses.
"	over	150	"	Pin-connected trusses.

Generally, "double-track through" bridges will have but two trusses, to avoid spreading the tracks at bridges.

LOADS.

23. All the structures shall be proportioned to carry the following loads:

Dead Load.—1st. The weight of metal in the structure and floor.

2d. The weight of rails, fastenings, ties, guards, footwalk, and ballast when used; the rails and fastenings being assumed at 100 pounds per foot of track, timber at 4½ pounds per foot B. M., and ballast at 100 pounds per cubic foot. Minimum will be assumed at 400 pounds per foot of track.

These two items, taken together, shall constitute the "dead load."

3d. *Live Loads.*—"A live load" on each track, supposed to be moving in either direction, consisting of two "consolidation" engines, coupled and followed by a train load, distributed as shown on diagram E......; or a special load equally distributed on two pairs of driving-wheels, spaced six feet centre to centre, of 100,000 pounds up to class E40, and of 120,000 pounds for all classes above E40.

(The diagram E40 for one rail is shown on page 69.)

NOTE.—As all the wheel loads in each diagram are made of the same percentages of the driving-wheel loads, the strains due to the different engine diagrams will be proportionate to the numerical classes of the engines.

Any intermediate numbers may be selected, with the understanding that this rule of proportion applies.

The maximum strains due to all positions of either of the above "live loads" of the required class, and of the "dead loads," shall be taken to proportion all the parts of the structure.

PROPORTION OF PARTS.

31. *Tensile Strain.*—All parts of the structures shall be proportioned in tension by the following allowed unit strains:

For Medium Steel.

	Lbs. per Sq. In.	
Floor-beam hangers and other similar members liable to sudden loading, net section	6,000	
Longitudinal, lateral, and sway bracing, for wind strains (§§ 8, 24, 25)*	18,000	
Longitudinal, lateral, and sway bracing, for live-load strains (§§ 27, 29, 105)*	12,000	
Solid rolled beams, used as cross floor-beams and stringers	10,000	
Bottom flanges of riveted cross girders, net section	10,000	
Bottom flanges of riveted longitudinal plate girders, used as track-stringers, net section	10,000	
	For Live Loads.	For Dead Loads
Bottom chords, main diagonals, counters, and long verticals	10,000	20,000

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* These articles refer to Cooper's specifications.

For swing bridges and other movable structures the dead-load unit strains, during motion, must not exceed three fourths of the above allowed unit strains for dead load on stationary structures.

The areas obtained by dividing the live-load strains by the live-load unit strains will be added algebraically to the areas obtained by dividing the dead-load strains by the dead-load unit strains to determine the required sectional area of any member. (§ 47.)

Soft Steel.—Soft steel may be used in tension with unit strains 10 per cent. less than those allowed for medium steel.

32. Angles subject to direct tension must be connected by both legs, or the section of one leg only will be considered as effective.

33. *Net Section.*—In members subject to tensile strains full allowance shall be made for reduction of section by rivet-holes, screw-threads, etc. (§ 60.)

34. *Compressive Strains.*—Compression members shall be proportioned by the following allowed unit strains:

For Medium Steel.

Chord segments:

$$P = 10{,}000 - 45\frac{l}{r} \text{ for live-load strains;}$$

$$P = 20{,}000 - 90\frac{l}{r} \text{ for dead-load strains.}$$

All posts of through bridges:

$$P = 8{,}500 - 45\frac{l}{r} \text{ for live-load strains;}$$

$$P = 17{,}000 - 90\frac{l}{r} \text{ for dead-load strains.}$$

All posts of deck bridges and trestles:

$$P = 9{,}000 - 40\frac{l}{r} \text{ for live-load strains;}$$

$$P = 18{,}000 - 80\frac{l}{r} \text{ for dead-load strains.}$$

End posts are not to be considered chord segments.

Lateral struts and rigid bracing:

$$P = 13{,}000 - 60\frac{l}{r} \text{ for wind strains.}$$

For live-load strains use two thirds of the above. (§§ 27, 29, 105.)

P = the allowed strain in compression per square inch of cross-section, in pounds.

l = the length of compression member, in inches, c. to c. of connections.

r = the least radius of gyration of the section, in inches.

No compression member, however, shall have a length exceeding 100 times its least radius of gyration for main members, or 120 times for laterals.

Soft Steel.—Soft steel may be used in compression with unit strains 15 per cent. less than those allowed for medium steel.

For swing bridges and other movable structures the dead-load unit strains during motion must not exceed $\frac{3}{4}$ of the above allowed unit strains for dead load on stationary structures.

35. For long-span bridges, when the ratio of the length and width of span is such that it makes the top chords acting as a whole a longer column than the segments of the chord, the chord will be proportioned for this greater length.

36. *Alternate Strains.*—All members and their connections subject to alternate strains of tension and compression

shall be proportioned to resist each kind of strain. Both of the strains shall, however, be considered as increased by an amount equal to $\frac{8}{10}$ of the least of the two strains, for determining the sectional areas by the above-allowed unit strains. (§§ 31, 34.)

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38. *Rivets, Bolts, and Pins.*—The rivets in all members, other than those of the floor and lateral systems, must be so spaced that the shearing strain per square inch shall not exceed 9000 pounds, nor the pressure on the bearing surface (diameter × thickness of the piece) of the rivet-hole exceed 15,000 pounds per square inch.

The rivets in all members of the floor system, including all hanger connections, must be so spaced that the shearing strains and bearing pressures shall not exceed 80 per cent. of the above limits.

The rivets in the lateral and sway bracing will be allowed 50 per cent. increase upon the above limits for lateral forces as per §§ 24, 25, but not per §§ 27, 29.

In the case of field riveting (and for bolts as per § 61) the above-allowed shearing strains and pressures shall be reduced one third.

Rivets and bolts must not be used in direct tension.

39. Pins shall be proportioned so that the shearing strain shall not exceed 9000 pounds per square inch; nor the pressure on the bearing surface of any member (other than forged eye-bars, see § 85) connected to the pin be greater per square inch than 15,000 pounds; nor the bending strain exceed 18,000 pounds, when the applied forces are considered as uniformly distributed over the middle half of the bearing of each member.

40. *Combined Strains.*—When any member is subjected to the action of both axial and bending strains, as in the case of end posts of through bridges (§ 37), or of chords

carrying distributed floor loads, it must be proportioned so that the greatest fibre strain will not exceed the allowed limits of tension or compression on that member.

If the fibre strain resulting from the weight only, of any member, exceeds 10 per cent. of the allowed unit strain on such member, such excess must be considered in proportioning the areas.

41. *Compression Flanges.*—In beams and plate girders the compression flanges shall be made of same *gross* section as the tension flanges.

42. *Depth of Girders.*—Riveted longitudinal girders shall have, preferably, a depth not less than $\frac{1}{10}$ of the span.

Rolled beams used as longitudinal girders shall have, preferably, a depth not less than $\frac{1}{12}$ of the span.

43. *Plate Girders, etc.*—Plate girders shall be proportioned upon the supposition that the bending or chord strains are resisted entirely by the upper and lower flanges, and that the shearing or web strains are resisted entirely by the web plate; no part of the web plate shall be estimated as flange area.

The distance between centres of gravity of the flange areas will be considered as the effective depth of all girders.

44. *Web Plates.*—The webs of plate girders must be stiffened at intervals, not exceeding the depth of the girders or a maximum of 5 feet, wherever the shearing strain per square inch exceeds the strain allowed by the following formula:

$$\text{Allowed shearing strain} = 10{,}000 - 75H,$$

where H = ratio of depth of web to its thickness; but no web plates shall be less than three eighths of an inch in thickness.

45. *Stiffeners.*—All stiffeners must be capable of carry-

ing the maximum vertical shear without exceeding the allowed unit strain,

$$P = 10{,}000 - 45\frac{l}{r}.$$

Each stiffener must connect to the webs by enough rivets to transfer the maximum shear to or from the webs.

46. *Rolled Beams.*—Rolled beams shall be proportioned (§§ 31, 41) by their moments of inertia.

*   *   *   *   *   *   *

DETAILS OF CONSTRUCTION.

49. All the connections and details of the several parts of the structures shall be of such strength that, upon testing, rupture will occur in the body of the members rather than in any of their details or connections.

50. Preference will be had for such details as shall be most accessible for inspection, cleaning, and painting; no closed sections will be allowed.

51. *Riveting.*—The pitch of rivets in all classes of work shall never exceed 6 inches, or sixteen times the thinnest outside plate, nor be less than three diameters of the rivet.

52. The rivets used shall generally be $\frac{3}{4}$ and $\frac{7}{8}$ inch diameter.

53. The distance between the edge of any piece and the centre of a rivet-hole must never be less than $1\frac{1}{4}$ inches, except for bars less than $2\frac{1}{2}$ inches wide; when practicable it shall be at least two diameters of the rivet.

54. For punching, the diameter of the die shall in no case exceed the diameter of the punch by more than $\frac{1}{16}$ of an inch, and all holes must be clean cuts without torn or ragged edges.

55. All rivet-holes must be so accurately spaced and punched that when the several parts forming one member

are assembled together, a rivet $\frac{1}{16}$ inch less in diameter than the hole can generally be entered, hot, into any hole, without reaming or straining the metal by "drifts"; occasional variations must be corrected by reaming.

* * * * * * *

66. *Abutting Joints.*—In compression members, abutting joints with untooled faces must be fully spliced, as no reliance will be placed on such abutting joints. The abutting ends must, however, be dressed straight and true, so there will be no open joints.

* * * * * * *

68. *Stiffeners.*—All web plates must have stiffeners over bearing points and at points of local concentrated loadings; such stiffeners must be fitted at their ends to the flange angles, at the bearing points.

* * * * * * *

72. *Flange Plates.*—In girders with flange plates, at least one half of the flange section shall be angles or else the largest-sized angles must be used. Flange plates must extend beyond their theoretical length, two rows of rivets at each end.

* * * * * * *

74. *Compression Flanges.*—The compression flanges of beams and girders shall be stayed against transverse crippling when their length is more than sixteen times their width.

75. *Width of Plates.*—The unsupported width (distance between rivets) of plates subject to compression shall not exceed thirty times their thickness; except cover-plates of top chords and end posts, which will preferably be limited to forty times their thickness; where a greater relative width is used in chords and end posts, however, only forty times the thickness shall be considered as effective section.

76. The flange plates of all girders must be limited in

width so as not to extend beyond the outer lines of rivets, connecting them with the angles, more than 5 inches or more than eight times the thickness of the first plate. Where two or more plates are used on the flanges, they shall either be of equal thickness or shall decrease in thickness outward from the angles.

77. Where the floor timbers are supported at their ends on the flange of one angle, such angle must have two rows of rivets in its vertical leg, spaced not over 4 inches apart.

78. *Thickness of Metal.*—For main members and their connections no material shall be used of a less thickness than $\frac{3}{8}$ of an inch; and for laterals and their connections, no material less than $\frac{5}{16}$ of an inch in thickness; except for lining or filling vacant spaces.

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91. The pitch of rivets at the ends of compression members shall not exceed four diameters of the rivets for a length equal to twice the width of the member.

92. The open sides of all compression members shall be stayed by batten-plates at the ends and diagonal lattice-work at intermediate points. The batten-plates must be placed as near the ends as practicable, and shall have a length not less than the greatest width of the member or one and one half times its least width. The size and spacing of the lattice bars shall be duly proportioned to the size of the member. They must not be less in width than 2 inches for members 9 inches or less in width, nor $2\frac{1}{4}$ inches for members 12 to 9 inches in width, nor $2\frac{1}{2}$ inches for members 15 to 12 inches in width. Single lattice bars shall have a thickness not less than one fortieth, or double lattice bars connected by a rivet at the intersection not less than one sixtieth, of the distance between the rivets connecting them to the members. They shall be inclined at an angle not less than 60° to the axis of the member

for single latticing, nor less than 45° for double latticing with riveted intersections. The pitch of the latticing must not exceed the width of the channel plus nine inches.

93. Where necessary, pin-holes shall be reinforced by plates, some of which must be of the full width of the member, so the allowed pressure on the pins shall not be exceeded, and so the strains shall be properly distributed over the full cross-section of the members. These reinforcing plates must contain enough rivets to transfer their proportion of the bearing pressure, and at least one plate on each side shall extend not less than 6 inches beyond the edge of the batten-plates. (§ 92.)

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108. *Friction Rollers.*—All bridges over 80 feet span shall have hinged bolsters on both ends, and at one end nests of turned friction rollers running between planed surfaces. These rollers shall not be less than 2⅝ inches diameter for spans 100 feet or less, and for greater spans this diameter shall be increased in proportion of 1 inch for 100 feet additional.

Nuts.—The rollers shall be so proportioned that the pressure per lineal inch of roller shall not exceed the product of the diameter in inches by 300 pounds.

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QUALITY OF MATERIAL.

128. *Steel.*—All steel must be made by the open-hearth process. The phosphorus must not exceed 0.06 of one per cent. for steel made by the acid method, or 0.04 for steel by the basic method.

129. The steel must be uniform in character for each specified kind. The finished bars, plates, and shapes must be free from cracks on the faces or corners, and have a clean, smooth finish. No work shall be put upon any steel

at or near the blue temperature or between that of boiling water and of ignition of hard-wood sawdust.

130. The tensile strength, elastic limit,* and ductility shall be determined by samples cut from the finished material after rolling. The samples to be at least 12 inches long, and to have a uniform sectional area not less than ½ square inch.

131. Material which is to be used without annealing or further treatment is to be tested in the condition in which it comes from the rolls. When material is to be annealed or otherwise treated before use, the specimen representing such material is to be similarly treated before testing, for tensile strength.

The elongation shall be measured on an original length of 8 inches. Two test pieces shall be taken from each melt or blow of finished material, one for tension and one for bending. (Art. 147.)

132. All samples or full-sized pieces must show uniform fine-grained fractures of a blue steel-gray color, entirely free from fiery lustre or a blackish cast.

133. *Medium Steel.*—Medium steel shall have an ultimate strength, when tested in samples of the dimensions above stated, of 60,000 to 68,000 pounds per square inch, an elastic limit of not less than one half of the ultimate strength, and a minimum elongation of 22 per cent. in 8 inches. Steel for pins may have a minimum elongation of 15 per cent.

134. Before or after heating to a low cherry red and cooling in water at 82 degrees Fah., this steel must stand

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* For the purpose of these specifications, the elastic limit will be considered the least strain producing a visible permanent elongation in a length of 8 inches, as shown by scribe marks of a pair of finely pointed dividers.

If the yield-point or drop of the beam can be calibrated for any machine and its speed to represent the elastic limit within 5 per cent., it may be used for general cases. Test reports must state by which method the elastic limit was determined.

bending to a curve whose inner radius is one and a half times the thickness of the sample, without cracking.

135. For all medium steel, ⅝ inch or less in thickness, rivet-holes punched as in ordinary practice (§§ 52, 53, 54), must stand drifting to a diameter one third greater than the original holes, without cracking either in the periphery of the holes or on the external edges of the piece, whether they be sheared or rolled.

136. *Soft Steel.*—Soft steel shall have an ultimate strength, on same size samples, of 54,000 to 62,000 pounds per square inch, an elastic limit not less than one half the ultimate strength, and a minimum elongation of 25 per cent. in 8 inches.

For soft steel the above drifting test (§ 135) shall apply to all material to be riveted.

137. Before or after heating to a light yellow heat and quenching in cold water, this steel must stand bending 180 degrees, to a curve whose inner radius is equal to the thickness of the sample, without sign of fracture.

138. *Rivet Steel.*—Rivet steel shall have an ultimate strength of 50,000 to 58,000 pounds per square inch, an elastic limit not less than one half the ultimate strength and an elongation of 26 per cent.

139. The steel for rivets must, under the above bending test (137), stand closing solidly together without sign of fracture.

140. *Eye-bars.*—Eye-bar material, 1½ inches and less in thickness, shall, on test pieces cut from finished material, fill the above requirements. For thicknesses greater than 1½ inches, there will be allowed a reduction in the percentage of elongation of 1 per cent. for each ⅛ of an inch increase of thickness, to a minimum of 20 per cent. (Art. 119.)

141. Full-size eye-bars shall show not less than 10 per cent. elongation in the body of the bar, and an ultimate strength not less than 56,000 pounds per square inch.

Should a bar break in the head, but develop 10 per cent. elongation and the ultimate strength specified, it shall not be cause for rejection, provided not more than one third of the total number of bars tested break in the head.

142. Pins over 7 inches in diameter shall be forged. Blooms for pins shall have at least three times the sectional area of the finished pins.

143. A variation of cross-section or weight in the finished members of 2½ per cent. from the specified size may be cause for rejection.

Steel Castings.

144. Steel castings will be used for drawbridge wheels, track segments and gearing. (Art. 1.)

They must be true to form and dimensions, of a workmanlike finish and free from injurious blowholes and defects. All castings must be annealed.

When tested in specimens of uniform sectional area of at least ½ square inch for a distance of 2 inches, they must show an ultimate strength of not less than 67,000 pounds per square inch, an elastic limit of one half the ultimate, and an elongation in 2 inches of not less than 10 per cent.

The metal must be uniform in character, free from hard or soft spots, and be capable of being properly tool finished.

Cast Iron.

145. Except where cast steel or chilled iron is required, all castings must be of tough, gray iron, free from cold shuts or injurious blowholes, true to form and thickness, and of a workmanlike finish. Sample pieces, 1 inch square, cast from the same heat of metal in sand molds, shall be capable of sustaining, on a clear span of 12 inches, a central load of 2400 pounds, when tested in the rough bar. A blow from a hammer shall produce an indentation on a rectangular edge of the casting without flaking the metal.

Art. 109.—General Specifications for Steel Railroad Bridges of the American Bridge Company, 1900.

The following principal clauses are taken from the latest issue of the General Specifications of the American Bridge Company:

General Description.

1. *Material.*—All structures to be wholly of rolled steel as specified below. Cast iron or cast steel will be permitted only in machinery of movable bridges and in special cases for shoes and bearings.

2. *Type of Bridges.*—The following types of bridges are recommended:

For spans up to 20 ft.—Rolled beams or longitudinal trough floors.

For spans from 20 to 100 ft.—Plate girders.

For spans from 100 to 140 ft.—Lattice girders.

For spans over 140 ft.—Pin-connected trusses.

3. *Clearance.*—On straight line a clear section shall be provided to conform to given requirements. The width must be increased so as to allow the same minimum clearance on curves and on double track.

4. *Spacing of Trusses.*—The width between centres of trusses shall in no case be less than $\frac{1}{20}$ of the span between centres of end pins.

5. *Spacing of Deck-plate Girders.*—Deck-plate girders shall be spaced generally $6\frac{1}{2}$ feet between centres.

6. *Floor Beams.*—All floor beams in through bridges shall be riveted between the posts, above or below the pin.

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15. *Effect of Impact.*—The effect of impact and vibration shall be added to the maximum strains resulting from the

above-mentioned live load, and is to be determined by the following formula,

$$I=S\left(\frac{300}{L+300}\right),$$

where I = impact to be added to the live-load strain;
S = calculated maximum live-load strain;
L = length of loaded distance in feet which produces the maximum strain in member.

16. *Wind Pressure.*—The wind pressure shall be assumed acting in either direction horizontally:

First. At 30 pounds per square foot on the exposed surface of all trusses and the floor as seen in elevation, in addition to a train of 10 feet average height, beginning 2 feet 6 inches above base of rail, moving across the bridge.

Second.—At 50 pounds per square foot on the exposed surface of all trusses and the floor system. The greatest result shall be assumed in proportioning the parts.

17. For determining the requisite anchorage for the loaded structure, the train shall be assumed to weigh 800 pounds per lineal foot.

18. *Momentum of Train.*—For longitudinal bracing of trestle towers and similar structures, the momentum produced by suddenly stopping the train shall be considered; the coefficient of friction of wheels sliding upon the rails being assumed as 0.2.

19. *Centrifugal Force of Train.*—When the structure is on a curve, the additional effects due to the centrifugal force of as many trains as there are tracks shall be considered and calculated by the following formula:

$$C=0.03\ WD \text{ for a curvature up to 5 degrees,}$$

where C = centrifugal force in pounds;
W = weight of train in pounds;
D = degree of curvature.

The coefficient for centrifugal force (0.03) shall be reduced 0.001 for every degree of curvature above 5 degrees.

PROPORTION OF PARTS.

20. *Least Thickness of Material.*—For main members and their connections no material shall be used less than $\frac{3}{8}$ of an inch thick, and for laterals and their connections not less than $\frac{5}{16}$ of an inch thick except for lining or filling vacant spaces.

21. *Permissible Tensile Strains.*—All parts of the structure shall be so proportioned that the sum of the maximum loads, together with the impact, shall not cause the tensile strain to exceed—

On soft steel, 15,000 pounds per square inch.

On medium steel, 17,000 pounds per square inch.

22. The same limiting unit strains shall also be used for members strained by wind pressure, centrifugal force, or momentum of train.

23. Net sections must be used in all cases in calculating tension members, and, in deducting rivet-holes, they must be taken $\frac{1}{8}$ of an inch larger than the size of the rivets.

24. Pin-connected riveted tension members shall have a net section through the pin-hole 25 per cent. in excess of the net section of the body of the member. The net section back of the pin-hole shall be at least 0.75 of the net section through the pin-hole.

25. *Permissible Compressive Strains.*—For compression members, these permissible strains of 15,000 and 17,000 pounds per square inch shall be reduced in proportion to the ratio of the length to the least radius of gyration of the section by the following formulæ:

$$\text{For soft steel,} \quad p = \frac{15,000}{1 + \frac{l^2}{13,500r^2}};$$

For medium steel, $$p = \frac{17,000}{1 + \frac{l^2}{11,000r^2}};$$

where p = permissible working strain per square inch in compression;

l = length of piece in inches, centre to centre of connection;

r = least radius of gyration of the section in inches.

26. No compression member, however, shall have a length exceeding 100 times its least radius of gyration, excepting those for wind bracing, which may have a length not exceeding 120 times the least radius of gyration.

27. The least width of posts in pin-connected bridges shall be limited to 10 inches.

28. *Alternate Strains.*—Members subject to alternate strains of tension and compression in immediate succession (as counter stresses in web members or chords in continuous trusses) shall be so proportioned that the total sectional area is equal to the sum of areas required for each strain.

29. *Combined Strains.*—In case the maximum strains due to wind and centrifugal force, added to the maximum strains due to vertical loading (including impact), shall exceed the following limits:

On soft steel, 19,000 pounds per square inch.

On medium steel 21,000 pounds per square inch, properly reduced for compression, addition must be made to such sections until these limits are not exceeded.

The permissible strains for the connections shall be increased proportionately.

30. Should the strains be reversed in any possible case, proper provision must be made for such strains in the opposite direction.

31. *Transverse Loading of Tension or Compression Members.*—When the floor system rests directly on the top or bottom chord, the latter must be so proportioned that the algebraic sum of the strains per square inch on the outer fibre, resulting from the direct compression or tension, and three fourths of the maximum bending moment (the chord being considered as a beam of one panel length, supported at the ends), shall not exceed the before-mentioned limiting strains in tension or compression, the proper amount of impact being added to each kind of loading.

32. The bending moment at panel points shall be assumed equal to that in the centre, but in opposite direction.

33. All other members which are subject to direct strain in addition to bending moment are to be similarly calculated.

34. *Shearing and Bearing Strains.*—The shearing strain on rivets, bolts, or pins, per square inch of section, shall not exceed 11,000 pounds for soft steel, and 12,000 pounds for medium steel; and the pressure upon the bearing surface of the projected semi-intrados (diameter × thickness) of the rivet-, bolt-, or pin-hole, shall not exceed 22,000 pounds per square inch for soft steel and 24,000 pounds for medium steel.

35. In field-riveting the number of rivets thus found shall be increased 25 per cent. if driven by hand, but 10 per cent. if driven by power.

36. *Bending Strains on Pins.*—The bending strain on the extreme fibre of pins shall not exceed 22,000 pounds per square inch for soft steel, and 25,000 per square inch for medium steel, when centres of bearings of the strained members are taken as the points of application of the strains.

37. *Plate Girders.*—Girders shall be proportioned on the assumption that ⅛ of the gross area of the web is available as flange area. The compressed flange shall have the same

sectional area as the tension flange; but the unsupported length of flange shall not exceed 16 times its width.

38. In calculating shearing strains and bearing strains on web rivets of plate girders, the whole of the shear action on the side next the abutment is to be considered as being transferred into the flange angles in a distance equal to the depth of the girder.

39. The shearing strain in web plates shall not exceed 9,000 pounds per square inch for soft steel, and 10,000 pounds per square inch for medium steel; but no web plate shall be less than $\frac{3}{8}$ of an inch in thickness.

40. The web shall have stiffeners riveted on both sides, with a close bearing against upper and lower flange angles at the ends and inner edges of bearing plates, and at all points of local and concentrated loads, and also, when the thickness of the web is less than $\frac{1}{60}$ of the unsupported distance between flange angles, at points throughout the length of the girder, generally not farther apart than the depth of the full web plate, with a maximum limit of 5 feet.

41. *Provision for Future Increase of Live Load.*—Wherever the live- and dead-load strains are of opposite character, only 70 per cent. of the dead-load strain shall be considered as effective in counteracting the live-load strain.

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53. *Bolsters and Expansion Rollers.*—All bridges exceeding 80 feet in length shall have hinge bolsters on both ends and at one end nests of turned friction rollers, running between planed surfaces. Rollers will not be less than 4 inches in diameter; and the pressure per lineal inch of roller, including impact, shall not exceed $1200\sqrt{d}$ for steel rollers between steel surfaces (d = diameter of roller in inches).

54. *Friction Plates.*—For bridges less than 80 feet in length one end shall be free to move upon planed surfaces.

55. *Bed-plates.*—Bed-plates shall be so proportioned that

the pressure upon masonry (including impact) will not exceed 400 pounds per square inch.

56. *Rivets.*—The pitch of rivets, in the direction of the strain, shall never exceed 6 inches, nor 16 times the thickness of the thinnest outside plate connected, and not more than 40 times that thickness at right angles to the strain.

57. At the ends of compression members the pitch shall not exceed four diameters of the rivet, for a length equal to twice the width of the member.

58. The distance from the edge of any piece to the centre of a rivet-hole must not be less than 1½ times the diameter of the rivet, nor exceed 8 times the thickness of the plate; and the distance between centres of rivet-holes shall not be less than 3 diameters of the rivet.

59. *Splices.*—All joints in riveted work, whether in tension or compression members, must be fully spliced. The sections of compression chords shall be connected at the abutting ends by splices sufficient to hold them truly in position.

60. *Tie-plates.*—All segments of compression members connected by latticing only, shall have tie-plates placed as near the end as practicable. They shall have a length of not less than the greatest depth or width of the member, and a thickness not less than $\frac{1}{50}$ of the distance between the rivets connecting them to the compressed members.

61. *Lacing.*—Single lattice bars shall have a thickness of not less than $\frac{1}{40}$, and double bars connected by a rivet at the intersection of not less than $\frac{1}{60}$ of the distance between the rivets connecting them to the member; and their width shall be:

For 15-inch channels, or built sections with 3½- and 4-inch angles..........	2½ inches (⅞-inch rivets.)
For 12- and 10-inch channels, or built sections with 3-inch angles.......................... . .	2¼ inches (¾-inch rivets).
For 9- and 8-inch channels, or built sections with 2½-inch angles..........................	2 inches (⅝-inch rivets).

62. *Pin-plates.*—All pin-holes shall be re-enforced by additional material when necessary, so as not to exceed the allowed pressure on the pins. These re-enforcing plates must contain enough rivets to transfer the proportion of pressure which comes upon them, and at least one plate on each side shall extend not less than 6 inches beyond the edge of the tie-plate.

63. *Web Splices.*—Web plates of girders must be spliced at all joints by a plate on each side of the web not less than ⅜ of an inch thick, capable of transmitting the full strain through splice-rivets.

64. *Flange Plates.*—The flange plates of all girders must be limited in width so as not to extend beyond the outer lines of rivets connecting them with the angles, more than five inches or more than eight times the thickness of the first plate. Where two or more plates are used on the flanges they shall either be of equal thickness or shall decrease in thickness outward from the angles.

Workmanship.

65. *Riveted Work.*—All riveted work shall be punched accurately with holes $\frac{1}{16}$ of an inch larger than the size of the rivet, and when the pieces forming one built member are put together the holes must be truly opposite; no drifting to distort the metal will be allowed; if the hole must be enlarged to admit the rivet, it must be reamed.

66. All holes for field rivets, excepting those in connections for lateral and sway bracing, shall be accurately drilled to an iron templet or reamed while the connecting parts are temporarily put together.

67. *Planing and Reaming.*—In medium steel over ⅝ of an inch thick all sheared edges shall be planed, and all holes shall be drilled or reamed to a diameter of ⅛ of an

inch larger than the punched holes, so as to remove all the sheared surface of the metal.

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78. All eye-bars shall be annealed.

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81. Pin-holes shall be bored truly parallel with one another and at right angles to the axis of the member unless otherwise shown in drawings; and in pieces not adjustable for length no variation of more than $\frac{1}{64}$ of an inch for every 20 feet will be allowed in the length between centres of pin-holes.

82. Bars which are to be placed side by side in the structure shall be bored at the same temperature, and shall be of such equal length that, upon being piled on each other, the pins shall pass through the holes at both ends at the same time without driving.

83. All pins shall be accurately turned to a gauge, and shall be straight and smooth.

84. The clearance between pin and pin-hole shall be $\frac{1}{32}$ of an inch for all lateral pins; and for truss pins the clearance shall be $\frac{1}{50}$ of an inch for pins $3\frac{1}{2}$ inches in diameter, which amount shall be gradually increased to $\frac{1}{32}$ of an inch for pins 6 inches in diameter and over.

85. All pins shall be supplied with steel pilot nuts, for use during erection.

86. All workmanship shall be first-class in every particular.

Steel.

87. *Process of Manufacture.*—All steel must be made by the open-hearth process, and if by acid process, shall contain not more than .08 per cent. of phosphorus, and if by basic process, not more than .05 per cent. of phosphorus, and must be uniform in character for each specified kind.

88. *Finish.*—The finished bars, plates, and shapes must

be free from injurious seams, flaws, or cracks, and have a clean, smooth finish.

89. *Test Pieces.*—The tensile strength, limit of elasticity, and ductility, shall be determined from a standard test-piece, cut from the finished material, of at least ½ square inch section. All broken samples must show a silky fracture of uniform color.

90. *Annealed Test-pieces.*—Material which is to be used without annealing or further treatment is to be tested in the condition in which it comes from the rolls. When material is to be annealed or otherwise treated before use the specimen representing such material is to be similarly treated before testing.

91. *Marking.*—Every finished piece of steel shall be stamped with the blow number identifying the melt.

92. *Physical Properties.*—Steel shall be of three grades: *Rivet*, *Soft*, and *Medium*.

93. *Rivet Steel.*—Rivet steel shall have: Ultimate strength, 48,000 to 58,000 pounds per square inch. Elastic limit, not less than one half the ultimate strength. Elongation, 26 per cent. Bending test, 180 degrees flat on itself, without fracture on outside of bent portion.

94. *Soft Steel.*—Soft steel shall have: Ultimate strength, 52,000 to 62,000 pounds per square inch. Elastic limit, not less than one half the ultimate strength. Elongation, 25 per cent. Bending test, 180 degrees flat on itself, without fracture on outside of bent portion.

95. *Medium Steel.*—Medium steel shall have: Ultimate strength, 60,000 to 70,000 pounds per square inch. Elastic limit, not less than one half the ultimate strength. Elongation, 22 per cent. Bending test, 180 degrees to a diameter equal to thickness of piece tested, without fracture on outside of bent portion.

96. *Full-size Test of Steel Eye-bars.*—Full-size test of

steel eye-bars shall be required to show not less than 10 per cent. elongation in the body of the bar, and tensile strength not more than 5000 pounds below the minimum tensile strength required in specimen tests of the grade of steel from which they are rolled. The bars will be required to break in the body, but should a bar break in the head, but develop 10 per cent. elongation and the ultimate strength specified, it shall not be cause for rejection, provided not more than one third of the total number of bars tested break in the head; otherwise the entire lot will be rejected.

97. *Pin-steel.*—Pins made of either of the above-mentioned grades of steel shall, on specimen test pieces cut from finished material, fill the requirements of the grade of steel from which they are rolled, excepting the elongation, which shall be decreased 5 per cent. from that specified.

98. *Drifting.*—Punched rivet-holes, pitched two diameters from a sheared edge, must stand drifting until the diameter is one third larger than the original hole, without cracking the metal.

99. *Slabs for Plates.*—The slabs for rolling plates shall be hammered or rolled from ingots of at least twice their cross-section.

100. *Pins.*—Pins up to 7 inches diameter shall be rolled.

101. Pins exceeding 7 inches diameter shall be forged under a steel hammer striking a blow of at least 5 tons. The blooms to be used for this purpose shall have at least three times the sectional area of the finished pins.

102. *Variation in Weight.*—A variation in cross-section or weight of rolled material of more than 2½ per cent. from that specified may be cause for rejection.

103. *Steel Castings.*—Steel castings shall be made of open-hearth steel containing from $\frac{25}{100}$ to $\frac{40}{100}$ per cent.

carbon and not over $\frac{8}{100}$ per cent. of phosphorus, and shall be practically free from blow-holes.

104. *Cast Iron.*—Except where chilled iron is specified, all castings shall be of tough, gray iron, free from injurious cold-shuts or blow-holes, true to pattern, and of workmanlike finish. Test bars one inch square, loaded in middle between supports 12 inches apart, shall bear 2500 pounds or over, and deflect 0.15 of an inch before rupture.

Art. 110.—Standard Specifications for Steel Bridges of the Pennsylvania Railroad Company.

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SECTION III.—DETERMINATION OF SECTIONS.

13. *Strain-sheet.*—The calculation of stresses produced by the above-mentioned live and dead loads will determine the following values for each member:

M = maximum calculated stress in member (compression or tension);

m = (1) minimum calculated stress in members subjected to one kind of stress only (all compression or all tension), or (2) maximum calculated stress of lesser kind, in members subjected to reversal of stress.

NOTE.—Minimum stress is understood to mean the absolute minimum; i.e., in a main diagonal or post of a single span m equals the calculated dead-load stress minus the maximum calculated counter-stress due to live load.

$$R = \frac{m}{M}.$$

14. *Stress-increment due to Impact, etc.*—The maximum calculated stress (M) in each member shall be multiplied by the coefficient $(1+k)$, and the resultant quantity, $M(1+k)$, shall be regarded as the equivalent static stress in the member. (For the value of k see Tables B and C, page 836.)

15. *Permissible Tensile Unit Stress.*—All members shall be so proportioned that the stress $M(1+k)$ shall not cause the tensile unit stress to exceed 15,000 pounds, nor the compressive unit stress to exceed 15,000 pounds properly reduced in accordance with Clause 16.

16. *Permissible Compressive Unit Stress.*—For compression members the unit stress of 15,000 pounds per square inch shall be reduced in proportion to the ratio of the length to the least radius of gyration of the section, by the following formula:

$$p = \frac{15,000}{1 + \frac{l^2}{13,500r^2}},$$

where p = permissible working stress per square inch in compression;

l = length of piece in inches between centres of connections;

r = least radius of gyration of section in inches.

17. *Section of Hip-verticals and Hangers.*—The net section of the long hip-verticals of through bridges shall be 25 per cent. in excess of the above requirements (see Clause No. 15), all details of these members being correspondingly strengthened. Short floor-beam hangers will be required to have 50 per cent. excess of strength.

18. *Unit Stresses for Wind Bracing.*—The same limiting unit stresses shall also be used for members strained by wind pressure or momentum of moving train, the stress-increment being neglected in these cases.

19. *Treatment of Stress due to Centrifugal Force.*—The stress due to centrifugal force shall be regarded as live load, and, when necessary, additions shall be made to the sections of truss chords or girder flanges until the unit

stress does not exceed 17,000 pounds in tension nor 17,000 pounds, properly reduced, compression.

In lateral bracing the stress due to centrifugal force shall be increased 50 per cent. and the unit stresses provided for in Clause 18 shall be used.

In the case of deck structures, when the curvature exceeds six degrees, the lower lateral bracing shall be designed to carry half the stress due to centrifugal force, and properly designed sway bracing shall be introduced to transfer the stress from the upper to the lower system.

20. *Limiting Value of* $\frac{l}{r}$.—No compression member shall have a length exceeding 100 times its least radius of gyration, excepting wind bracing, in which the length may be 120 times the least radius of gyration.

21. *Increase of Section required by Wind Stress, etc.*—In case the maximum stresses in chords of bridges or posts of trestle towers due to wind and momentum of train, added to the maximum stresses from vertical loading and centrifugal force, properly increased, shall exceed 19,000 pounds per square inch (properly reduced for buckling in the case of compression members), additions must be made to the sections until this limit is not exceeded.

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25. *Stability under Increased Live-loads.*—To insure the stability of bridges under increased live loads, a live load shall be assumed 100 per cent. greater than that previously provided for in this specification. If the resultant stress, $M(1+k)$, produces a stress per square inch in any member more than twice the permissible unit stress previously specified, additions must be made to the sections until that limit is not exceeded. Counters, having in no case less than one and one-half square inches of section, must be provided where required by the increased live load; and in case of

reversal of stress the member must be properly designed to resist such reversal.

26. *Bending Moment in Pins and Bolts.*—Pins and bolts are to be so proportioned that the maximum stress in the extreme fibres due to bending moment shall not exceed 22,000 pounds per square inch for soft steel nor 25,000 pounds per square inch for pin-steel, the centres of bearings of the members connected by the pin or bolt being taken as points of application of bending strains.

27. *Bearing Pressures, Pins, and Rivets.*—The bearing pressure of pins, bolts, or rivets upon the projected semi-intrados (diameter × thickness) of the hole shall not exceed 26,000 pounds per square inch for rivets nor 22,000 pounds per square inch for pins and bolts.

28. *Shear in Rivets, Pins, and Bolts.*—The shearing stress per square inch of section on rivets, bolts, or pins shall not exceed 11,000 pounds for soft steel nor 12,000 pounds for pin-steel.

29. *Net Sections. Tension Members.*—Net sections must be used in all cases in proportioning tension members, and in deducting rivet-holes they shall be taken one eighth of an inch larger in diameter than the nominal size of the rivets.

30. *Field-riveted Connections.*—In the field-riveted connections of track-stringers and floor-beams the number of rivets determined by the foregoing rules must be increased 33⅓ per cent. in all cases. In all other field-riveted joints a 25 per cent. excess of rivets will be required if hand-driven and a 10 per cent. excess if satisfactory power rivets are used.

31. *Value of Countersunk Rivets.*—Rivets with countersunk heads shall be assumed to have three fourths the value of corresponding rivets with full heads.

32. *Riveted Tension Members and Section at Pin-holes.*—Riveted tension members shall have a section through the pin-hole 25 per cent. in excess of the net section of the body of the member. The section back of the pin-hole shall be at least 75 per cent. of the section through the pin-hole.

33. *Same Gross Section in Both Flanges of Plate Girders.*—No allowance shall in general be made for the web as resisting any of the bending moment in plate girders. The compression flange shall have the same gross sectional area as the tension flange. The unsupported length of the compression flange shall not exceed twelve times its width.

34. *Shear in Web Rivets. Plate Girders.*—In calculating the shearing stresses and bearing stresses in the web rivets of plate girders, the whole of the shear, with its proper increment, acting on the side of the panel next the abutment is to be considered as transferred into the flange angles in a distance equal to the depth of the girder.

35. *Extra Rivets. Top Flanges of Plate Girders.*—In the case of the rivets connecting the upper flange angles with the web in deck girders carrying the floor directly on the top flanges, allowance must be made for the concentrated load of a 12½-ton driver, which shall be considered as distributed over three ties.

36. *Thickness of Web Plates.*—The thickness of web plates shall be such that the maximum calculated shear, with its proper increment, shall not cause the shearing stress per square inch of net section of the web to exceed 13,000 pounds; but no web plate shall be less than three eighths of an inch in thickness.

37. *Rolled Beams and Channels as Girders.*—Rolled beams and channels subjected to bending stresses shall be so proportioned that the stress per square inch in the outer fibres, deduced from the calculated bending moment (with

its proper increment) and the moment of inertia of the section, shall not be more than 14,000 pounds. The unsupported length of the top flanges of such beams and channels shall in no case be greater than twelve times the flange width.

38. *Minimum Sections.*—For main members and their connections no material shall be used of less thickness than three eighths of an inch; and for laterals and their connections no materials shall be used of less thickness than five sixteenths of an inch. Material of less thickness will be permitted only for lining or filling vacant spaces.

No material used in compression shall have an unsupported width of more than fifty times its thickness.

No lateral or sway rod shall be used having less than one square inch of section.

* * * * * * *

47. *Expansion Rollers.*—All bridges 80 feet or more in length shall have at one end nests of turned friction rollers running between planed surfaces. Rollers shall not be less than 3 inches in diameter, and the pressure per lineal inch of roller, including the proper increment, shall not exceed $1200\sqrt{d}$ (d = diameter of rollers in inches).

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48. *Sliding Bearings.*—Bridges less than 80 feet in length shall be free to move at one end on planed surfaces.

* * * * * * *

50. *Assumed Temperature Variation.*—Provision shall be made for a free expansion and contraction of the completed structure corresponding to a variation of temperature of 150 degrees Fahrenheit.

51. *Pressure on Masonry.*—Bed-plates shall be so proportioned that the pressure upon masonry, including the

proper increment, shall not exceed 400 pounds per square inch.

* * * * * * *

55. *Web Splices. Plate Girders.*—Web plates of girders must be spliced at all joints with a plate on each side of the web capable of transmitting the full shearing stress through the splice-rivets.

56. *Rivet Spacing. Compression Members.*—In members subject to compression, rivets shall be so spaced that they shall not be farther apart in the direction of the stress than sixteen times the thickness of the thinnest external plate connected, and not more than fifty times that thickness at right angles to the direction of stress.

57. *Rivet Pitch at Ends of Compression Members.*—At the ends of compression members the pitch shall not exceed four diameters of the rivet, for a distance equal to twice the greatest width of the member.

58. *Splices.*—All joints of riveted work, whether in tension or compression members, must be fully spliced.

59. *Limiting Pitch of Rivets, Distance from Edge of Plate, etc.*—The distance from the edge of any piece to the centre of a rivet-hole must not be less than one and one half times the diameter of the rivet, nor exceed eight times the thickness of the plate; the distance between centres of rivet-holes shall not be less than three diameters of the rivet.

60. *Tie-plates.*—All segments of compression members connected by lacing only shall have tie-plates placed as near the ends as practicable. The tie-plates shall have a length not less than the greatest width of the member, and a thickness not less than one fortieth of the distance between the lines of connecting rivets, measured at right angles to the length of the member.

61. *Lattice Bars.*—Single lattice bars shall have a thickness of not less than one fortieth and double bars connected

by a rivet at the intersection of not less than one sixtieth of the distance between the rivets connecting them to the member; and their width shall be:

For 15-inch channels, or built sections with 3½- or 4-inch angles........................	2½ inches (⅞-inch rivets).
For 12- and 10-inch channels, or built sections with 3-inch angles........................	2¼ inches (¾-inch rivets).
For 9- and 8-inch channels, or built sections with 2½-inch angles........................	2 inches (⅝-inch rivets).

The distance between connections of the lattice bars shall not exceed eight times the least width of the segments connected.

62. *Pin-plates.*—All pin-holes shall be reinforced with additional material when necessary, so that the permissible pressure on pins shall not be exceeded. These reinforcing plates must contain enough rivets to transfer the proportion of pressure which comes upon them in accordance with the previously stated rules for proportioning rivets.

63. *Wide Girder-flanges requiring Four Rows of Rivets.*—Flanges of plate girders running over 14 inches in width, or projecting more than 3 inches beyond the edge of flange angles, shall have at least four lines of rivets.

64. *Length of Cover-plates. Plate Girders.*—In all plate girders having cover-plates at least one plate on each flange shall extend from end to end of the same, and, in general, cover-plates shall be made of such lengths as to allow of at least two rows of rivets of the regular pitch being placed at each end of the plate, in addition to those theoretically required.

65. *Web-stiffeners. Plate Girders.*—The webs of plate girders shall have stiffeners riveted on both sides, with a close bearing against upper and lower flange angles, at the ends and inner edges of bearing-plates, and at all points of local and concentrated loads; and also when the thickness of web is less than one sixtieth of the unsupported distance

between flange angles, at points throughout the length of the girder, generally not farther apart than the depth of the full web plate, with a maximum limit of 5 feet.

PART SECOND.—MATERIALS AND WORKMANSHIP.

SECTION I.—MATERIALS.

A. *Rolled Steel.*

1. *Character of Rolled Steel.*—In general, soft steel will be used in all parts of the work. For pins, lateral bolts, and expansion rollers, however, medium steel will be used. All steel must be made by the open-hearth process, and may be either basic or acid, at the discretion of the chief engineer.

2. *Phosphorus Limit. Acid Open-hearth Steel.*—If made in an acid furnace, the maximum allowable amount of phosphorus in the finished product shall be six hundredths of one per cent

3. *Phosphorus Limit. Basic Open-hearth Steel.*—If made in a basic furnace, the maximum allowable amount of phosphorus in the finished product shall be four hundredths of one per cent.

4. *Surface Requirements.*—The finished product shall be perfect in all parts and free from irregularities and surface imperfections of all kinds. All steel must be free from piping.

5. *Permissible Excess of Weight.*—No difference of more than 2½ per cent. from the section shown on the plans will be permitted, except in the case of extra-wide plates.

6. *Marking Finished Pieces.*—Every finished plate, bar, or angle shall be plainly stamped on one side, near the middle, with a number identifying the melt. Steel for pins shall have the melt numbers stamped on the end. Rivet-

steel, and small pieces not forming part of the calculated section of members, may be shipped in bundles wired together, with the melt number on a metal tag attached.

B. *Cast Steel.*

7. *Cast Steel. General.*—Cast steel shall be made in an open-hearth furnace, and shall fulfil the following requirements:

(*a*) *Annealing of Steel Castings.*—All steel castings shall be annealed.

(*b*) *Tests of Steel Castings.*—Every steel casting shall be made with a coupon for testing, which coupons shall be cut off after annealing, and the test shall be made from a ¾-inch round cut from the coupon. The test piece shall show an ultimate strength of at least 65,000 pounds, an elastic limit of not less than 33,000 pounds, an elongation of at least 15 per cent. in two inches, and a reduction of area of 20 per cent. at the point of fracture.

(*c*) *Soundness of Steel Castings.*—When the bearing surface of any steel casting is finished there shall be no blowholes visible exceeding one inch in either dimension, nor exceeding one half a square inch in area. The length of blow-holes cut by any straight line laid in any direction shall never exceed one inch in any one foot.

C. *General Tests.*

8. *Sample Bar.*—A sample bar not more than 2 inches wide, and having a sectional area of not less than ½ square inch, shall be cut from the finished product of every melt. When taken from metal more than two inches thick the sample may be a turned, round bar. The laboratory tests shall be made on this sample bar in its natural state, without annealing.

9. *Varieties of Material to be Tested.*—When a melt is rolled into several varieties of material, each variety shall be separately tested. A variety shall consist of one of the following shapes: Sheared Plates, Universal Mill Plates, Beams, Angles, Channels, Z Bars, Flats, Rounds, Pin-steel, Eye-bar Steel.

10. *Measured Length of 8 inches for Elongation Test.*—In the laboratory tests, measurements to determine elongation shall be made on an original length of eight inches.

11. *Bending Test.*—A piece of each sample bar shall be bent cold 180°, and closed up against itself. In the case of "pin-steel" the test shall be considered satisfactory if no crack nor flaw appear on the outside of the bent portion until the diameter of the circle around which the specimen is bent has become less than the thickness of the sample bar. Samples of "soft steel" will be further required to close up on themselves without developing any crack or flaw on the outside of the bent portion before the test shall be considered satisfactory.

12. *Drifting Test.*—The ductility of the metal must be such that a punched hole seven eighths inch in diameter, the centre of which is not more than one and one-half inches from the sheared or rolled edge of the piece, may be enlarged by drifting to a diameter 50 per cent. greater than the original hole without cracking the specimen at any point.

13. *Ultimate Strength, Elastic Limit, etc.*—The sample bar shall be tested in a lever machine, and shall fulfil the following requirements:

	Ultimate Strength.	Elastic Limit.	Elongation.	Reduction of Area.
Pin steel......	62,000 to 70,000 lbs.	33,000 lbs.	17 per cent.	40 per cent.
Soft steel......	52,000 to 62,000 "	28,000 "	25 " "	50 " "
Rivet-steel.....	48,000 to 56,000 "	28,000 "	28 " "	56 " "

14. *Fracture.*—The entire fracture shall be silky.

15. *Minimum Limits.*—The requirements for Elastic Limit, Elongation, and Reduction of Area are minima, and no steel will be accepted which fails to meet these requirements, except as provided in Clause 16.

16. *Duplicate Tests, when Allowed.*—Duplicate tests may be made when the sample tested fulfils five of the six requirements. If the second test and also the average of both tests meet all the requirements, the melt may be accepted.

17. *Chemical Analyses.*—Analyses shall be made showing the amount of phosphorus, carbon, sulphur, silicon, and manganese whenever required, the drillings for these analyses being taken directly from the finished material.

D. *Full-sized Eye-bar Tests.*

18. *Selection of Eye-bars for Test.*—The eye-bars required for full-sized tests and those required for the structure shall be made at one time. The test bars shall be selected by the inspector and must be fair average specimens of those which would be classed as good bars, acceptable for the work. No bar that is known to be defective in any way shall be selected for testing.

19. *Elastic Limit and Ultimate Strength of Test Bars.*—The test bars shall show an elastic limit of not less than 27,000 pounds and an ultimate strength of not less than 48,000 pounds per square inch of section.

20. *Elongation of Test Bars.*—The test bars will be required to develop an average stretch of 16 per cent. and a minimum stretch of 14 per cent. before breaking, the elongation to be measured on a gauged length of 10 feet, including the fracture.

21. *Minimum Limit of Elongation.*—The specified elongations are minima, and a failure in these requirements will

TABLE B.

COEFFICIENTS OF STRESS-INCREMENTS.

Case 1.—Stresses of one kind only—all compression or all tension.

$$k=\frac{1-R}{1+R}.$$

R	k	R	k	R	k	R	k	R	k
1.00	0.000	0.79	0.117	0.58	0.266	0.37	0.460	0.16	0.724
.99	.005	.78	.124	.57	.274	.36	.471	.15	.739
.98	.010	.77	.130	.56	.282	.35	.482	.14	.754
.97	.015	.76	.136	.55	.290	.34	.493	.13	.770
.96	.020	.75	.143	.54	.299	.33	.504	.12	.786
.95	.026	.74	.149	.53	.307	.32	.515	.11	.802
.94	.031	.73	.156	.52	.316	.31	.527	.10	.818
.93	.036	.72	.163	.51	.325	.30	.538	.09	.835
.92	.042	.71	.170	.50	.333	.29	.550	.08	.852
.91	.047	.70	.176	.49	.342	.28	.563	.07	.869
.90	.053	.69	.183	.48	.351	.27	.575	.06	.887
.89	.058	.68	.190	.47	.361	.26	.587	.05	.905
.88	.064	.67	.198	.46	.370	.25	.600	.04	.923
.87	.070	.66	.205	.45	.379	.24	.613	.03	.942
.86	.075	.65	.212	.44	.389	.23	.626	.02	.961
.85	.081	.64	.220	.43	.399	.22	.639	.01	.980
.84	.087	.63	.227	.42	.408	.21	.653	.00	1.000
.83	.093	.62	.235	.41	.418	.20	.667	—	—
.82	.099	.61	.242	.40	.429	.19	.681	—	—
.81	.105	.60	.250	.39	.439	.18	.695	—	—
.80	.111	.59	.258	.38	.449	.17	.709	—	—

TABLE C.

COEFFICIENTS OF STRESS-INCREMENTS.

Case 2.—Stresses subject to reversal.

$$k=\frac{2+R}{2-R}.$$

R	k	R	k	R	k	R	k	R	k
0.00	1.000	0.21	1.235	0.42	1.532	0.63	1.920	0.84	2.448
.01	1.010	.22	1.247	.43	1.548	.64	1.941	.85	2.478
02	1.020	.23	1.260	.44	1.564	.65	1.963	.86	2.509
.03	1.030	.24	1.273	.45	1.581	.66	1.985	.87	2.540
.04	1.041	.25	1.286	.46	1.597	.67	2.007	.88	2.571
.05	1.051	.26	1.290	.47	1.614	.68	2.030	.89	2.603
.06	1.062	.27	1.312	.48	1.632	.69	2.053	.90	2.636
.07	1.073	.28	1.325	.49	1.649	.70	2.077	.91	2.670
.c8	1.083	.29	1.339	.50	1.667	.71	2.101	.92	2.704
.09	1.094	.30	1.353	.51	1.685	.72	2.125	.93	2.738
.10	1.105	.31	1.367	.52	1.703	.73	2.150	.94	2.773
.11	1.116	.32	1.381	.53	1.721	.74	2.175	.95	2.809
.12	1.128	.33	1.395	.54	1.740	.75	2.200	.96	2.846
.13	1.139	.34	1.409	.55	1.759	.76	2.226	.97	2.884
.14	1.150	.35	1.424	.56	1.778	.77	2.252	.98	2.922
.15	1.162	.36	1.439	.57	1.797	.78	2.279	.99	2.960
.16	1.174	.37	1.454	.58	1.817	.79	2.306	1.00	3.000
.17	1.186	.38	1.460	.59	1.837	.80	2.333	—	—
.18	1.198	.39	1.484	.60	1.857	.81	2.361	—	—
.19	1.210	.40	1.500	.61	1.878	.82	2.390	—	—
.20	1.222	.41	1.516	.62	1.899	.83	2.419	—	—

be sufficient cause for condemning the bars represented by the test.

22. *Location of Fractures.*—In general bars will be required to break in the body. When a bar breaks in the head, but develops 14 per cent. elongation before breaking, a second bar shall be selected from the same lot. If this bar breaks in the body and the average elongation of the two bars is not less than 16 per cent., the bars of this lot may be accepted.

23. *Condemnation if Bars Break in Head.*—If more than one third of all the bars tested break in the head, this shall be deemed sufficient cause for the rejection of the entire lot of eye-bars.

Art. 111.—Specifications for the Steel Cables, Suspenders, etc., of the New East River Bridge at New York City.

* * * * * * *

Wire.

57. *Ultimate Strength and Elongation.*—The wire for the cables and for the suspenders and ties must have an ultimate strength of 200,000 pounds or more to the square inch, and must have an elongation under test of at least 2½ per cent. in five feet of observed length and of at least 5 per cent. in eight inches of observed length.

58. *Bending Test.*—It must be capable of being coiled cold around a rod of its own diameter without cracking.

59. *Size.*—All wire will be bright and, for the cables, it will be of size No. 6, Roebling wire gauge, not straightened by machine, but drawn absolutely straight and free from any tendency to coil when unrolled from the drums.

60. *Lengths of Wire and Size of Drums.*—It shall be made in lengths of not less than 4000 feet and shipped to

the site on drums of such diameters as not to cause any tendency to coil.

61. *Splices.*—All the wires of each cable strand will be spliced so as to form one continuous wire. The splices must be made so as to have a strength of at least 95 per cent. of the strength of the wire. The splice which will accomplish this result with the least increase in size above that of the wire will be approved by the engineer. The right and left sleeve-nut splice, shown on the drawings and used for the cables of the New York and Brooklyn Bridge, is the only one known to the engineer which will give the required strength.

62. *Distribution of Splices.*—The lengths of the wires must be arranged so that the splices will be uniformly distributed throughout the cables.

Steel for Wire.

24. *Furnace.*—All steel for wire shall be made in an open-hearth furnace lined with silica.

25. *No Scrap Allowed.*—This steel shall be made entirely from pig iron, without the admixture of scrap of any kind or form and without the use of any other stock.

26. *Quality of Pig.*—No portion of the pig iron used shall contain more than .06 of 1 per cent. of phosphorus nor more than .05 of 1 per cent. of sulphur.

27. *Iron Ore.*—The use of iron ore for the reduction of carbon in the furnace charge will be allowed according to usual and good practice.

28. *Recarbonization.*—The recarbonization of steel and the addition of manganese shall be accomplished by the use of ferro-manganese or spiegeleisen only, and shall be performed carefully in a manner most likely to give good results.

29. *Decarbonization.*—During the reduction of the steel

in the open-hearth furnace it shall not be decarbonized below .10 of 1 per cent.

30. *Chemical Limits.*—The finished steel shall not contain to exceed the following limits of the elements named:

Phosphorus	.04 of one per cent.
Sulphur	.03 of one per cent.
Manganese	.50 of one per cent.
Silicon	.10 of one per cent.
Copper	.02 of one per cent.

31. *Size of Ingots.*—The finished steel shall be made into bottom-cast ingots, not larger than 16 inches square in greatest cross-section, weighing not more than 5000 pounds each, and cast in groups of not less than six ingots to each group.

* * * * * * *

Art. 112.—Standard Specifications for Cast-iron Pipe and Special Castings; Department of Water-supply, Gas, and Electricity, City of New York, 1903.

Allowable Variations in Diameter.

5. Especial care shall be taken to have the sockets of the required size. The seat and shoulder of the socket and the end of the spigot must be straight and even, so as to make a smooth joint. The sockets and spigots will be tested by circular gauges, and no pipe or special casting will be received which is defective in joint room from any cause, or whose eccentricity at the spigot and socket ends, or either, exceeds one-eighth ($\frac{1}{8}$) of an inch. The diameters of the sockets and the outside diameters of the spigot ends of the pipes shall not vary from the standard dimensions by more than one-sixteenth ($\frac{1}{16}$) of an inch for pipes sixteen (16) inches or less in diameter, and one-eighth ($\frac{1}{8}$) of an inch for all larger diameters.

Allowable Variation in Thickness.

6. For all pipes and special castings whose standard thickness is less than one (1) inch, the thickness of metal in the barrel shall not be more than eight one-hundredths (.08) of an inch less than the standard thickness, and for pipes and special castings whose standard thickness is one (1) inch or more, the variation shall not exceed one-tenth ($\frac{1}{10}$) of an inch, except that for spaces not exceeding eight (8) inches in length in any direction, the total variation shall not exceed one-eighth ($\frac{1}{8}$) of an inch. The thickness of the metal of the pipes and special castings will be tested by calipers, after the castings have been freed from sand and cleaned.

Allowable Variation in Weight.

9. No straight pipe will be received which weighs less than the standard weights herein mentioned by more than four (4) per cent. of said weights, and no excess of weight in any one straight pipe of more than four (4) per cent. above standard weights will be paid for. No special casting will be accepted the weight of which may be less than the standard weight by more than eight (8) per cent., and no excess above the standard weight of more than eight (8) per cent. will be paid for.

Quality of Iron.

10. The metal of which the castings are to be cast (which must be remelted in a cupola or air-furnace) shall be pig iron made without any admixture of cinder, iron or other inferior metal, and shall be of such character as to make a pipe strong, tough and of even grain, entirely free from uncombined carbon when seen under the microscope, and such as will bear satisfactorily drilling and cutting, and the iron in the pipe shall have a tensile strength of at least eighteen thousand (18,000) pounds to the square inch.

Specimen Bars.

11. Specimen bars of the metal used, each being twenty-six (26) inches long by two (2) inches wide and (1) inch thick, shall be made at the cost and expense of the contractor, as often as the engineer may direct, and the inspector shall test at least one bar from each heat or run of metal. The bars when placed flatwise upon knife-edge supports, twenth-four (24) inches apart and loaded in the centre, shall support a load of two thousand (2000) pounds and show a deflection of not less than thirty-two one-hundredths (.32) of an inch. The inspector may break three bars from each heat or run of metal, and the test shall be based upon the average results of the three bars. Should the dimensions of the bars differ from those herein given, a proper allowance therefor shall be made in the results of the tests.

Cast Vertically.

12. All straight pipes shall be cast vertically, in such moulding sand or loam as will leave the surface clean and smooth. Pipes sixteen (16) inches or less in diameter shall be cast with the socket end up or down. Pipes eighteen (18) inches or more in diameter shall be cast with the socket end down. The pipes shall not be stripped or taken from the pit while showing color of heat, but shall be left in the flasks a sufficient length of time to prevent unequal contraction by subsequent exposure.

Quality of Castings.

13. The pipes and special castings shall be smooth, free from scales, lumps, blisters, sand-holes and defects of every nature which, in the opinion of the engineer, unfit them for the use for which they are intended. No plugging or filling will be allowed.

Coating.

15. Every pipe and special casting shall be coated inside and out with coal-tar pitch varnish. The varnish shall be made from coal-tar distilled until the naphtha is entirely removed and the material deodorized. To this material sufficient oil shall be added to make a smooth coating, tough and tenacious when cold and not brittle, nor with any tendency to scale off.

Heating.

16. Each casting shall be heated to a temperature of three hundred (300) degrees Farenheit immediately before it is dipped, and shall possess not less than this temperature at the time it is put in the vat. The ovens in which the castings are heated shall be so arranged that all portions of the casting shall be heated to an even temperature. Each casting shall remain in the bath at least five (5) minutes. The varnish shall be heated to a temperature of three hundred (300) degrees Farhenheit, and shall be maintained at this temperature during the time the casting is immersed.

Hydrostatic Test.

18. When the coating has become hard the straight pipes shall be subjected to a proof by hydrostatic pressure, and, if required by the engineer, they shall also be subjected to a hammer test under this pressure, and any casting which shows any defect by leaking, sweating or otherwise will be rejected. This proof will be made at the foundry and at the expense of the contractor.

Pressures.

19. The light or "L" class pipe shall be subjected to a pressure of three hundred (300) pounds per square inch, and the heavy or "H" class pipe to a pressure of three hundred and fifty (350) pounds per square inch. Special castings will not receive the hydrostatic test.

CHAPTER XVIII.

THE FATIGUE OF METALS.

Art. 113.—Woehler's Law.

In all the preceding pages, that force or stress which, by a single or gradual application, will cause the failure or rupture of a piece of material has been called its "ultimate resistance." It has long been known, however, that a stress less than the ultimate resistance *may* cause rupture if its application be repeated (without shock) a sufficient number of times. Preceding 1859 no experiments had been made for the purpose of establishing any law connecting the number of applications with the stress requisite for rupture, or with the variation between the greatest and least values of the applied stress.

During the interval between 1859 and 1870, A. Wöhler, under the auspices of the Prussian Government, undertook the execution of some experiments, at the completion of which he had established the following law:

Rupture may be caused not only by a force which exceeds the ultimate resistance, but by the repeated action of forces alternately rising and falling between certain limits, the greater of which is less than the ultimate resistance; the number of repetitions requisite for rupture being an inverse function both of this variation of the applied force and its upper limit.

This phenomenon of the decrease in value of the breaking load with an increase of repetitions is known as "*the fatigue of materials.*"

Although the experimental work requisite to give Wöhler's law complete quantitative expression in the various conditions of engineering constructions can scarcely be considered more than begun, yet enough has been done by Wöhler and Spangenberg to establish the *fact* of metallic fatigue, and a few simple formulæ, provisional though they may be. The importance of the subject in its relation to the durability of all iron and steel structures is of such a high character that a synopsis of some of the experimental results of Wöhler and Spangenberg will be given in the next article.

Art. 114.—Experimental Results.

The experiments of Wöhler are given in "Zeitschrift für Bauwesen," Vols. X., XIII., XVI., and XX., and those of Spangenberg may be consulted in "Fatigue of Metals," translated from the German of Prof. Ludwig Spangenberg, 1876.

These results show in a very marked manner the effect of repeated vibrations on the intensity of stress required to produce rupture.

Spangenberg states that "the experiments show that vibrations may take place between the following limits with equal security against rupture by tearing or crushing:

Wrought iron	+17,600 and −17,600	lbs. per sq. in.
	+33,000 and— 0	" " " "
	+48,400 and +26,400	" " " "
Axle cast steel	+30,800 and −30,800	" " " "
	+52,800 and 0	" " " "
	+88,000 and +38,500	" " " "
Spring-steel not hardened	+55,000 and 0	" " " "
	+77,000 and +27,500	" " " "
	+88,000 and +44,000	" " " "
	+99,000 and +66,000	" " " "

And for axle cast steel in shearing·

+24,200 and −24,200 lbs. per sq. in.
+41,800 and 0 " " " "

PHŒNIX IRON IN TENSION.

Pounds Stress per Square Inch.	Number of Repetitions.	Pounds Stress per Square Inch.	Number of Repetitions.
0 to 52,800	800 rupture	0 to 39,600	480,852 rupture
0 to 48,400	106,910 rupture	0 to 35,200	10,141,645 rupture
0 to 44,000	340,853 rupture	22,000 to 48,400	2,373,424 rupture
0 to 39,600	409,481 rupture	26,400 to 48,400	4,000,000 not broken

WESTPHALIA IRON IN TENSION.

0 to 52,800	4,700 rupture	0 to 39,600	180,800 rupture
0 to 48,400	83,199 rupture	0 to 39,600	596,089 rupture
0 to 48,400	33,230 rupture	0 to 39,600	433,572 rupture
0 to 44,000	136,700 rupture	0 to 35,200	280,121 rupture
0 to 44,000	159,639 rupture	0 to 35,200	566,344 rupture

FIRTH & SONS' STEEL IN TENSION.

0 to 66,000	83,319 rupture	0 to 55,000	103,540 rupture
0 to 60,500	168,396 rupture	0 to 53,900	12,200,000 not broken
0 to 55,000	133,910 rupture	0 to 53,900	229,230 rupture
0 to 55,000	185,680 rupture	0 to 52,800	692,543 rupture
0 to 55,000	360,235 rupture	0 to 52,800	12,200,000 not broken
0 to 55,000	186,005 rupture	0 to 50,600	——

KRUPP'S AXLE-STEEL IN TENSION.

0 to 88,000	18,741 rupture	0 to 55,000	473,766 rupture
0 to 77,000	46,286 rupture	0 to 52,800	13,600,000 not broken
0 to 66,000	170,000 rupture	0 to 50,600	12,200,000 not broken
0 to 60,500	123,770 rupture	——	——

PHOSPHOR-BRONZE (UNWORKED) IN TENSION

0 to 27,500	147,850 rupture	0 to 13,750	1,548,920 rupture
0 to 22,000	408,350 rupture	0 to 13,750	2,340,000 rupture
0 to 16,500	2,731,161 rupture	——	——

PHOSPHOR-BRONZE (WROUGHT) IN TENSION.

0 to 22,000	53,900 rupture	0 to 13,750	1,621,300 rupture
0 to 16,500	2,600,000 not broken	——	——

COMMON BRONZE IN TENSION.

0 to 22,000	4 200 rupture	0 to 11,000	5,447,600 rupture
0 to 16,500	6,300 rupture	—	—

PHŒNIX IRON IN FLEXURE (ONE DIRECTION ONLY).

Pounds Stress per Square Inch.	Number of Repetitions.	Pounds Stress per Square Inch.	Number of Repetitions.
0 to 60,500	169,750 rupture	0 to 39,600	4,035,400 rupture
0 to 55,000	420,000 rupture	0 to 35,200	3,420,000 rupture
0 to 49,500	481,975 rupture	0 to 33,000	4,820,000 not broken
0 to 44,000	1,320,000 rupture	—	—

WESTPHALIA IRON IN FLEXURE (ONE DIRECTION ONLY).

0 to 52,250	612,065 rupture	0 to 44,000	1,493,511 rupture
0 to 49,500	457,229 rupture	0 to 39,600	3,587,509 rupture
0 to 46,750	799,543 rupture	—	—

HOMOGENEOUS IRON IN FLEXURE (ONE DIRECTION ONLY).

0 to 60,500	169,750 rupture	0 to 39,600	4,035,400 rupture
0 to 55,000	420,000 rupture	0 to 35,020	3,420,000 not broken
0 to 49,500	481,975 rupture	0 to 33,000	48,200,000 not broken
0 to 44 000	1,320,000 rupture	—	—

FIRTH & SONS' STEEL IN FLEXURE (ONE DIRECTION ONLY).

0 to 63,250	281,856 rupture	0 to 52,250	578,323 rupture
0 to 60,500	266,556 rupture	0 to 49,500	5,640,596* rupture
0 to 55,000	1,479,908 rupture	0 to 49,500	13,700,000 not broken

* Accidental.

KRUPP'S AXLE-STEEL IN FLEXURE (ONE DIRECTION ONLY).

0 to 77,000	104,300 rupture	0 to 55,000	729,400 rupture
0 to 66,000	317,275 rupture	0 to 55,000	1,499,600 rupture
0 to 60,500	612,500 rupture	0 to 49,500	43,000,000 not broken

KRUPP'S SPRING-STEEL IN FLEXURE (ONE DIRECTION ONLY).

0 to 110,000	39,950 rupture	72,600 to 110,000	19,673,300 not broken
0 to 88,000	117,000 rupture	66,000 to 99,000	33,600,000 not broken
0 to 66,000	468,200 rupture	44,000 to 88,000	35,800,000 not broken
0 to 55,000	40,600,000 not broken	44,000 to 88,000	38,000,000 not broken
0 to 49,500	32,942,000 not broken	61,600 to 88,000	36,000,000 not broken
88,000 to 132,000	35,600,000 not broken	27,500 to 77,000	36,600,000 not broken
99,000 to 132,000	33,478,700 not broken	33,000 to 77,000	31,152,000 not broken

PHOSPHOR-BRONZE IN FLEXURE (ONE DIRECTION ONLY).

Pounds Stress per Square Inch.	Number of Repetitions.	Pounds Stress per Square Inch.	Number of Repetitions.
0 to 22,000	862,980 rupture	0 to 16,500	5,075,169 rupture
0 to 19,800	8,151,811 rupture	0 to 13,200	10,000,000 not broken

COMMON BRONZE IN FLEXURE (ONE DIRECTION ONLY).

0 to 22,000	102,659 rupture	0 to 16,500	837,760 rupture
0 to 19,800	151,310 rupture	0 to 13,200	10,400,000 not broken

PHŒNIX IRON IN TORSION (BOTH DIRECTIONS).

− 35,200 to + 35,200	56,430 rupture	− 24,200 to + 24,200	3,632,588 rupture
− 33,000 to + 33 000	99,000 rupture	− 22,000 to + 22,000	4,917,992 rupture
− 28,600 to + 28 600	479,490 rupture	− 19,800 to + 19,800	19 186,791 rupture
− 26,400 to + 26,400	909,810 rupture	− 17,600 to + 17,600	132,250,000 not broken

ENGLISH SPINDLE-IRON IN TORSION (BOTH DIRECTIONS).

− 37,400 to + 37,400	204,400 rupture	− 30,800 to + 30,800	079,100 rupture
− 37,400 to + 37,400	147 800 rupture	− 28 600 to + 28,600	1,142,600 rupture
− 35,200 to + 35,200	911,100 rupture	− 28,600 to + 28,600	595,910 rupture
− 35,200 to + 35,200	402,900 rupture	− 26,400 to + 26,400	3,823,200 rupture
− 33,000 to + 33,000	1,064,700 rupture	− 26,400 to + 26,400	6 100,000 not broken
− 33 000 to + 33 000	384,800 rupture	− 22,000 to + 22,000	8,800 000 not broken
− 30 800 to + 30,800	1,337,700 rupture	− 22,000 to + 22,000	4,000,000 not broken

KRUPP'S AXLE-STEEL IN TORSION (BOTH DIRECTIONS)

− 44,000 to + 44,000	367,400 rupture	− 46,200 to + 46,200	55,100 rupture
− 39,600 to + 39 600	925,800 rupture	− 37,400 to + 37,200	797,525 rupture
− 37,400 to + 37,400	4,000,000 not broken	− 35 200 to + 35,200	1 665,580 rupture
− 35,200 to + 35,200	4,800,000 not broken	− 33,000 to + 33,000	4,163 375 rupture
− 33,000 to + 33 000	5,000 000 not broken	− 33,000 to + 33,000	45,050,640 rupture

In Art. 42 will be found some experiments by the late Capt. Rodman, U.S.A., on the fatigue of cast iron, but they are sufficient in number and character to show the general effect only, and give no quantitative results.

The specimens used in all the preceding experiments were small.

During 1860, '61, and '62 Sir Wm. Fairbairn constructed a built beam of plates and angles with a depth of 16 inches, clear span of 20 feet, and estimated centre breaking load of 26,880 pounds.

This beam was subjected to the action of a centre load of 6643 pounds, alternately applied and relieved eight times per minute; 596,790 continuous applications produced no visible alterations.

The load was then increased from one fourth to two sevenths the breaking weight, and 403,210 more applications were made without apparent injury.

The load was next increased to two fifths the breaking weight, or to 10,486 pounds; 5175 changes then broke the beam in the tension flange near the centre.

The total number of applications was thus 1,005,175.

The beam was then repaired and loaded with 10,500 pounds at centre 158 times, then with 8025 pounds 25,900 times, and finally with 6643 pounds enough times to make a total of 3,150,000.

In these experiments the load was completely removed each time.

It is thus seen that vibrations (without shock) with one-fourth the calculated breaking centre load produced no apparent effect on the resistance of the beam, but that two fifths of that load caused failure after a comparatively small number of repetitions.

It is probable that the breaking centre load was calculated too high, in which case the ratios $\frac{1}{4}$ and $\frac{2}{5}$ should be somewhat increased.

Art. 115.—Formulæ of Launhardt and Weyrauch.

Let R represent the intensity (stress per square unit of section) of ultimate resistance for any material in tension, compression, shearing, torsion, or bending; R will cause rupture at a single, gradual application. But the material may also be ruptured if it is subjected a sufficient number of times, and alternately, to the intensities P and Q, Q being less than P and both less than R, *while all are of the same kind*. When $Q=0$ let $P=W$, and let $D=P-Q$. W is called the "primitive safe resistance," since the bar returns to its primitive unstressed condition at each application. In the general case P is called the "working ultimate resistance."

By the notation adopted:

$$P=Q+D. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

But by Wöhler's law, P is a function of D, or

$$P=f(D). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

A sufficient number of experiments have not yet been made in order to completely determine the form of the function $f\ (D)$.

It is known, however, that

$$\text{for } Q=0, \qquad P=D=W;$$
$$\text{and for } D=0, \qquad P=Q=R.$$

Provisionally, Launhardt satisfies these two extreme conditions by taking

$$P=\frac{R-W}{R-P}D=\frac{R-W}{R-P}(P-Q). \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

Even at these limits this is not thoroughly satisfactory, for when $D=0$, $P=\frac{0}{0}(R-W)$, or is indeterminate.

By solving eq. (3),

$$P=W\left(1+\frac{R-W}{W}\cdot\frac{Q}{P}\right). \quad . \quad . \quad . \quad . \quad (4)$$

But if the least value of the total stress to which any member of a structure is subjected is represented by *min B*, and its greatest value by *max B*, there will result $\frac{min\ B}{max\ B}=\frac{Q}{P}$.

Hence

$$P=W\left(1+\frac{R-W}{W}\frac{min\ B}{max\ B}\right), \quad . \quad . \quad . \quad . \quad (5)$$

which is Launhardt's formula. In the preceding article some values of W are shown. In applying eq. (5) it is only necessary to take the primitive safe resistance, W, for the total number of times which the structure will be subjected to loads. Since bridges are expected to possess an indefinite duration of life, in such structures that number should be indefinitely large.

Eq. (5), it is to be borne in mind, is to be applied when the piece is *always subjected to stress of one kind, or in one direction only*. It agrees well with some experiments by Wöhler on Krupp's untempered cast spring steel.

If the stress in any piece varies from one kind to another, as from tension to compression, or *vice versa*, or from one direction to another, as in torsion on each side of a state of no stress, Weyrauch has established the following formula by a course of reasoning similar to that used by Launhardt.

If the opposite stresses, which will cause rupture by a certain number of applications, are equal in intensity, and if that intensity is represented by S, then will S be called the "vibration resistance"; this was established by Wöhler for some cases, and some of its values are given in the preceding article.

Let $+P$ and $-P'$ represent two intensities of opposite kinds or in opposite directions, of which P *is numerically the greater*. Then if $D=P+P'$,

$$P=D-P'.$$

The two following limiting conditions will hold:

$$\text{For } P'=0, \quad P=D=W;$$
$$\text{For } P'=S; \quad P=S=\tfrac{1}{2}D.$$

But by Wöhler's law $P=f(D)$, and the two limiting conditions just given will be found to be satisfied by the provisional formula

$$P=\frac{W-S}{2W-S-P}D=\frac{W-S}{2W-S-P}(P+P'). \quad . \quad . \quad (6)$$

By the solution of eq. (6),

$$P=W\left(1-\frac{W-S}{W}\cdot\frac{P'}{P}\right). \quad . \quad . \quad . \quad . \quad . \quad (7)$$

If, without regard to kind or direction, *max* B is numerically the greatest total stress which the piece has to carry, while *max* B' is the greatest total stress of the other kind or direction, then will $\frac{P'}{P}=\frac{max\ B'}{max\ B}$. Hence there will result the following, which is the formula of Weyrauch:

$$P=W\left(1-\frac{W-S}{W}\,\frac{max\ B'}{max\ B}\right). \quad . \quad . \quad . \quad . \quad (8)$$

Eqs. (5) and (8) give values of the intensity P which are to be used in determining the cross-section of pieces designed to carry given amounts of stress. If n is the safety factor and F the total stress to be carried, the area of section desired will be

$$A = \frac{nF}{P},$$

in which $\frac{P}{n}$ is the greatest working stress permitted.

If for wrought iron in tension $W = 30{,}000$ and $R = 50{,}000$, eq. (5) gives

$$P = 30{,}000\left(1 + \frac{2}{3}\frac{min\ B}{max\ B}\right).$$

Hence, if the total stress due to fixed and moving loads in the web member of a truss is $max\ B = 80{,}000$ pounds, while that due to the fixed load alone is $min\ B = 40{,}000$, there will result

$$P = 30{,}000\left(1 + \frac{2}{3}\cdot\frac{40{,}000}{80{,}000}\right) = 40{,}000.$$

In such a case the greatest permissible working stress with a safety factor of 3 would be about 13,300 pounds.

For steel in tension, if $W = 50{,}000$ and $R = 75{,}000$,

$$P = 50{,}000\left(1 + \frac{1}{2}\frac{min\ B}{max\ B}\right).$$

For wrought iron in torsion, if $S = 18{,}000$ and $W = 24{,}000$, eq. (8) will give

$$P = 24{,}000\left(1 - \frac{1}{4}\frac{max\ B'}{max\ B}\right).$$

Other methods based on Wöhler's experiments have been deduced by Müller, Gerber, and Schäffer, of which synopses may be found in Du Bois' translation of Weyrauch's "Structures of Iron and Steel."

Art. 116.—Influence of Time on Strains.

In the section "*Elevation of Ultimate Resistance and Elastic Limit*," in Art. 41, the effect of prolonged tensile stress and subsequent rest between the elastic limit and ultimate resistance was shown to be the elevation of both those quantities. It is a matter of common observation, however, that if a piece of wrought iron be subjected to a tensile stress nearly equal to its ultimate resistance, and held in that condition, the stretch will increase as the time elapses.

Experiments are still lacking which may show that a piece of metal can be ruptured by a tensile stress much below its ultimate resistance. It may be indirectly inferred, however, from experiments on flexure, that such failure may be produced, as the following by Prof. Thurston will show.

A bar 10 parts tin and 90 parts copper, 1 × 1 × 22 inches and supported at each end, sustained about 65 per cent. of its breaking load at the centre for five minutes. During that time its deflection increased 0.021 inch. The same bar sustained 1485 pounds at centre for 13 minutes and then failed.

A second bar of the same size, but 90 parts tin and 10 parts copper, was loaded at the centre with 160 pounds, causing a deflection of 1.294 inches. After 10 minutes the deflection had increased 0.025 inch; after one day, 1.00 inch; after two days, 2.00 inches; and after three days, 3.00 inches, when the bar failed under the load of 160 pounds.

Another bar of the same size showed remarkable results;

it was composed of 90 parts zinc and 10 parts copper. It gave the same general increase of deflection with time, but eventually broke under a centre load which ran down from 1233 to 911 pounds, after holding the latter about three minutes.

A bar of the same size and 96 parts copper with 4 parts tin, after it had carried 700 pounds at centre for sixty minutes was loaded with 1000 pounds, with the following results:

After.	Deflection.
0 minute	3.118 inches.
5 minutes	3.540 "
15 minutes	3.660 "
45 minutes	4.102 "
75 minutes	7.634 "

Broke under 1000 pounds.

A wrought-iron bar of the same size gave, under a centre load of 1600 pounds:

After.	Deflection.
0 minute	0.489 inch.
3 minutes	0.632 "
6 minutes	0.650 "
16 minutes	0.660 "
344 minutes	0.660 "

It subsequently carried 2589 pounds with a deflection of 4.67 inches.

During 1875 and 1876 Prof. Thurston made a number of other similar experiments with the same general results.

Metals like tin and many of its alloys showed an increasing rate of deflection and final failure, far below the so-called "ultimate resistance." The wrought-iron bars, however, showed a decreasing increment of deflection, which finally became zero, leaving the deflection constant.

Whether there may be a point for every metal, beyond which, with a given load, the increment of deflection may retain its value or go on increasing until failure, and below which this increment decreases as the time elapses, and finally becomes zero, is yet undetermined, but seems probable.

It does not follow, therefore, that the principle enunciated in the section named at the beginning of this article is to be taken without qualification. If "rest" under stress, too near the ultimate resistance, be sufficiently prolonged, it has been seen that it is possible that failure may take place.

In verifying some experimental results by Herman Haupt, determined over forty years ago, Prof. Thurston tested three seasoned pine beams about 1⅛ inches square and 40 inches length of span, and found that 60 per cent. of the ordinary "breaking load" caused failure at the end of 8, 12, and 15 months. In these cases the deflection slowly and steadily increased during the periods named.

Two other sets of three pine beams each broke under 80 and 95 per cent. of the usual "breaking load," after much shorter intervals of time.

In all these instances it is evident that the molecules under the greatest stress "flow" over each other to a greater or less extent. In the cases of decreasing increments of strain, the new positions afford capacity of increased resistance; in the others, those movements are so great that the distances between some of the molecules exceed the reach of molecular action, and failure follows.

In many cases strained portions of material recover partially or wholly from permanent set. In such cases a portion of the material has been subjected to intensities of stress high enough to produce true "flow" of the molecules, while the remaining portion has not. The internal elastic stresses in the latter portion, after the removal of the external forces, produce in time a reverse flow in consequence of the elastic endeavor to resume the original shape.

It is altogether probable that the phenomena of fatigue and flow of metals are very intimately associated. Some of the prominent characteristics of the latter will be given in the next chapter.

CHAPTER XIX.

THE FLOW OF SOLIDS.

Art. 117.—General Statements.

ALTHOUGH there is no reason to suppose that true solids may not retain a definite shape for an indefinite length of time if subjected to no external force other than gravity,* many phenomena resulting both from direct experiment for the purpose, and incidentally from other experiments involving the application of external stress of considerable intensity, show that a proper intensity of internal stress (in many cases comparatively low) will cause the molecules of a solid to flow at ordinary temperatures like those of a liquid. And this flow, moreover, is entirely different from, and independent of, the elastic properties of the material; for it arises from a permanent and considerable relative displacement of the molecules. Nor is it to be confounded with that internal "friction" which, if an elastic body is subjected to oscillations, causes the amplitudes to gradually decrease and finally disappear, even in vacuo. This latter motion is typically elastic and the retarding cause may be considered a kind of elastic friction.

It is evident that if a mass of material be enclosed on all its faces, or outer surfaces, but one or a portion of one, and if external pressure be brought to bear on those faces, the material will be forced to move to and through the free surface; in other words, *the flow of the material will take place in the direction of least resistance.*

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* This, perhaps, may be considered a definition of a true solid.

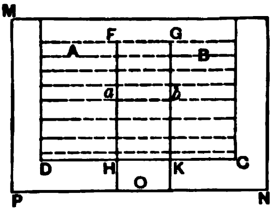

FIG. 1.

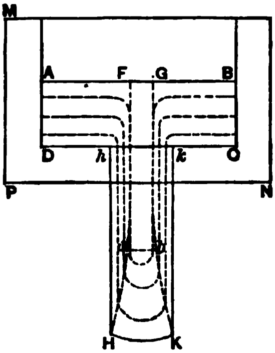

FIG. 2.

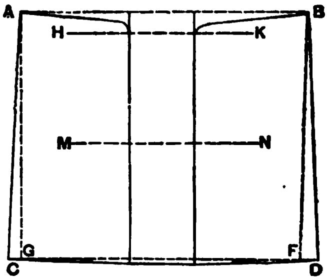

FIG. 3.

The theory of the flow of solids to be given is that developed by Mons. H. Tresca in his "Mémoire sur l'Écoulement des Corps Solides," 1865. He made a large number of experiments on hard and soft metals, ceramic pastes, sand, and shot.

These different materials all manifested the same characteristics of flow, which are well shown in Fig. 2. *ABCD*, Fig. 1, is supposed to be a cylindrical mass of lead with circular horizontal section, confined in a circular cylinder, *MN*, closed at one end with the exception of the orifice *O*.

This cylinder is supported on the base *PN*, while the face *AB* of the lead receives external pressure from a close-fitting piston. When the pressure is sufficiently increased, the face *AB* in Fig. 1 sinks to *AB* in Fig. 2, while the column *hkHK*, in the latter figure, is forced to flow through the orifice *O*.

In Tresca's experiments with lead, the diameter *AB* was about 3.9 inches; the diameter *HK* of the orifice, from 0.75 in. to 1.5 ins., while the length of the column or jet *hK* varied from 0.4 in. to about 24 ins.

The total pressure on the face *AB* varied from 119,000 to 198,000 pounds. The initial thickness *AD* varied from 0.24 inch to 2.4 inches.

Some experiments exhibiting in a remarkably clear manner the flow of metals in cold punching were made by David Townsend in 1878, and the results were given by him in the "Journal of the Franklin Institute" for March of that year. If the dotted rectangle *ABFG*, Fig. 3, shows the original outline of the middle section of a nut before punching, he found that the final outline of the same section would be represented by the full lines. The top and bottom faces were depressed by the punching, as shown; the upper width *AB* remained about the same, but the lower, *GF*, was increased to *CD*. Although the depth of the nut, *AC*, was 1.75 inches, the length of the core punched out was only 1.063 inches. The density of this core was then examined and found to be the same as that of the original nut. Hence a portion of the core equal in length to $1.75 - 1.063 = 0.687$ inch was forced, or flowed, back into the body of the nut. Subsequent experiments showed that this flow did not take place at the immediate upper surface *AB*, nor very much in the lower half of the nut, but that it was chiefly confined to a zone equal in depth to about half that of the nut, the upper surface of which lies a very short distance below the upper face of the nut. The location of this zone is shown by the lines *HK* and *MN* in Fig. 3.

Tresca's experiments on punching showed essentially the same result.

Art. 118.—Tresca's Hypotheses.

The central cylinder *FGKH*, Fig. 1 of Art. 117, was called by Tresca the "primitive central cylinder." As the metal flows, this cylinder will be drawn out into the volume of revolution, whose axis is that of the orifice and whose meridian section is *FGkKIlh*, Fig. 2, the diameter *FG* being gradually decreased.

It was found by experiment that if the original mass *AC*, Fig. 1, was composed of horizontal layers of uniform thickness, the reduced mass in Fig. 2 was also composed of the same number of layers of uniform thickness, except in the immediate vicinity of the central cylinder.

Tresca then assumed these three hypotheses:

1°.—*The density of the material remains the same whether in the cylinder or in the jet;* in other words, the volume of the material in the jet and in the cylinder remains constant.

Let R = radius of the cylinder;
R_1 = radius of the orifice;
y = variable length of the jet (i.e., hH);
D = original depth of material ($BC = AD$, Fig. 1) in the cylinder;
d = variable depth of material ($BC = AD$, Fig. 2) in the cylinder;

then by the hypothesis just stated

$$R^2d = R^2D - R_1^2y. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

2°.—*The rate of compression along any and all lines parallel to the axis of the primitive central cylinder, and taken outside of that limit, is constant.*

If, then, the material lying outside of the central cylinder be divided into horizontal layers of equal thickness, a very small decrease in the variable depth equal to d (a) will cause the same amount of material to move or flow from each of these layers into the space originally occupied by the central cylinder, thus causing a portion of the material previously resting over the orifice to flow through the latter. If $d(d)$ is the indefinitely small change of depth, and dR_1 the indefinitely small change in the radius of the cylindrical portion resting over the orifice, then the equality of volumes expressing this hypothesis is the following:

$$\pi(R^2 - R_1^2) . d(d) = 2\pi R_1 d . dR_1,$$

or

$$\frac{d(d)}{d} = \frac{2R_1 dR_1}{R^2 - R_1^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

3°.—*The rate of decrease of the radius of the primitive central cylinder is constant throughout its length at any given instant during flow.*

Let r be any radius less than R_1, then if the latter is decreased by the very small amount dR_1, the former will be shortened by the amount dr; and by the last hypothesis there must result

$$\frac{dR_1}{R_1} = \frac{dr}{r}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

This is a perfectly general equation, in which r may or may not be the variable value of the radius of that portion of the primitive central cylinder remaining above the orifice at any instant during flow.

These are the three hypotheses on which Tresca based his theory of the flow of solids. It is thus seen to be put upon a purely geometrical basis, entirely independent of the elastic or other properties of the material.

Art. 119.—The Variable Meridian Section of the Primitive Central Cylinder.

The meridian curve *haH*, or *hbK*, Fig. 2 of Art. 117, may now easily be determined.

Eq. (1) of Art. 118 may take the first of the following forms, while its differential, considering d and y variable, may take the second:

$$d = D - \frac{R_1^2}{R^2} y,$$

$$d(d) = -\frac{R_1^2}{R^2} dy.$$

Dividing the second by the first,

$$\frac{d(d)}{d} = \frac{dy}{y - \frac{R^2}{R_1^2} D} = \frac{2R_1 dR_1}{R^2 - R_1^2}.$$

The last member of this equation is simply eq. (2) of Art. 118; and if the value of dR_1, in eq. (3) of the same article, be inserted in the third member of this equation, there will result

$$\frac{2R_1^2}{R^2 - R_1^2} \cdot \frac{dr}{r} = \frac{dy}{y - \frac{R^2}{R_1^2} D}.$$

Integrating between the limits of r and R_1, and remembering that r will be restricted to the representation of the radius of that portion of the primitive central cylinder which remains, at any instant, over the orifice, by taking $y = 0$ for $r = R_1$,

$$\frac{2R_1^2}{R^2 - R_1^2} \log \frac{r}{R_1} = \log \left(\frac{y - \frac{R^2}{R_1^2} D}{- \frac{R^2}{R_1^2} D} \right);$$

"*log*" indicates a Napierian logarithm.

Passing from logarithms to the quantities themselves, and reducing,

$$y=\frac{R^2D}{R_1^{\ 2}}\left[1-\left(\frac{r}{R_1}\right)^{\frac{2R_1^{\ 2}}{R^2-R_1^{\ 2}}}\right]. \quad . \quad . \quad . \quad . \quad (1)$$

This is the desired equation of the line, in which r is measured normal to the axis of the cylinder or jet, while y is measured along that axis from the extremity of the jet. When the material is wholly expelled,

$$y=\frac{R^2}{R_1^{\ 2}}D, \text{ and } r=0.$$

Eq. (2) is applicable to the jet only. For the line hF or Gk, resort will be had to the equation

$$\frac{d(d)}{d}=\frac{2R_1^{\ 2}}{R^2-R_1^{\ 2}}\frac{dr}{r}.$$

Again integrating between the limits d and D, or r and R_1, and reducing,

$$r=R_1\left(\frac{d}{D}\right)^{\frac{R^2-R_1^{\ 2}}{2R_1^{\ 2}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

This value of r is the radius of that portion of the primitive central cylinder which remains over the orifice when D is reduced to d.

Art. 120.—Positions in the Jet of Horizontal Sections of the Primitive Central Cylinder.

That portion of the primitive central cylinder below ab, in Fig. 1 of Art. 117, will be changed to $abKH$ in Fig. 2 of the same article.

If, in the latter Fig., y' is the distance from HK to ab, measured along the axis, then the volume of $HKab$ will have the value

$$\int_0^{y'} \pi r^2 dy.$$

If d' is the distance $aF = bG$, in Fig. 1, the equality of volumes will give

$$\int_0^{y'} r^2 dy = R_1{}^2(D - d').$$

Eq. (1) of Art. 102 gives

$$r_2 = R_1{}^2\left(1 - \frac{R_1{}^2 y}{R^2 D}\right)^{\frac{R^2 - R_1{}^2}{R_1{}^2}};$$

$$\therefore \int_0 r^2 dy = R_1{}^2 D - R_1{}^2 D\left(1 - \frac{R_1{}^2 y'}{R^2 D}\right)^{\frac{R^2}{R_1{}^2}} = R_1{}^2(D - d');$$

$$\therefore y' = \frac{R^2}{R_1{}^2}\left[1 - \left(\frac{d'}{D}\right)^{\frac{R_1{}^2}{R^2}}\right]D. \quad . \quad . \quad . \quad . \quad (1)$$

If N is the number of horizontal layers required to compose the total thickness D, and n the number in the depth d',

$$\frac{d'}{D} = \frac{n}{N}.$$

Hence

$$y' = \frac{R^2}{R_1{}^2}\left[1 - \left(\frac{n}{N}\right)^{\frac{R_1}{R^2}}\right]D. \quad . \quad . \quad . \quad . \quad (2)$$

Tresca computed values of y' for some of his experiments and compared the results with actual measurements. The agreement, though not exact, was very satisfactory. Within limits not extreme, the longer the jet the more satisfactory was the agreement.

Art. 121.—Final Radius of a Horizontal Section of the Primitive Central Cylinder.

Let it be required to determine what radius the section situated at the distance d' from the upper surface of the primitive central cylinder will possess in the jet.

It will only be necessary to put for y in eq. (1) of Art. 119 the value of y' taken from eq. (1) of Art. 120. This operation gives

$$\left(\frac{d'}{D}\right)^{\frac{R_1^2}{R^2}} = \left(\frac{r'}{R_1}\right)^{\frac{2R_1^2}{R^2-R_1^2}}.$$

Hence

$$r' = R_1\left(\frac{d'}{D}\right)^{\frac{R^2-R_1^2}{2R^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If R_1 is small, as compared with R, there will result approximately

$$r' = R_1\left(\frac{d'}{D}\right)^{\frac{1}{2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Art. 122.—Path of Any Molecule.

The hypotheses on which the theory of flow is based enable the hypothetical path of any molecule to be easily established.

In consequence of the nature of the motion there will be three portions of the path, each of which will be represented by its characteristic equation, as follows:

First, *let the molecule lie outside of the primitive central cylinder.*

Let R' and H be the original co-ordinates of the molecule considered, measured normal to and along the axis of the cylinder, respectively, from the centre of the orifice HK (Fig. 1, Art. 117) as an origin, while r and h are the variable co-ordinates.

The first hypothesis, by which the density remains constant, then gives the following equation:

$$\pi(R^2-R'^2)H=\pi(R^2-r^2)h,$$

or

$$hR^2-hr^2=(R^2-R'^2)H. \quad . \quad . \quad . \quad . \quad (1)$$

This is the equation to the path of the molecule, in which r must always exceed R_1.

As this equation is of the third degree, the curve cannot be one of the conic sections.

Second, *let the molecule move in the space originally occupied by the central cylinder.*

While h and r now vary, the volume $\pi r^2(D-h)$ must remain constant. When $r=R_1$ let $h=h_1$. Hence

$$r^2(D-h)=R_1^2(D-h_1). \quad . \quad . \quad . \quad . \quad . \quad (2)$$

But if $h=h_1$ and $r=R_1$ in eq. (1),

$$h_1=\left(\frac{R^2-R'^2}{R^2-R_1^2}\right)H.$$

Placing this value in eq. (2).

$$r^2(D-h)=R_1^2\left(D-H\frac{R^2-R'^2}{R^2-R_1^2}\right). \quad . \quad . \quad . \quad (3)$$

Third, *let the molecule move in the jet.*

After the molecule passes the orifice, its path will evidently be a straight line parallel to the axis of the jet. Its distance r_1 from that axis will be found by putting $h=0$ in eq. (3). Hence

$$r_1 = R_1\left(1 - \frac{H}{D}\,\frac{R^2 - R'^2}{R^2 - R_1^2}\right)^{\frac{1}{2}}. \quad . \quad . \quad . \quad . \qquad (4)$$

APPENDIX I.

ELEMENTS OF THEORY OF ELASTICITY IN AMORPHOUS SOLID BODIES.

CHAPTER I.

GENERAL EQUATIONS.

Art. 1.—Expressions for Tangential and Direct Stresses in Terms of the Rates of Strains at Any Point of a Homogeneous Body.

LET any portion of material perfectly homogeneous be subjected to any state of stress whatever. At any point as O, Fig. 1, let there be assumed any three rectangular co-ordinate planes; then consider any small rectangular parallelopiped whose faces are parallel to those planes. Finally let the stresses on the three faces nearest the origin be resolved into components normal and parallel to their planes of action, whose directions are parallel to the co-ordinate axis.

The intensities of these tangential and normal components will be represented in the usual manner, i.e., p_{xy} signifies a tangential intensity on a plane normal to the axis of X (plane ZY), whose direction is parallel to the axis of Y, while p_{xx} signifies the intensity of a normal stress on a plane normal to the axis of X (plane ZY) and in the direction of the axis of X. Two unlike subscripts, therefore, indicate a tangential stress, while two of the same kind signify a normal stress.

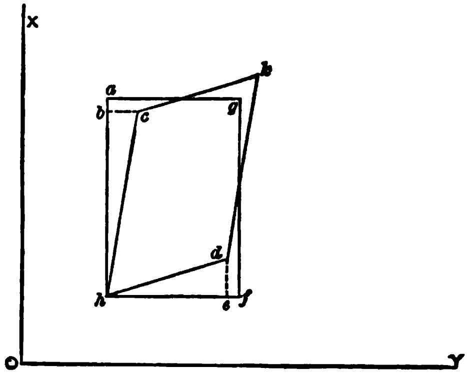

FIG. 1.

From eq. (3) on page 5 and eq. (7) on page 12 there is at once deduced

$$S = \frac{E}{2(1+r)}\phi = G\phi. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Now when the material is subjected to stress the lines bounding the faces of the parallelopiped will no longer be at right angles to each other. It has already been shown in Art. 2 that the angular changes of the lines from right angles are the characteristic shearing strains, which, multiplied by Gs give the shearing intensities.

Let ϕ_1 be the change of angle of the boundary lines parallel to X and Y.

Let ϕ_2 be the change of angle of the boundary lines parallel to Y and Z.

Let ϕ_3, be the change of angle of the boundary line parallel to Z and X.

Eq. (1) will then give the following three equations:

$$p_{xy} = \frac{E}{2(1+r)}\phi_1; \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$p_{yz} = \frac{E}{2(1+r)}\phi_2; \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$p_{zx} = \frac{E}{2(1+r)}\phi_3. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

In Fig. 1 let the rectangle *agfh* represent the right projection of the indefinitely small parallelopiped $dx\,dy\,dz$. If u, v, and w are the strains parallel to the axis of x, y, and z of the original point h, the rates of variation of strain $\frac{du}{dx}, \frac{dv}{dy}, \frac{dw}{dz}$, etc., may be considered constant throughout this parallelopiped; consequently the rectangular faces will change to oblique parallelograms. The oblique parallelogram *dhck*, whose diagonals may or may not coincide with those of *agfh*, therefore, may represent the strained condition of the latter figure.

Then, by Art. 2, the difference petween *dhc* and the right angle at h will represent the strain ϕ_1. But, from Fig. 1, ϕ_1 has the following value:

$$\phi_1 = dhe + bhc. \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

But the limiting values of the angles in the second member are coincident with their tangents; hence

$$\phi_1 = \frac{de}{dy} + \frac{bc}{dx}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

But, again, de is the distortion parallel to OX *found by moving parallel to OY only;* hence it is a partial differential of u, or it has the value

$$de=\frac{du}{dy}dy. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

In precisely the same manner bc is the partial differential of v in respect to x, or

$$bc=\frac{dv}{dx}dx.$$

By the aid of these considerations, eq. (6) takes the form

$$\phi_1=\frac{du}{dy}+\frac{dv}{dx}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

If XY be changed to YZ, and then to ZX, there may be at once written by the aid of eq. (8)

$$\phi_2=\frac{dv}{dz}+\frac{dw}{dy}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$\phi_3=\frac{dw}{dx}+\frac{du}{dz}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Eqs. (2), (3), and (4) now take the following form:

$$p_{xy}=G\left(\frac{du}{dy}+\frac{dv}{dx}\right); \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

$$p_{yz}=G\left(\frac{dv}{dz}+\frac{dw}{dy}\right); \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

$$p_{zx}=G\left(\frac{dw}{dx}+\frac{du}{dz}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

The direct stresses are next to be given in terms of the displacements u, v, and w. Again, let the rectangular parallelopiped $dx\ dy\ dz$ be considered. Eq. (1), on page 3, shows that the strain per unit of length is found by dividing the intensity of stress by the coefficient of elasticity, *if a single stress only exists.* But in the present instance, any state of stress whatever is supposed. Consequently the strain caused by p_{xx}, for example, acting alone must be combined with the lateral strains induced by p_{yy} and p_{zz}. Denoting the actual rates of strain along the axes of X, Y, and Z by l_1, l_2, and l_3, therefore, the following equations may be at once written by the aid of the principles given on pages 9 and 10:

$$\frac{p_{xx}}{E} = l_1 + (p_{yy} + p_{zz})\frac{r}{E}; \quad . \quad . \quad . \quad . \quad (14)$$

$$\frac{p_{yy}}{E} = l_2 + (p_{xx} + p_{zz})\frac{r}{E}; \quad . \quad . \quad . \quad . \quad (15)$$

$$\frac{p_{zz}}{E} = l_3 + (p_{yy} + p_{xx})\frac{r}{E}. \quad . \quad . \quad . \quad . \quad (16)$$

Eliminating between these three equations,

$$p_{xx} = \frac{E}{1+r}\left[l_1 + \frac{r}{1-2r}(l_1 + l_2 + l_3)\right]; \quad . \quad . \quad (17)$$

$$p_{yy} = \frac{E}{1+r}\left[l_2 + \frac{r}{1-2r}(l_1 + l_2 + l_3)\right]; \quad . \quad . \quad (18)$$

$$p_{zz} = \frac{E}{1+r}\left[l_3 + \frac{r}{1-2r}(l_1 + l_2 + l_3)\right]. \quad . \quad . \quad (19)$$

But if u, v, and w are the actual strains at the point where these stresses exist, the rates of strain l_1, l_2, and l_3 will evidently be equal to $\frac{du}{dx}$, $\frac{dv}{dy}$, and $\frac{dw}{dz}$, respectively. The volume of the parallelopiped will be changed by those strains to

$$dx(1+l_1)dy(1+l_2)dz(1+l_3)=dx\,dy\,dz(1+l_1+l_2+l_3)$$

if powers of l_1, l_2, and l_3 above the first be omitted. The quantity $(l_1+l_2+l_3)$ is, then, *the rate of variation of volume*, or *the amount of variation of volume for a cubic unit.* If there be put

$$\theta=\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}, \quad \text{and} \quad G=\frac{E}{2(1+r)},$$

eqs. (17), (18), and (19) will take the forms

$$p_{xx}=\frac{2Gr}{1-2r}\theta+2G\frac{du}{dx}; \quad . \quad . \quad . \quad . \quad (20)$$

$$p_{yy}=\frac{2Gr}{1-2r}\theta+2G\frac{dv}{dy}; \quad . \quad . \quad . \quad . \quad (21)$$

$$p_{zz}=\frac{2Gr}{1-2r}\theta+2G\frac{dw}{dz}. \quad . \quad . \quad . \quad . \quad (22)$$

The form in which eqs. (14), (15), and (16) are written shows that if p_{xx}, p_{yy}, or p_{zz} is positive, the stress is tension, and compression if it is negative. Consequently a positive value for any of the intensities in eqs. (20), (21), or (22) will indicate a tensile stress, while a negative value will show the stress to be compressive.

The eqs. (14) to (19), together with the elimination involved, also show that the coefficients of elasticity for tension and compression have been taken equal to each other, and that the ratio r is the same for tensile and compressive strains.

Further, in eqs. (11), (12), and (13), it has been assumed that G is the same for all planes.

Hence eqs. (11,) (12), (13), (20), (21), and (22) *apply only to bodies perfectly homogeneous in all directions.*

It is to be observed that the co-ordinate axes have been taken perfectly arbitrarily.

Art. 2.—General Equations of Internal Motion and Equilibrium.

In establishing the general equations of motion and equilibrium, the principles of dynamics and statics are to be applied to the forces which act upon the parallelopiped represented in Fig. 1, the edges of which are dx, dy, and dz. The notation to be used for the intensities of the stresses acting on the different faces will be the same as that used in the preceding article.

Let the stresses which act on the faces nearest the origin be considered negative, while those which act on the other three faces are taken as positive.

The stresses which act in the direction of the axis of X are the following:

On the face normal to X, nearest to O, $-p_{xx}\,dy\,dz$;

" " " " " " farthest from O, $\left(p_{xx}+\frac{dp_{xx}}{dx}dx\right)dy\,dz$;

" " " $dy\,dx$ nearest to O, $-p_{zx}\,dy\,dx$;

" " " " " farthest from O, $\left(p_{zx}+\frac{dp_{zx}}{dz}dz\right)dy\,dx$;

" " " $dz\,dx$ nearest to O, $-p_{yx}\,dz\,dx$;

" " " " " farthest from O. $\left(p_{yx}+\frac{dp_{yx}}{dy}dy\right)dz\,dx$.

The differential coefficients of the intensities are the rates of variation of those intensities for each unit of the variable, which, multiplied by the differentials of the variables, give the amounts of variation for the different edges of the parallelopiped.

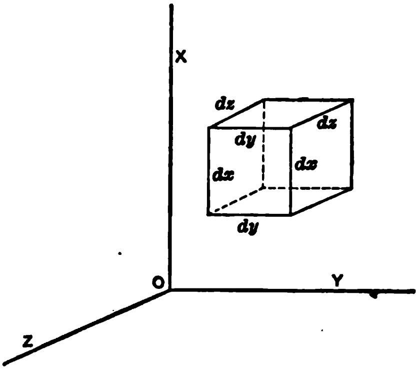

FIG. 1.

Let X_0 be the external force acting in the direction of X on a unit of volume at the point considered; then $X_0\,dx\,dy\,dz$ will be the amount of external force acting on the parallelopiped.

These constitute all the forces acting on the parallelopiped in the direction of the axis of X, and their sum, if unbalanced, must be equal to $m\frac{d^2u}{dt^2}dx\,dy\,dz$; in which m is the mass or inertia of a unit of volume, and dt the differential of the time. Forming such an equation, therefore, and dropping the common factor $dx\,dy\,dz$, there will result

$$\frac{dp_{xx}}{dx}+\frac{dp_{yx}}{dy}+\frac{dp_{zx}}{dz}+X_0=m\frac{d^2u}{dt^2}. \quad . \quad . \quad . \qquad (1)$$

Changing x to y, y to z, and z to x, eq. (1) will become

$$\frac{dp_{xy}}{dx}+\frac{dp_{yy}}{dy}+\frac{dp_{zy}}{dz}+Y_0=m\frac{d^2v}{dt^2}. \quad . \quad . \quad . \qquad (2)$$

Again, in eq. (1), changing x to z, z to y, and y to x,

$$\frac{dp_{xz}}{dx}+\frac{dp_{yz}}{dy}+\frac{dp_{zz}}{dz}+Z_0=m\frac{d^2w}{dt^2}. \quad . \quad . \quad . \qquad (3)$$

The line of action of the resultant of all the forces which act on the indefinitely small parallelopiped, at its limit, passes through its centre of gravity, *consequently it is subjected to the action of no unbalanced moment.* The parallelopiped, therefore, can have no rotation about an axis passing through its centre of gravity, whether it be in motion or equilibrium. Hence, let an axis passing through its centre of gravity and parallel to the axis of X, be considered. The only stresses, which, from their direction can possibly have moments about that axis, are those with the subscripts (*yz*), (*zy*), (*yy*), or (*zz*). But those with the last two subscripts act directly through the centre of the parallelopiped, consequently their moments are zero. The stresses $\frac{dp_{yz}}{dy}dy.dx\,dz$ and $\frac{dp_{zy}}{dz}dz.dx\,dy$ are two of six forces whose resultant is directly opposed to the resultant of those three forces which represent the increase of the intensities of the normal, or direct, stresses on three of the faces of the parallelopiped; these, therefore, have no moments about the assumed axis. The only stresses remaining are those whose intensities are p_{zy} and p_{yz}. The resultant moment, which must be equal to zero, then, has the following value:

$$p_{yz}dx\,dz.dy + p_{zy}dx\,dy.dz = 0; \quad . \quad . \quad . \quad (4)$$

$$\therefore \; p_{yz} = -p_{zy}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Hence the two intensities are equal to each other.

The negative sign in eq. (5) simply indicates that their *moments* have opposite signs or directions; consequently, that the shears themselves, on adjacent faces, act toward or from the edge between those faces. In eqs. (1), (2), and (3), the tangential stresses, or shears, are all to be affected by the same sign, since direct, or normal, stresses only can have different signs.

The eq. (5) is perfectly general, hence there may be written:

$$p_{xy}=p_{yx}, \text{ and } p_{zx}=p_{xz}. \quad . \quad . \quad . \quad . \quad (6)$$

Adopting the notation of Lamé, there may be written:

$$p_{xx}=N_1, \quad p_{yy}=N_2, \quad p_{zz}=N_3;$$
$$p_{zy}=T_1, \quad p_{xz}=T_2, \quad p_{xy}=T_3;$$

by which eqs. (1), (2), and (3) take the following forms:

$$\frac{dN_1}{dx}+\frac{dT_3}{dy}+\frac{dT_2}{dz}+X_0=m\frac{d^2u}{dt^2}; \quad . \quad . \quad . \quad (7)$$

$$\frac{dT_3}{dx}+\frac{dN_2}{dy}+\frac{dT_1}{dz}+Y_0=m\frac{d^2v}{dt^2}; \quad . \quad . \quad . \quad (8)$$

$$\frac{dT_2}{dx}+\frac{dT_1}{dy}+\frac{dN_3}{dz}+Z_0=m\frac{d^2w}{dt^2}. \quad . \quad . \quad . \quad (9)$$

The equations (11), (12), (13), (20), (21), and (22) of the preceding article are really kinematical in nature; in order that the principles of dynamics may hold, they must satisfy eqs. (7), (8), and (9). As the latter stand, by themselves, they are applicable to rigid bodies as well as elastic ones; but when the values of N and T, in terms of the strains u, v, and w, have been inserted, they are restricted, in their use, to elastic bodies only. With those values so inserted, they form the equations on which are based the mathematical theory of sound and light vibrations, as well as those of elastic rods, membranes, etc. In general, they are the equations of motion which the different parts of the body can have in reference to each other, in consequence of the elastic nature of the material of which the body is composed.

If all parts of the body are in equilibrium under the action of the internal stresses, the rates of variation of the strains $\frac{d^2u}{dt^2}$, $\frac{d^2v}{dt^2}$, and $\frac{d^2w}{dt^2}$, will each be equal to zero. Hence, eqs. (7), (8), and (9) will take the forms

$$\frac{dN_1}{dx}+\frac{dT_3}{dy}+\frac{dT_2}{dz}+X_0=0; \quad . \quad . \quad . \quad . \quad (10)$$

$$\frac{dT_3}{dx}+\frac{dN_2}{dy}+\frac{dT_1}{dz}+Y_0=0; \quad . \quad . \quad . \quad . \quad (11)$$

$$\frac{dT_2}{dx}+\frac{dT_1}{dy}+\frac{dN_3}{dz}+Z_0=0. \quad . \quad . \quad . \quad . \quad (12)$$

These are the general equations of equilibrium. As they stand, they apply to a rigid body. For an elastic body, the values of N and T from the preceding article, in terms of the strains u, v, and w, must satisfy these equations.

The eqs. (10), (11), and (12) express the three conditions of equilibrium that the sums of the forces acting on the small parallelopiped, taken in three rectangular co-ordinate directions, must each be equal to zero. The other three conditions, indicating that the three component moments about the same co-ordinate axes must each be equal to zero, are fulfilled by eqs. (5) and (6). The latter conditions really eliminate three of the nine unknown stresses. The remaining six consequently appear in both the equations of motion and equilibrium.

The equations (7) to (12), inclusive, belong to the interior of the body. At the exterior surface, only a portion of the small parallelopiped will exist, and that portion will be a tetrahedron, the base of which forms a part of the exterior surface of the body, and is acted upon by external forces. Let $\frac{da}{2}$ be the area of the base of this tetrahedron, and let p, q, and r be the angles which a normal to it forms with the three axes of X, Y, Z, respectively. Then will

$$da \cos p = dy\, dz,\ da \cos q = dz\, dx, \text{ and } da \cos r = dx\, dy.$$

Let P be the known intensity of the external force acting on da, and let π, χ, and ρ be the angles which its direction makes with the co-ordinate axes. Then there will result:

$$X_0 = P\, da . \cos \pi,\ Y_0 = P\, da . \cos \chi, \text{ and } Z_0 = P\, da . \cos \rho.$$

The origin is now supposed to be so taken that the apex of the tetrahedron is located between it and the base; hence that part of the parallelopiped in which acted the stresses involving the derivatives, or differential coefficients, is wanting; consequently those stresses are also wanting.

The sums of the forces, then, which act on the tetrahedron, in the co-ordinate directions, are the following:

$$\begin{aligned} &-(N_1\, dy\, dz + T_3\, dz\, dx + T_2\, dy\, dx) + Pda \cos \pi = 0; \\ &-(T_3\, dz\, dy + N_2\, dz\, dx + T_1\, dy\, dx) + Pda \cos \chi = 0; \\ &-(T_2\, dz\, dy + T_1\, dz\, dx + N_3\, dy\, dx) + Pda \cos \rho = 0. \end{aligned}$$

Substituting from above,

$$N_1 \cos p + T_3 \cos q + T_2 \cos r = P \cos \pi; \quad . \quad . \quad (13)$$

$$T_3 \cos p + N_2 \cos q + T_1 \cos r = P \cos \chi; \quad . \quad . \quad (14)$$

$$T_2 \cos p + T_1 \cos q + N_3 \cos r = P \cos \rho. \quad . \quad . \quad (15)$$

These equations must always be satisfied at the exterior surface of the body; and since the external forces must always be known, in order that a problem may be determinate, they will serve to determine constants which arise from the integration of the general equations of motion and equilibrium.

Art. 3.—Equations of Motion and Equilibrium in Semi-polar Co-ordinates.

For many purposes it is convenient to have the conditions of motion and equilibrium expressed in either semi-polar or polar co-ordinates; the first form of such expression will be given in this article.

The general analytical method of transformation of co-ordinates may be applied to the equations of the preceding article, but the direct treatment of an indefinitely small portion of the material, limited by co-ordinate surfaces, possesses many advantages. In Fig. 1 are shown both the small portion of material and the co-ordinates, semi-polar as well as rectangular. The angle made by a plane normal to ZY, and containing OX, with the plane XY is represented by ϕ; the distance of any point from OX, measured parallel to ZY, is called r; the third co-ordinate, normal to

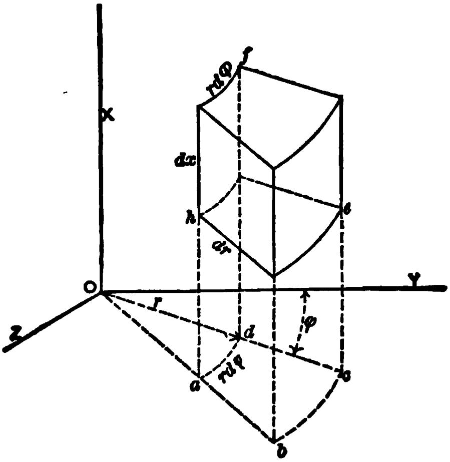

FIG. 1.

r and ϕ, is the co-ordinate x, as before. It is important to observe that the co-ordinates x, r, and ϕ, at any point, are *rectangular.*

The indefinitely small portion of material to be considered will, as shown in Fig. 1, be limited by the edges dx, dr, and $r\,d\phi$. The faces $dx\,dr$ are inclined to each other at the angle $d\phi$.

The intensities of the normal stresses in the directions of X and r will be indicated by N_1 and R, respectively. The remainder of the notation will be of the same general character as that in the preceding article; i.e., T_{xr} will represent a shear on the face $dr.r\,d\phi$ in the direction of r, while $N_{\phi\phi}$ is a normal stress, in the direction of ϕ, on the face $dx\,dr$.

The strains or displacements, in the directions of x, r, and ϕ, will be represented by u, ρ, and w; consequently the unbalanced forces in those directions, per unit of mass, will be

$$m\frac{d^2u}{dt^2},\quad m\frac{d^2\rho}{dt^2},\quad \text{and}\quad m\frac{d^2w}{dt^2}. \quad . \quad . \quad . \quad . \qquad (1)$$

Those forces acting on the faces hf, fe, and he, will be considered negative; those acting on the other faces, positive.

Forces Acting in the Direction of r.

$-R.r\,d\phi\,dx$, and

$$+Rr\,d\phi\,dx+\left(\frac{d(Rr)}{dr}dr=r\frac{dR}{dr}dr+R\,dr\right)d\phi\,dx.$$

$-T_{\phi r}dr\,dx$, and

$$+T_{\phi r}dr\,dx+\frac{dT_{\phi r}}{d\phi}d\phi\,.dr\,dx.$$

$-T_{xr}.r\,d\phi\,dr$, and

$$+T_{xr}.r\,d\phi\,dr+\frac{dT_{xr}}{dx}dx\,.r\,d\phi\,dr.$$

On the face $dr\,dx$, nearest to ZOX, there acts the normal stress $\left(N_{\phi\phi}dr\,dx+\frac{dN_{\phi\phi}}{d\phi}d\phi\,.\,dr\,dx\right)=N'$. Now N' has a component acting parallel to the face fe and toward OX, equal to $N'\sin(d\phi)=N'\frac{r\,d\phi}{r}=N'd\phi$. But the second term of this product will hold $(d\phi)^2$, hence it will disappear, at the limit, in the first derivative of $N'd\phi$ $\therefore$ $N'd\phi=N_{\phi\phi}d\phi\,dr\,dx$. Since this force must be taken as acting toward OX, it acts with the normal forces on hf, and, consequently, must be given the negative sign.

If R_0 is the external force acting on a unit of volume, another force (external) acting along r will be $R_0\,.\,r\,d\phi\,dr\,dx$.

The sum of all these forces will be equal to

$$m\,.\,r\,d\phi\,dr\,dx\,.\,\frac{d^2\rho}{dt^2}.$$

Forces Acting in the Direction of ϕ.

$-N_{\phi\phi}dr\,dx$, and

$+N_{\phi\phi}dr\,dx+\frac{dN_{\phi\phi}}{d\phi}d\phi\,.\,dr\,dx$.

$-T_{r\phi}\,.\,r\,d\phi\,dx$, and

$+T_{r\phi}\,.\,r\,d\phi\,dx+\left(\frac{d(rT_{r\phi})}{dr}dr=r\frac{dT_{r\phi}}{dr}dr+T_{r\phi}dr\right)d\phi\,dx$.

$-T_{x\phi}\,.\,r\,d\phi\,dr$, and

$+T_{x\phi}\,.\,r\,d\phi\,dr+\frac{dT_{x\phi}}{dx}dx\,.\,r\,d\phi\,dr$.

As in the case of $N_{\phi\phi}$, in connection with the forces along r, so the force $T_{\phi r}\,dr\,dx$ has a component along ϕ (normal to fe) equal to $T_{\phi r}\,dr\,dx\,.\,\sin(d\phi)=T_{\phi r}\,d\phi\,dr\,dx$. It will have a positive sign, because it acts from OX.

The external force is $\Phi_0\,.\,r\,d\phi\,dr\,dx$.

Forces Acting in the Direction of x.

$-N_1 . r\, d\phi\, dr$, and

$+N_1 r\, d\phi\, dr + \frac{dN_1}{dx} dx . r\, d\phi\, dr.$

$-T_{rx} . dx\, r\, d\phi$, and

$+T_{rx} . dx\, r\, d\phi + \left(\frac{d(rT_{rx})}{dr} dr = r \frac{dT_{rx}}{dr} dr + T_{rx}\, dr \right) dx\, d\phi.$

$-T_{\phi x} dx\, dr$, and

$+T_{\phi x} dx\, dr + \frac{dT_{\phi x}}{d\phi} d\phi . dx\, dr.$

The external force is $X_0 . r\, d\phi\, dx\, dr$.

Putting each of these three sums equal to the proper rates of variation of momentum, and dropping the common factor $r\, d\phi\, dx\, dr$:

$$\frac{dN_1}{dx} + \frac{dT_{rx}}{dr} + \frac{dT_{\phi x}}{r\, d\phi} + \frac{T_{rx}}{r} \qquad + X_0 = m \frac{d^2 u}{dt^2}; \quad . \quad (2)$$

$$\frac{dT_{xr}}{dx} + \frac{dR}{dr} \quad + \frac{dT_{\phi r}}{r\, d\phi} + \frac{R - N_{\phi\phi}}{r} + R_0 = m \frac{d^2 \rho}{dt^2}; \quad . \quad (3)$$

$$\frac{dT_{x\phi}}{dx} + \frac{dT_{r\phi}}{dr} + \frac{dN_{\phi\phi}}{r\, d\phi} + \frac{T_{r\phi} + T_{r\phi}}{r} + \Phi_0 = m \frac{d^2 w}{dt^2}. \quad . \quad (4)$$

These are the general equations of motion (vibration) in terms of semi-polar co-ordinates; if the second members are made equal to zero, they become equations of equilibrium. Eqs. (2), (3), and (4), are not dependent upon the nature of the body.

Since x, r, and ϕ are rectangular, it at once follows that

$$T_{rx} = T_{xr}, \; T_{r\phi} = T_{\phi r}, \text{ and } T_{x\phi} = T_{\phi x}. \quad . \quad . \quad (5)$$

In order that eqs. (2), (3), and (4) may be restricted to elastic bodies, it is necessary to express the six intensities of stresses involved, in terms of the rates of variation of the strains in the rectangular co-ordinate directions of x, r, and ϕ. Since these co-ordinates are rectangular, the eqs. (11), (12), (13), (20), (21), and (22) of Article 1, may be made applicable to the present case by some very simple changes dependent upon the nature of semi-polar co-ordinates.

For the present purpose the strains in the co-ordinate directions of x, y, and z will be represented by u', v', and w'. Since the axis of x remains the same in the two systems, evidently

$$\frac{du'}{dx}=\frac{du}{dx}.$$

From Fig. 1 it is clear that the axis of y corresponds exactly to the co-ordinate direction r; hence

$$\frac{dv'}{dy}=\frac{d\rho}{dr}.$$

From the same Fig. it is seen that the axis of z corresponds to ϕ, or $r\phi$. Bu the total differential, dw', must be considered as made up of two parts; consequently the rate of variation $\frac{dw'}{dz}$ will consist of two parts also. If there is no distortion in the direction of r, or if the distance of a molecule from the origin remains the same, one part will be $\frac{dw}{d(r\phi)}=\frac{dw}{r\,d\phi}$. If, however, a unit's length of material be removed from the distance r to $r+\rho$ from the centre O, Fig. 1, while ϕ remains constant, its length will be changed from 1 to $1.\left(1+\frac{\rho}{r}\right)$, in which ρ may be implicitly positive or negative. Consequently there will result

$$\frac{dw'}{dz}=\frac{dw}{r\,d\phi}+\frac{\rho}{r}.$$

For the reason already given, there follow

$$\frac{du'}{dy}=\frac{du}{dr} \quad \text{and} \quad \frac{dv'}{dx}=\frac{d\rho}{dx}.$$

In Fig. 2 let dc be the side of a distorted small portion of the material, the original position of which was $d'e$. Od is the distance r from the origin, $ad=dr$ and $ac=dw$, while $dd'=w$. The angular change in position of dc is $\frac{ac}{ad}=\frac{dw}{dr}$; but an amount equal to $\frac{ab}{ad}=\frac{w}{r}$ is due to the movement of r, and is not a movement of dc relatively to the material immediately adjacent to d.

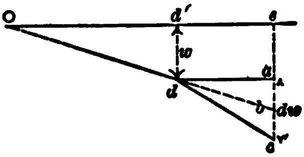

FIG. 2.

Hence

$$\frac{dw'}{dy}=\frac{dw}{dr}-\frac{w}{r}, \quad \text{also} \quad \frac{dv'}{dz}=\frac{d\rho}{r\,d\phi}.$$

There only remain the following two, which may be at once written

$$\frac{dw'}{dx}=\frac{dw}{dx} \quad \text{and} \quad \frac{du'}{dz}=\frac{du}{r\,d\phi}.$$

The rate of variation of volume takes the following form in terms of the new co-ordinates:

$$\theta=\frac{du'}{dx}+\frac{dv'}{dy}+\frac{dw'}{dz}=\frac{du}{dx}+\frac{d\rho}{dr}+\frac{dw}{r\,d\phi}+\frac{\rho}{r}. \quad . \quad . \quad (6)$$

Accenting the intensities which belong to the rectangular system x, y, z, the eqs. (11), (12), (13), (20), (21), and (22). of Art. 1, take the following form:

$$N_1 = N_1' = \frac{2G\mathfrak{r}}{1-2\mathfrak{r}}\theta + 2G\frac{du}{dx}; \quad . \; . \; . \; . \; . \; . \quad (7)$$

$$R = N_2' = \frac{2G\mathfrak{r}}{1-2\mathfrak{r}}\theta + 2G\frac{d\rho}{dr}; \quad . \; . \; . \; . \; . \; . \quad (8)$$

$$N_{\phi\phi} = N_3' = \frac{2G\mathfrak{r}}{1-2\mathfrak{r}}\theta + 2G\left(\frac{dw}{r\,d\phi} + \frac{\rho}{r}\right); \quad . \; . \quad (9)$$

$$T_{xr} = T_3' = G\left(\frac{du}{dr} + \frac{d\rho}{dx}\right); \quad . \; . \; . \; . \; . \; . \; . \quad (10)$$

$$T_{r\phi} = T_1' = G\left(\frac{d\rho}{r\,d\phi} + \frac{dw}{dr} - \frac{w}{r}\right); \quad . \; . \; . \; . \quad (11)$$

$$T_{\phi x} = T_2' = G\left(\frac{dw}{dx} + \frac{du}{r\,d\phi}\right). \quad . \; . \; . \; . \; . \; . \; . \quad (12)$$

If these values are introduced in eqs. (2), (3), and (4), those equations will be restricted in application to bodies of homogeneous elasticity only.

The notation $\mathfrak{r}$ is used to indicate that the r involved is the ratio of lateral to direct strain, and that it has no relation whatever to the co-ordinate r.

The limiting equations of condition, (13), (14), and (15) of Art. 2, remain the same, except for the changes of notation, shown in eqs. (7) to (12), for the intensities N and T.

Art. 4.—Equations of Motion and Equilibrium in Polar Co-ordinates.

The relation, in space, existing between the polar and rectangular systems of co-ordinates is shown in Fig. 1. The angle ϕ is measured in the plane ZY and from that of XY; while ψ is measured normal to ZY in a plane which contains OX. The analytical relation existing between the two systems is, then, the following:

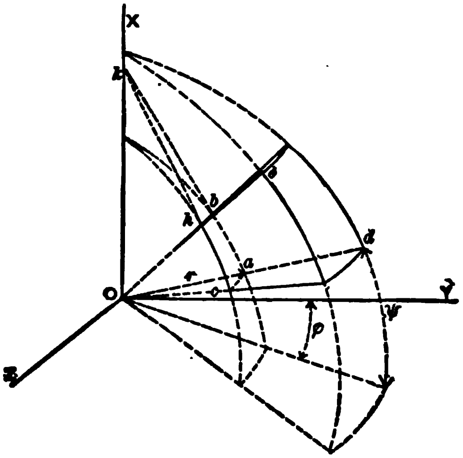

FIG. 1.

$$x = r\sin\psi, \quad y = r\cos\psi\cos\phi, \quad \text{and} \quad z = r\cos\psi\sin\phi.$$

The indefinitely small portion of material to be considered is $a\,h\,e\,d$. It is limited by the co-ordinate planes located by ϕ and ψ, and concentric spherical surfaces with radii r and $r+dr$. The directions r, ϕ, and ψ, at any point, are rectangular; hence the sums of the forces acting on the small portion of the material, taken in these directions, must be found and put equal to

$$m\frac{d^2\rho}{dt^2}, \quad m\frac{d^2\eta}{dt^2}, \quad \text{and} \quad m\frac{d^2\omega}{dt^2},$$

in which expressions, ρ, η, and ω represent the strains in the direction of r, ϕ, and ψ respectively.

Those forces which act on the faces *ah*, *bd*, and *cd* will be considered negative, and those which act on the other faces positive.

The notation will remain the same as in the preceding articles, except that the three normal stresses will be indicated by N_r, N_ϕ, and N_ψ.

Forces Acting Along r.

$$-N_r . r\, d\psi\, r \cos\psi\, d\phi.$$

$$+N_r . r^2 \cos\psi\, d\psi\, d\phi$$

$$+\left(\frac{d(N_r r^2)}{dr} dr = r^2 \frac{dN_r}{dr} dr + 2r N_r dr\right) \cos\psi\, d\psi\, d\phi.$$

$$-T_{\phi r} . r\, d\psi\, dr.$$

$$+T_{\phi r} . r\, d\psi\, dr + \frac{dT_{\phi r}}{d\phi} d\phi . r\, d\psi\, dr.$$

$$-T_{\psi r} . r \cos\psi\, d\phi\, dr.$$

$$+T_{\psi r} . r \cos\psi\, d\phi\, dr$$

$$+\left(\frac{d(T_{\psi r} \cos\psi)}{d\psi} d\psi = \cos\psi \frac{dT_{\psi r}}{d\psi} d\psi - T_{\psi r} \sin\psi\, d\psi\right) r\, d\phi\, dr.$$

$-N_\phi . r\, d\psi\, dr . \sin aOc = -N_\phi . r\, d\psi\, dr . \cos\psi\, d\phi$, on face *ce*.

$-N_\psi . r \cos\psi\, d\phi\, dr . \sin aOb = -N_\psi . r \cos\psi\, d\phi\, dr . d\psi$,
on face *be*.

Forces Acting Along φ.

$$-T_{r\phi} . r \cos\psi\, d\phi\, r\, d\psi.$$

$$+T_{r\phi} . r^2 \cos\psi\, d\phi\, d\psi$$

$$+\left(\frac{d(T_{r\phi} r^2)}{dr} dr = r^2 \frac{dT_{r\phi}}{dr} dr + 2r\, T_{r\phi} dr\right) \cos\psi\, d\phi\, d\phi.$$

$-N_\phi . r\, d\psi\, dr.$

$+N_\phi . r\, d\psi\, dr + \frac{dN_\phi}{d\phi} d\phi\, r\, d\psi\, dr.$

$-T_{\psi\phi} . r \cos \psi\, d\phi\, dr.$

$+T_{\psi\phi} \cos \psi . r\, d\phi\, dr$

$$+\left(\frac{d(T_{\psi\phi} \cos \psi)}{d\psi} d\psi = \cos \psi \frac{dT_{\phi\phi}}{d\psi} d\psi - T_{\phi\phi} \sin \psi\, d\psi\right) r\, d\phi\, dr.$$

$+T_{\phi r}\, r\, d\psi\, dr . \cos \psi\, d\phi,$ on face *ce*.

$-T_{\phi\psi}\, r\, d\psi\, dr\left(\sin akc = \frac{r \cos \psi\, d\phi}{r \cot \psi}\right) = -T_{\phi\psi}\, r\, d\psi\, dr . \sin \psi\, d\phi,$ on face *ce*.

The lines *ak* and *ck* are drawn normal to *Oc* and *Oa*,

Forces Acting Along ψ.

$-T_{r\phi} . r \cos \psi\, d\phi . r\, d\psi.$

$+T_{r\phi}\, r^2 \cos \psi\, d\phi\, d\psi$

$$+\left(\frac{d(T_{r\phi} r^2)}{dr} dr = r^2 \frac{dT_{r\phi}}{dr} ar + 2r\, T_{r\phi}\, dr\right) \cos \psi\, d\psi\, d\phi.$$

$-T_{\phi\psi} . r\, d\psi\, dr.$

$+T_{\phi\psi}\, r\, d\psi\, dr + \frac{dT_{\phi\psi}}{d\phi} d\phi . r\, d\psi\, dr.$

$-N_\psi . r \cos \psi\, d\phi\, dr.$

$+N_\psi . r \cos \psi\, d\phi\, dr$

$$+\left(\frac{d\,(N_\psi \cos \psi)}{d\psi} d\psi = \cos \psi \frac{dN_\psi}{d\psi} d\psi - N_\psi \sin \psi\, d\psi\right) r\, d\phi\, dr.$$

$+T_{\psi r} . r \cos \psi\, d\phi\, dr . d\psi,$ on face *be*.

$+N_\phi . r\, d\psi\, dr . \sin akc = +N_\phi\, r\, d\psi\, dr . \sin \psi\, d\phi,$ on face *ce*.

The volume of the indefinitely small portion of the material is (omitting second powers of indefinitely small quantities)

$$r \cos \psi \, d\phi . r \, d\psi . dr = \Delta V,$$

and its mass is m multiplied by this small volume. The latter may be made a common factor in each of the three sums to be taken.

The external forces acting in the directions R, ϕ, and ψ will be represented by

$$R_0 \Delta V, \quad \Phi_0 \Delta V, \quad \text{and} \quad \Psi_0 \Delta V,$$

respectively.

Taking each of the three sums, already mentioned, and dropping the common factor ΔV, there will result

$$\frac{dN_r}{dr} + \frac{dT_{\phi r}}{r \cos \psi . d\phi} + \frac{dT_{\psi r}}{r \, d\psi} + \frac{2N_r - N_\phi - N_\psi - T_{\psi r} \tan \psi}{r} + R_0 = m \frac{d^2 \rho}{dt^2}; \quad (1)$$

$$\frac{dT_{r\phi}}{dr} + \frac{dN_\phi}{r \cos \psi . d\phi} + \frac{dT_{\psi\phi}}{r \, d\psi} + \frac{2T_{r\phi} + T_{\phi r} - T_{\psi\phi} \tan \psi - T_{\phi\psi} \tan \psi}{r} + \Phi_0 = m \frac{d^2 \eta}{dt^2}; \quad (2)$$

$$\frac{dT_{r\psi}}{dr} + \frac{dT_{\phi\psi}}{r \cos \psi \, d\phi} + \frac{dN_\psi}{r \, d\psi} + \frac{2T_{r\psi} + T_{\psi r} - N_\phi \tan \psi + N_\psi \tan \psi}{r} + \Psi_0 = m \frac{d^2 \omega}{dt^2}. \quad (3)$$

Since r, ϕ, and ψ are rectangular at any point,

$$T_{\phi r} = T_{r\phi}, \quad T_{r\psi} = T_{\psi r}, \quad \text{and} \quad T_{\psi\phi} = T_{\phi\psi}.$$

Hence

$$\frac{2T_{r\phi}+T_{\phi r}-\tan\psi(T_{\psi\phi}+T_{\phi\psi})}{r}=\frac{3T_{r\phi}-2\tan\psi.T_{\psi\phi}}{r},$$

$$\frac{2T_{r\psi}+T_{\psi r}-\tan\psi(N_{\psi}-N_{\phi})}{r}=\frac{3T_{r\psi}-\tan\psi(N_{\psi}-N_{\phi})}{r},$$

These relations somewhat simplify the first members of eqs. (2) and (3).

Eqs. (1), (2), and (3) are entirely independent of the nature of the material; also, they apply to the case of equilibrium, if the second members are made equal to zero.

The rectangular rates of strain, at any point, in terms of r, ϕ, and ψ are next to be found. As in the preceding article, the rates of strain in the rectangular directions of r, ϕ, and ψ will be indicated by

$$\frac{dv'}{dy'},\ \frac{dw'}{dz'},\ \frac{du'}{dx'},\ \frac{dv'}{dx'},\ \frac{du'}{dy'},\ \text{etc.}$$

Remembering the reasoning in connection with the value of $\frac{dw'}{dz}$, in the preceding article, and attentively considering Fig. 1, there may at once be written,

$$\frac{du'}{dx'}=\frac{d\omega}{r\,d\psi}+\frac{\rho}{r}.$$

In Fig. 1, if $ac=1$ and $ab=\omega$, while $ak=r\cot.\psi$ (ak is perpendicular to aO), the difference in length between ac and bh will be

$$-\frac{\omega}{r\cot\psi}=-\frac{\omega\tan\psi}{r}.$$

This expression is negative because a *decrease* in length takes place in consequence of a movement in the *positive* direction of $r\psi$.

Again, a consideration of Fig. 1, and the reasoning connected with the equation above, will give

$$\frac{dw'}{dz'} = \frac{d\eta}{r\cos\psi\, d\phi} + \frac{\rho}{r} - \frac{\omega\tan\psi}{r}.$$

Without explanation there may at once be written:

$$\frac{dv'}{dy'} = \frac{d\rho}{dr}.$$

Fig. 1 of this, and Fig. 2 of the preceding article, give

$$\frac{du'}{dy'} = \frac{d\omega}{dr} - \frac{\omega}{r} \quad \text{and} \quad \frac{dv'}{dx'} = \frac{d\rho}{r\, d\psi}.$$

These are to be used in the expression for $T_{\psi r}$. Precisely the same figures and method give

$$\frac{dv'}{dz'} = \frac{d\rho}{r\cos\psi\, d\phi} \quad \text{and} \quad \frac{dw'}{dy'} = \frac{d\eta}{dr} - \frac{\eta}{r},$$

which are to be used in finding $T_{\phi r}$.

The expression for $\frac{dw'}{dx'}$ will be composed of the *sum* of two parts. In Fig. 2, *ab* is the original position of $r\, d\psi$, and after the strain η exists it takes the position *ec*. Consequently *ac* (equal and parallel to *bd* and perpendicular to *ak*) represents the strain η, while *ed* represents $d\eta$. Since, also, *fc* is perpendicular to *ck*, the strains of the kind η change the right angle *fck* to the angle *fce*; or the angle *eck* is equal to

$$\frac{dw'}{dx'} = ecd + dck = \frac{ed}{dc} + \frac{ca}{ak}$$

$$= \frac{d\eta}{r\, d\psi} + \frac{\eta}{r\cot\psi}.$$

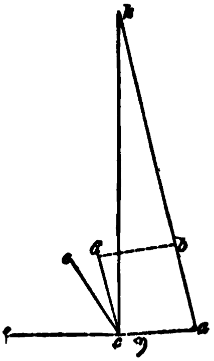

FIG. 2.

In Fig. 2, the points *a*, *b*, and *k* are identical with the points similarly lettered in Fig. 1. The expression for $\frac{du'}{dz'}$ may be at once written from Fig. 1. There may, then, finally be written,

$$\frac{dw'}{dx'}=\frac{d\eta}{r\,d\psi}+\frac{\eta\tan\psi}{r}\quad\text{and}\quad\frac{du'}{dz'}=\frac{d\omega}{r\cos\psi\,d\phi}.$$

These equations will give the expression for $T_{\phi\phi}$.

The value of

$$\theta=\frac{du'}{dx'}+\frac{dv'}{dy'}+\frac{dw'}{dz'}$$

now takes the following form:

$$\theta=\frac{d\rho}{dr}+\frac{d\eta}{r\cos\psi\,d\phi}+\frac{d\omega}{r\,d\psi}+\frac{2\rho}{r}-\frac{\omega\tan\psi}{r}. \quad . \quad . \quad (4)$$

The last two terms are characteristic of the spherical co-ordinates.

The eqs. (20), (21), (22), (11), (12), and (13), of Art. 1, take the forms

$$N_r=\frac{2G\mathfrak{r}}{1-2\mathfrak{r}}\theta+2G\frac{d\rho}{dr}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$N_\phi=\frac{2G\mathfrak{r}}{1-2\mathfrak{r}}\theta+2G\left(\frac{d\eta}{r\cos\psi\,d\phi}+\frac{\rho}{r}-\frac{\omega\tan\psi}{r}\right); \quad . \quad (6)$$

$$N_\psi=\frac{2G\mathfrak{r}}{1-2\mathfrak{r}}\theta+2G\left(\frac{d\omega}{r\,d\psi}+\frac{\rho}{r}\right); \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$T_{\phi\phi}=G\left(\frac{d\eta}{r\,d\psi}+\frac{d\omega}{r\cos\psi\,d\phi}+\frac{\eta\tan\psi}{r}\right); \quad . \quad . \quad . \quad . \quad (8)$$

$$T_{r\phi}=G\left(\frac{d\omega}{dr}-\frac{\omega}{r}+\frac{d\rho}{r\,d\phi}\right); \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$T_{r\phi}=G\left(\frac{d\rho}{r\cos\psi\,d\phi}+\frac{d\eta}{dr}-\frac{\eta}{r}\right); \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

If these values are inserted in eqs. (1), (2), and (3), the resulting equations will be applicable to isotropic material only.

As in the preceding article, $\mathbf{r}$ is used to express the ratio between direct and lateral strains, and has no relation whatever to the co-ordinate r.

It is interesting and important to observe that the equations of motion and equilibrium for elastic bodies are only special cases of equations which are entirely independent of the nature of the material, of equations, in fact, which express the most general conditions of motion or equilibrium.

CHAPTER II.

THICK, HOLLOW CYLINDERS AND SPHERES, AND TORSION.

Art. 5.—Thick, Hollow Cylinders.

IN Fig. 1 is represented a section, taken normal to its axis, of a circular cylinder whose walls are of the appreciable thickness t. Let p and p_1 represent the interior and exterior intensities of pressures, respectively. The material will not be stressed with uniform intensity throughout the thickness t. Yet if that thickness, comparatively speaking, is small, the variation will also be small; or, in other words, the intensity of stress throughout the thickness t may be considered constant. This approximate case will first be considered.

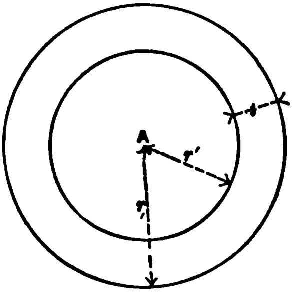

FIG. 1.

The interior intensity p will be considered greater than the exterior p_1, consequently the tendency will be toward rupture along a diametral plane. If, at the same time, the ends of the cylinder are taken as closed, as will be done, a tendency to rupture through the section shown in the figure will exist.

The force tending to produce rupture of the latter kind will be

$$F = \pi(pr'^2 - p_1 r_1^2). \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If N_1 represents the intensity of stress developed by this force,

$$N_1 = \frac{F}{\pi(r_1{}^2 - r'^2)} = \frac{pr'^2 - p_1 r_1{}^2}{r_1{}^2 - r^2}. \quad . \quad . \quad . \quad . \quad (2)$$

If the exterior pressure is zero, and if r' is nearly equal to

$$\frac{r_1 + r'}{2},$$

$$N_1 = \frac{pr'}{2(r_1 - r')} = \frac{pr'}{2t}. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

In this same approximate case, the tendency to split the cylinder along a diametral plane, for unit of length, will be

$$F' = pr' - p_1 r_1.$$

If N' is the intensity of stress developed by F',

$$N' = \frac{F'}{t} = \frac{pr' - p_1 r_1}{t}. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

N' is thus seen to be *twice* as great as N_1 when $p_1 = 0$. If, therefore, the material has the same ultimate resistance in both directions the cylinder will fail longitudinally when the interior intensity is only *half* great enough to produce transverse rupture, *the thickness being assumed to be very small and the exterior pressure zero.*

N_1 and N' are tensile stresses, because the interior pressure was assumed to be large compared with the exterior. If the opposite assumption were made, they would be found to be compression, while the general forms would remain exactly the same.

The preceding formulas are too loosely approximate for many cases. The exact treatment requires the use of the general equations of equilibrium, and the forms which they take in Art. 3 are particularly convenient. As in that article, the axis of x will be taken as the axis of the cylinder.

Since all external pressure is uniform in intensity and normal in direction, no shearing stresses will exist in the material of the cylinder. This condition is expressed in the notation of Art. 3 by putting

$$T_{\phi x} = T_{rx} = T_{r\phi} = 0.$$

Again the cylinder will be considered closed at the ends, and the force F, eq. (1), will be assumed to develop a stress of *uniform* intensity throughout the transverse section shown in Fig. 1. This condition, in fact, is involved in that of making all the tangential stresses equal to zero.

Since this case is that of equilibrium, the equations (2), (3), and (4) of Art. 3 take the following form, after neglecting X_0, R_0, and Φ_0:

$$\frac{dN_1}{dx} = 0; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$\frac{dR}{dr} + \frac{R - N_{\phi\phi}}{r} = 0; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$\frac{dN_{\phi\phi}}{r\,d\phi} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

These equations are next to be expressed in terms of the strains u, ρ, and w.

In consequence of the manner of application of the external forces, all movements of indefinitely small portions of the material will be along the radii and axis of the cylinder. Hence

u will be independent of r and ϕ;
ρ " " " " ϕ " x;
$w=0$.

The rate of change, therefore, of volume will be (eq. (6) of Art. 3)

$$\theta=\frac{du}{dx}+\frac{d\rho}{dr}+\frac{\rho}{r}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

As ρ is independent of x, $\frac{d\theta}{dx}=\frac{d^2u}{dx^2}$; hence if the value of N_1 be taken from eq. (7) of Art. 3 and put in eq. (5) of this article,

$$\frac{dN_1}{dx}=\frac{2G\mathfrak{r}}{1-2\mathfrak{r}}\frac{d^2u}{dx^2}+2G\frac{d^2u}{dx^2}=0;$$

$$\therefore \frac{d^2u}{dx^2}=0 \quad \text{and} \quad u=ax+a'.$$

But the transverse section in which the origin is located may be considered fixed. Consequently if $x=0$, $u=0$ and thus $a'=0$. The expression for u is then $u=ax$.

The ratio $u\div x$ is the l of eq. (1), on page 3, while the p of the same equation is simply N_1 of eq. (2), given above. Hence

$$a=\frac{u}{x}=\frac{N_1}{E}=\frac{pr'^2-p_1r_1^2}{E(r_1^2-r'^2)}. \quad . \quad . \quad . \quad . \quad (9)$$

Again, eq. (8) of Art. 3, in connection with eqs. (8) and (6) of this, gives

$$\frac{2G\mathfrak{r}}{1-2\mathfrak{r}}\left(\frac{d^2\rho}{dr^2}+\frac{d\rho}{r\,dr}-\frac{\rho}{r^2}\right)+2G\left(\frac{d^2\rho}{dr^2}+\frac{d\rho}{r\,dr}-\frac{\rho}{r^2}\right)=0.$$

$$\therefore\ \frac{d^2\rho}{dr^2}+\frac{d\rho}{r\,dr}-\frac{\rho}{r^2}=\frac{d^2\rho}{dr^2}+\frac{d\left(\frac{\rho}{r}\right)}{dr}=0.$$

$$\therefore\ \frac{d\rho}{dr}+\frac{\rho}{r}=c,\ \text{or}$$

$$r\,d\rho+\rho\,dr=d(\rho r)=cr\,dr.$$

$$\therefore\ \rho r=\frac{cr^2}{2}+b,\ \text{or}\ \ \rho=\frac{cr}{2}+\frac{b}{r}.\quad .\ .\ .\quad (10)$$

This value of ρ in eqs. (8) and (9) of Art. 3 will give

$$R=2G\left\{\frac{\mathfrak{r}(a+c)}{1-2\mathfrak{r}}+\frac{c}{2}-\frac{b}{r^2}\right\};\quad .\ .\ .\ .\quad (11)$$

$$N_{\phi\phi}=2G\left\{\frac{\mathfrak{r}(a+c)}{1-2\mathfrak{r}}+\frac{c}{2}+\frac{b}{r^2}\right\}.\quad .\ .\ .\ .\ .\quad (12)$$

At the interior surface R must be equal to the internal pressure, and at the exterior surface to the external pressure. Or since negative signs indicate compression,

If $r=r'$ . . . . . . $R=-p$.
If $r=r_1$ . . . . . . $R=-p_1$.

Either of these equations is the simple result of applying eqs. (13), (14), and (15) to the present case, for which

$$\cos p=\cos r=\cos\pi=\cos\rho=0,$$
$$\cos q=\cos\chi=1,\ \text{and}\ P=-p\ \text{or}\ -p_1.$$

Applying eq. (11) to the two surfaces,

$$-p = 2G\left\{\frac{\mathfrak{r}(a+c)}{1-2\mathfrak{r}} + \frac{c}{2} - \frac{b}{r'^2}\right\}; \quad . \quad . \quad . \quad (13)$$

$$-p_1 = 2G\left\{\frac{\mathfrak{r}(a+c)}{1-2\mathfrak{r}} + \frac{c}{2} - \frac{b}{r_1^2}\right\}. \quad . \quad . \quad . \quad (14)$$

Subtracting (14) from (13),

$$2Gb = \frac{(p_1 - p)\, r_1^2 r'^2}{r'^2 - r_1^2}.$$

Inserting this value in eq. (13),

$$2G\left\{\frac{\mathfrak{r}(a+c)}{1-2\mathfrak{r}} + \frac{c}{2}\right\} = \frac{p_1 r_1^2 - p r'^2}{r'^2 - r_1^2}.$$

The general expressions of R and $N_{\phi\phi}$, freed from the arbitrary constants of integration, can now be easily written by inserting these last two values in eqs. (11) and (12). By making the insertions there will result

$$R = \frac{p_1 r_1^2 - p r'^2}{r'^2 - r_1^2} - \frac{(p_1 - p)\, r_1^2 r'^2}{r'^2 - r_1^2} \cdot \frac{1}{r^2}; \quad . \quad . \quad (15)$$

$$N_{\phi\phi} = \frac{p_1 r_1^2 - p r'^2}{r'^2 - r_1^2} + \frac{(p_1 - p)\, r_1^2 r'^2}{r'^2 - r_1^2} \cdot \frac{1}{r^2}. \quad . \quad . \quad (16)$$

The stress $N_{\phi\phi}$ is a tension directed *around* the cylinder, and has been called "hoop tension." Eq. (16) shows that the hoop tension will be greatest at the interior of the cylinder. An expression for the thickness, t, of the annulus in terms of the greatest hoop tension (which will be called h) can easily be obtained from eq. (16).

If $r = r'$ in that equation,

$$h = \frac{2p_1 r_1^2 - p(r'^2 + r_1^2)}{r'^2 - r_1^2}.$$

$$\therefore \frac{r_1}{r'} = \left(\frac{h+p}{2p_1 - p + h}\right)^{\frac{1}{2}}.$$

$$\therefore r_1 - r' = t = r'\left\{\left(\frac{h+p}{2p_1 - p + h}\right)^{\frac{1}{2}} - 1\right\}. \quad . \quad . \quad (17)$$

Eq. (17) will enable the thickness to be so determined that the hoop tension shall not exceed any assigned limit h. If p_1 is so small in comparison with p that it may be neglected, t will become

$$t = r'\left\{\left(\frac{h+p}{h-p}\right)^{\frac{1}{2}} - 1\right\}. \quad . \quad . \quad . \quad . \quad . \quad (18)$$

If p_1 is greater than p, $N_{\phi\phi}$ becomes compression, but the equations are in no manner changed.

The values of the constants b and c may easily be found from the two equations immediately preceding eq. (15).

It is interesting to notice that the rate of change of volume, θ, is equal to $(a+c)$ and therefore constant for all points.

Art. 6.—Torsion in Equilibrium.

The formulas to be deduced in this article are those first given by Saint-Venant, but, with one or two exceptions, established in a different manner.

It will in all cases, except that of the final result for a rectangular cross-section, be convenient to use those equations of Art. 3 which are given in terms of semi-polar coordinates.

Let Fig. 1 represent a cylindrical piece of material, with any cross-section, fixed in the plane ZY, and let the origin of co-ordinates be taken at O. Let it be twisted also by a couple

$$P.ab = Pl,$$

the plane of which is parallel to ZY. The material will thus be subjected to no bending, but to pure torsion.

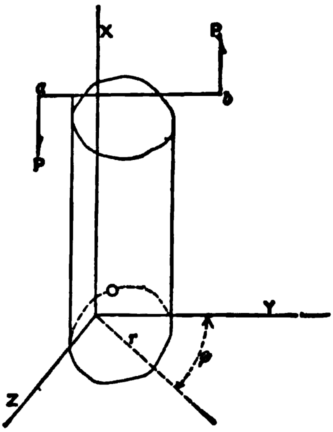

Fig. 1.

The axis of the piece is supposed to be parallel to the axis of X as well as the axis of the couple. Normal sections of the piece, originally parallel to ZOY, will not remain plane after torsion takes place. But the tendency to twist any elementary portion of the piece about an axis passing through its centre and parallel to the axis of X will be very small compared with the tendency to twist it about either the axis of r or ϕ; consequently the first will be neglected. In the notation of Art. 3, this condition is equivalent to making $T_{r\phi} = 0$.

As the piece is acted upon by a couple only, all normal stresses will be zero.

Eqs. (7), (8), (9), and (11) of Art. 3 then become

$$N_1 = \frac{2G\mathfrak{r}}{1-2\mathfrak{r}}\theta + 2G\frac{du}{dx} = 0; \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$R = \frac{2G\mathfrak{r}}{1-2\mathfrak{r}}\theta + 2G\frac{d\rho}{dr} = 0; \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$N_{\phi\phi} = \frac{2G\mathfrak{r}}{1-2\mathfrak{r}}\theta + 2G\left(\frac{dw}{r\,d\phi} + \frac{\rho}{r}\right) = 0; \quad . \quad . \quad (3)$$

$$T_{r\phi} = G\left(\frac{d\rho}{r\,d\phi} + \frac{dw}{dr} - \frac{w}{r}\right) = 0. \quad . \quad . \quad . \quad . \quad (4)$$

After introducing the values of T_{rx} and $T_{\phi x}$, from eqs. (10) and (12) of Art. 3, in eqs. (2), (3), and (4) of the same article, at the same time making the external forces and second members of those equations equal to zero, and bearing in mind the conditions given above, there will result

$$\frac{dT_{rx}}{dr}+\frac{dT_{\phi x}}{r\,d\phi}+\frac{T_{rx}}{r}$$
$$=G\left(\frac{d^2u}{dr^2}+\frac{d^2\rho}{dr\,dx}+\frac{d^2w}{r\,d\phi\,dx}+\frac{d^2u}{r^2d\phi^2}+\frac{du}{r\,dr}+\frac{d\rho}{r\,dx}\right)=0; \quad (5)$$

$$\frac{dT_{rx}}{dx}=G\left(\frac{d^2u}{dr\,dx}+\frac{d^2\rho}{dx^2}\right)=0; \quad . \quad . \quad . \quad . \quad (6)$$

$$\frac{dT_{x\phi}}{dx}=G\left(\frac{d^2w}{dx^2}+\frac{d^2u}{r\,d\phi\,dx}\right)=0. \quad . \quad . \quad . \quad . \quad (7)$$

Also by eq. (6) of Art. 3,

$$\theta=\frac{du}{dx}+\frac{d\rho}{dr}+\frac{dw}{r\,d\phi}+\frac{\rho}{r}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The cylindrical piece of material is supposed to be of such length that the portion to which these equations apply is not affected by the manner of application of the couple. This portion is, therefore, twisted uniformly from end to end; consequently the strain u will not vary with any change in x. Hence

$$\frac{du}{dx}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Eq. (1) then shows that $\theta=0$. This was to be anticipated, since a pure shear cannot change the volume or density. Because $\theta = 0$, eqs. (2) and (3) at once give

$$\frac{d\rho}{dr} - \frac{dw}{r\,d\phi} + \frac{\rho}{r} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

As the torsion is uniform throughout the portion considered,

$$\frac{d\rho}{dx} = 0 = \frac{d\rho}{r\,dx}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Eq. (11), in connection with eq. (10), gives

$$\frac{d^2w}{r\,dx\,d\phi} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Eqs. (11) and (12), in connection with eq. (10), reduce eq. (5) to the following form:

$$\frac{d^2u}{r^2\,d\phi^2} + \frac{d^2u}{dr^2} + \frac{du}{r\,dr} = 0 = \frac{d^2u}{d\phi^2} + r\frac{d\left(r\frac{du}{dr}\right)}{dr}. \quad . \quad . \quad . \quad (13)$$

Both terms of the second member of eq. (6) reduce to zero by eqs. (9) and (11), and give no new condition. The second term of the second member of eq. (7) is zero by eq. (9); the remaining term therefore gives

$$\frac{d^2w}{dx^2} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

As the stress is all shearing, ρ will not vary with ϕ. Hence

$$\frac{d\rho}{r\,d\phi} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Eqs. (10), (11), and (15) show that $\rho=0$, and reduce eq. (4) to

$$\frac{dw}{dr}-\frac{w}{r}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Eq. (10) now becomes $\frac{dw}{r\,d\phi}=0$, and shows that w does not contain ϕ; while eq. (14) shows that w does not contain x^2 or any higher power of x. The strain w, in connection with these conditions, is to be so determined as to satisfy eq. (16).

If α is a constant, the following form fulfils all conditions:

$$w=\alpha rx. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Eq. (17) shows that *the strain w, in the direction of ϕ*, i.e., *the angular strain at any point, varies directly as the distance from the axis of X, and as the distance from the origin measured along that axis.* This is a direct consequence of making $T_{r\phi}=0$.

The quantity α is evidently the *angle of torsion*, or the angle through which one end of a unit of fibre, situated at unit's distance from the axis, is twisted; for if

$$r=x=1, \quad w=\alpha.$$

An equation of condition relative to the exterior surface of the twisted piece yet remains to be determined; and that is to be based on the supposition that no external force whatever acts on the outer surface of the piece. In eqs. (13), (14), and (15) of Art. 2, consequently, $P=0$. The conditions of the problem also make all the stresses except

$$T_3=T_{xr} \quad \text{and} \quad T_2=T_{\phi x}$$

equal to zero, while the cylindrical character of the piece makes

$$p = 90^\circ; \quad \therefore \cos p = 0.$$

If $\cos t$ be written for $\cos r$,

$$\cos t = \sin q.$$

Eq. (13), just cited, then gives

$$T_{xr} \cos q + T_{\phi x} \sin q = 0. \quad . \quad . \quad . \quad . \quad (18)$$

But since $\rho = 0$ and $w = \alpha r x$,

$$T_{xr} = G\frac{du}{dr} \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

and

$$T_{x\phi} = G\left(\frac{du}{r\,d\phi} + \alpha r\right). \quad . \quad . \quad . \quad . \quad (20)$$

Eq. (18) now becomes

$$\frac{\dfrac{du}{dr}}{\dfrac{du}{r\,d\phi} + \alpha r} = -\tan q = -\frac{dr_0}{r_0\,d\phi}, \quad . \quad . \quad . \quad (21)$$

in which r_0 is the value of r for the perimeter of any normal section.

Eqs. (13) and (21) are all that are necessary and all that exist for the determination of the strain u. Eq. (13) must be fulfilled at all points in the interior of the twisted piece, while eq. (21) must at the same time hold true at all points of the exterior surface.

After u is determined, T_{xr} and $T_{x\phi}$ at once result from eqs. (19) and (20). The resisting moment of torsion then becomes

$$M = \int\int T_{x\phi} r^2 \, d\phi . dr = G\int\int \frac{du}{d\phi} . r \, dr \, d\phi + G\alpha I_p. \quad (22)$$

In this equation $I_p = \int\int r^2 . r \, d\phi \, dr$ is the polar moment of inertia of the normal section of the piece about the axis of X, and the double integral is to be extended over the whole section.

According to the old or common theory of torsion

$$M = G\alpha I_p.$$

The third member of eq. (22) shows, however, that such an expression is not correct unless u is equal to zero; i.e., unless all normal sections remain plane while the piece is subjected to torsion. It will be seen that this is true for a circular section only.

It may sometimes be convenient to put eq. (22) in the following form:

$$M = G\int\int r \, dr . \frac{du}{d\phi} d\phi + G\alpha I_p = G\int u . r \, dr + G\alpha I_p. \quad (23)$$

In this equation u is to be considered as

$$\int_0^\phi \frac{du}{d\phi} d\phi,$$

while the remaining integration in r is to be so made that the whole section shall be covered.

It is very important to observe that the equations of condition for the determination of u, and consequently the general values of T_{xr} and $T_{x\phi}$, are wholly independent of any considerations regarding the position of the axis of torsion, or the axis of X. It follows from this that *the resistance of pure torsion is precisely the same wherever may be the axis about which the piece is twisted.* It is to be borne in mind, however, that if the axis of the twisting is not the axis of the cylindrical piece the latter will be subjected to combined bending and torsion, the bending being produced by a force sufficient to cause the piece to take the helical position necessitated by the torsion. The cylindrical axis is the straight line locus of the centres of gravity of all the normal sections.

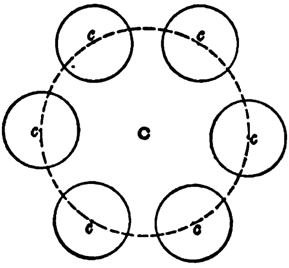

FIG. 2.

If, as in Fig. 2, there are n cylinders whose centres c are all at the same distance $Cc = l$ from the centre C of twisting or motion, and if M is the total moment of torsion of the system, while m is the moment of torsion of each cylinder about its own axis or centre c, then will $M = nm$, and each cylinder will be subject to a bending moment whose amount can be determined from the condition that the diameter of each piece lying along Cc before torsion must pass through C after, and during, torsion also.

Since T_{xr} and $T_{x\phi}$ act at right angles to each other, the resultant intensity of shear at any point in an originally normal section of the twisted piece will be

$$T = \sqrt{T_{xr}^2 + T_{x\phi}^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

According to the ordinary methods of the calculus, the co-ordinates of the point at which T has its greatest value must satisfy the equations

$$\frac{dT}{d\phi}=\frac{dT}{dr}=0; \quad \ldots \ldots \quad (25)$$

$$\frac{d^2T}{d\phi^2}<0; \quad \frac{d^2T}{dr^2}<0; \quad \left(\frac{d^2T}{d\phi\, dr}\right)^2-\frac{d^2T}{d\phi^2}\cdot\frac{d^2T}{dr^2}\leqq 0.$$

After the solution of eqs. (25) it will usually be necessary only to inspect the resulting value of T, in order to determine whether it is a maximum or minimum, without applying the tests which follow those equations.

Equations of Condition in Rectangular Co-ordinates.

In the case of a rectangular normal section, the analysis is somewhat simplified by taking some of the quantities used in terms of rectangular co-ordinates.

In the notation of Art. 2 all stresses will be zero except T_3 and T_2. Hence eqs. (10), (11), and (12) of that article reduce to

$$\frac{dT_3}{dy}+\frac{dT_2}{dz}=0;$$

$$\frac{dT_3}{dx}=0;$$

$$\frac{dT_2}{dx}=0.$$

The strains in the directions of x, y, and z are, respectively, u, v, and w. Introducing the values of T_3 and T_2 in the equations above, in terms of these strains, from eqs. (11) and (13) of Art. 1, and then doing the same in reference to the conditions,

$$N_1=N_2=N_3=T_1=0,$$

the following equations will result:

$$\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}=0; \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

$$\frac{dv}{dz}+\frac{dw}{dy}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

The operations by which these results are reached are identical with those used above in connection with semi-polar co-ordinates, and need not be repeated.

Eq. (27) is satisfied by taking

$$v= \alpha xz;$$
$$w=-\alpha xy;$$

in which α is the angle of torsion, as before.

Eqs. (11) and (13) of Art. 5 then give

$$T_3=G\left(\frac{du}{dy}+\frac{dv}{dx}\right)=G\left(\frac{du}{dy}+\alpha z\right); \quad . \quad . \quad . \quad (28)$$

$$T_2=G\left(\frac{du}{dz}+\frac{dw}{dx}\right)=G\left(\frac{du}{dz}-\alpha y\right). \quad . \quad . \quad . \quad (29)$$

The element of a normal section is $dz\,dy$. Hence the moment of torsion is

$$M=\int\int(T_3z-T_2y)dy\,dz;$$

$$\therefore\ M=G\int\int\left(\frac{du}{dy}z-\frac{du}{dz}y\right)dy\,dz+G\alpha I_p; \quad . \quad (30)$$

$$\therefore\ M=G\int(zu\,dz-yu\,dy)+G\alpha I_p. \quad . \quad . \quad . \quad . \quad (31)$$

$$I_p=\int\int(z^2+y^2)dy\,dz$$

is the polar moment of inertia of any section about the axis of X.

The integrals are to be extended over the whole section; hence, in eq. (31), $zu\,dz$ is to be taken as

$$z\,dz\,.\int_{-y_0}^{+y_0}\frac{du}{dy}dy$$

and $yu\,dy$ as

$$y\,dy\int_{-z_0}^{+z_0}\frac{du}{dz}dz,$$

in which expressions y_0 and z_0 are general co-ordinates of the perimeter of the normal section.

Eq. (26) is identical with eq. (13), and can be derived from it, through a change in the independent variables, by the aid of the relations

$$z=r\cos\phi \quad \text{and} \quad y=r\sin\phi.$$

Solutions of Eqs. (13) *and* (21).

It has been shown that the function u, which represents the strain parallel to the axis of the piece, must satisfy eq. (13) [or eq. (26)] for all points of any normal section, and eq. (21) (or a corresponding one in rectangular co-ordinates) at all points of the perimeter; and those two are the only conditions to be satisfied.

It is shown by the ordinary operations of the calculus that an indefinite number of functions u, of r and ϕ, will satisfy eq. (13); and, of these, that some are algebraic and some transcendental.

It is further shown that the various functions u which satisfy both eqs. (13) and (21) differ only by constants.

If u is first supposed to be algebraic in character, and if c_1, c_2, c_3, etc., represent constant coefficients, the following general function will satisfy eq. (13):

$$u = \alpha \left\{ \begin{array}{l} c_1 r \sin \phi + c_2 r^2 \sin 2\phi + c_3 r^3 \sin 3\phi + \ldots \\ + c'_1 r \cos \phi + c'_2 r^2 \cos 2\phi + c'_3 r^3 \cos 3\phi + \ldots \end{array} \right\} \quad (32)$$

and the following equation, which is supposed to belong to the perimeter of a normal section only, will be found to satisfy eq. (21):

$$\begin{array}{l} \frac{r^2}{2} + c_1 r \cos \phi + c_2 r^2 \cos 2\phi + c_3 r^3 \cos 3\phi + \ldots \\ \quad - c'_1 r \sin \phi - c'_2 r^2 \sin 2\phi - c'_3 r^3 \sin 3\phi - \ldots = C. \end{array} \quad (33)$$

C is a constant which changes only with the form of section.

If $\frac{du}{dr}$ and $\frac{du}{r\,d\phi}$ be found from eq. (32), while $\frac{dr_0}{r_0\,d\phi}$ be taken from eq. (33), and if these quantities be then introduced in eq. (21), it will be found that that equation is satisfied.

The only form of transcendental function needed, among those to which the integration of eq. (13) or eq. (26) leads, will be given in connection with the consideration of pieces with rectangular section, where it will be used.

Elliptical Section about its Centre.

Let a cylindrical piece of material with elliptical normal section be taken, and let a be the semi-major and b the semi-minor axis, while the angle ϕ is measured from a with the centre of the ellipse as the origin of co-ordinates, since the cylinder will be twisted about its own axis. The polar equation of the elliptical perimeter may take the following shape:

$$\frac{r^2}{2}+\frac{r^2}{2}\cdot\frac{b^2-a^2}{a^2+b^2}\cos 2\phi=\frac{a^2b^2}{a^2+b^2}. \quad . \quad . \quad . \quad (34)$$

By a comparison of eqs. (33) and (34), it is seen that

$$c_2=\frac{b^2-a^2}{2(a^2+b^2)} \quad \text{and} \quad C=\frac{a^2b^2}{a^2+b^2},$$

and that all the other constants are zero. Hence eq. (32) gives

$$u=\alpha\frac{b^2-a^2}{2(a^2+b^2)}r^2\sin 2\phi=\frac{\alpha}{2}fr^2\sin 2\phi. \quad . \quad . \quad (35)$$

The quantity represented by f is evident.

By eqs. (19) and (20)

$$T_{xr}=G\alpha\frac{b^2-a^2}{a^2+b^2}r\sin 2\phi; \quad . \quad . \quad . \quad . \quad . \quad (36)$$

$$T_{x\phi}=G\alpha\left(\frac{b^2-a^2}{a^2+b^2}r\cos 2\phi+r\right). \quad . \quad . \quad . \quad (37)$$

Since $\frac{r_0 . r_0\,d\phi}{2}=dA$, A being the area of the ellipse, or πab, the second member of eq. (22), by the aid of eq. (37), may take the form

$$M=G\alpha\int d\phi\int_0^r\left(\frac{b^2-a^2}{a^2+b^2}r^3\cos 2\phi+r^3\right)dr;$$

$$\therefore\ M=G\alpha\int\left(\frac{b^2-a^2}{a^2+b^2}\frac{r^4}{4}\cos 2\phi+\frac{r^4}{4}\right)d\phi.$$

Then using eq. (34),

$$M = G\alpha\frac{a^2b^2}{a^2+b^2}\int dA = G\alpha\frac{\pi a^3b^3}{a^2+b^2}. \quad . \quad . \quad . \quad (38)$$

If I_p is the polar moment of inertia of the ellipse (i.e., about an axis normal to its plane and passing through its centre), so that

$$I_p = \frac{\pi ab(a^2+b^2)}{4};$$

then

$$M = G\alpha\frac{A^4}{4\pi^2 I_p}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

Using f in the manner shown in eq. (35), the resultant shear at any point becomes, by eq. (24),

$$T = G\alpha r\sqrt{f^2+2f\cos 2\phi+1}.$$

$$\therefore \frac{dT}{d\phi} = 0$$

gives

$$\sin 2\phi = 0, \quad \text{or} \quad \phi = 90° \quad \text{or} \quad 0°.$$

Since f is negative, T will evidently take its maximum when ϕ has such a value that $2f \cos 2\phi$ is positive, or ϕ must be 90°.

Hence the greatest intensity of shear will be found somewhere along the minor axis. But the preceding expression shows that T varies directly as the distance from the centre. Hence *the greatest intensity of shear is found at the extremities of the minor axis.*

Making $\phi = 90°$ and $r = b$ in the value of T,

$$T = T_m = G\alpha b(1 - f) = G\alpha \frac{2a^2b}{a^2 + b^2}. \quad . \quad . \quad . \quad (40)$$

Taking $G\alpha$ from eq. (40) and inserting it in eq. (38),

$$M = T_m \frac{\pi ab^2}{2} = 2T_m \frac{I_a}{b}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

in which

$$I_a = \frac{\pi ab^3}{4},$$

or the moment of inertia of the section about the major axis.

Equilateral Triangle about its Centre of Gravity.

This case is that of a cylindrical piece whose normal cross-section is an equilateral triangle, and the torsion will be supposed about an axis passing through the centres of gravity of the different normal sections. The cross-section is represented in Fig. 3, G being the centre of gravity as well as the origin of co-ordinates.

Let $GH = \frac{1}{2}GD = a$. Then from the known properties of such a triangle,

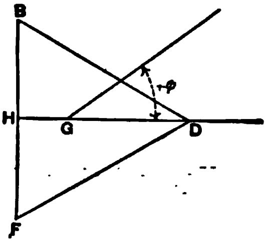

Fig. 3.

$$FD = DB = BF = 2a\sqrt{3}.$$

Hence the equation for DB is; $r \sin \phi - \frac{2a - r \cos \phi}{\sqrt{3}} = 0$.

Hence the equation for BF is; $r \cos \phi + a = 0$.

Hence the equation for FD is; $r \sin \phi + \frac{2a - r \cos \phi}{\sqrt{3}} = 0$.

Taking the product of these three equations and reducing, there will result for the equation to the perimeter

$$\frac{r^2}{2}-\frac{r^3}{6a}\cos 3\phi=\frac{2a^2}{3}. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (42)$$

Comparing this equation with eq. (33),

$$c_3=-\frac{1}{6a} \quad \text{and} \quad C=\frac{2a^2}{3}.$$

Hence

$$u=-\alpha\frac{r^3 \sin 3\phi}{6a}. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (43)$$

And by eqs. (19) and (20)

$$T_{xr}=-G\alpha\frac{r^2 \sin 3\phi}{2a}; \quad . \quad . \quad . \quad . \quad . \quad . \qquad (44)$$

$$T_{x\phi}=G\alpha\left(r-\frac{r^2 \cos 3\phi}{2a}\right). \quad . \quad . \quad . \quad . \quad . \qquad (45)$$

Eq. (22) then gives

$$M=G\alpha I_p-G\alpha\int\int\frac{r^4 \cos 3\phi}{2a}dr\,d\phi;$$

$$=G\alpha I_p-G\alpha\int\frac{r^4 \sin 3\phi}{6a}dr;$$

$$=G\alpha\left(I_p-\frac{6}{5}a^4\sqrt{3}\right)=0.6\,G\alpha I_p=1.8\,G\alpha a^4\sqrt{3}; \quad . \qquad (46)$$

since I_p = polar moment of inertia $=3a^4\sqrt{3}$.

By eq. (24)

$$T = G\alpha\sqrt{r^2 - \frac{r^3 \cos 3\phi}{a} + \frac{r^4}{4a^2}}; \quad . \quad . \quad . \quad (47)$$

$$\therefore \frac{dT}{d\phi} = 0 \quad \text{gives} \quad \sin 3\phi = 0,$$

or

$$\phi = 0°, 60°, 120°, 180°, 240°, 300°, \text{ or } 360°.$$

The values 0°, 120°, 240°, and 360° make

$$\cos 3\phi = +1;$$

hence, for a given value of r, these make T a minimum. The values 60°, 180°, and 300° make,

$$\cos 3\phi = -1;$$

hence, for a given value of r, these make T a maximum. Putting $\cos 3\phi = -1$ in eq. (47),

$$T = G\alpha\left(r + \frac{r^2}{2}\right). \quad . \quad . \quad . \quad . \quad . \quad (48)$$

This value will be the greatest possible when r is the greatest. But $\phi = 60°$, 180°, and 300° correspond to the normal a dropped on each of the three sides of the triangle from G. Hence $r = a$, in eq. (48), gives the greatest intensity of shear T_m, or

$$T_m = \frac{3}{2} G\alpha a. \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

Or *the greatest intensity of shear exists at the middle point of each side.* Those points are the nearest of all, in the perimeter, to the axis of torsion.

The value of $G\alpha$, from eq. (49), inserted in eq. (46), gives

$$M = 0.4\frac{I_p}{a}T_m = \frac{l^3 T_m}{20}, \quad . \quad . \quad . \quad . \quad (50)$$

in which $l =$ side of section $= 2a\sqrt{3}$.

Rectangular Section about an Axis passing through its Centre of Gravity.

In this case it will be necessary to consider one of the transcendental forms to which the integration of eq. (13) [or (26)] leads; for if the polar equation to the perimeter be formed, as was done in the preceding case, it will be found to contain r^4, to which no term in eq. (33) corresponds.

If e is the base of the Napierian system of logarithms (numerically $e = 2.71828$, nearly) and A any constant whatever, it is known that the general integral of the partial differential eq. (13) may be expressed as follows:

$$u = Ae^{nr\cos\phi}\, e^{n'r\sin\phi}, \quad . \quad . \quad . \quad . \quad . \quad (51)$$

when $n^2 + n'^2 = 0$; for

$$\frac{d^2u}{dr^2} + \frac{d^2u}{r^2\,d\phi^2} + \frac{du}{r\,dr} = A(n^2 + n'^2)\, e^{nr\cos\phi}\, e^{n'r\sin\phi}.$$

But the second member of this equation is evidently equal to zero if

$$(n^2 + n'^2) = 0 \quad \text{or} \quad n' = \sqrt{-n^2}.$$

These relations make it necessary that neither n or n' shall be imaginary.

It will hereafter be convenient to use the following notation for hyperbolic sines, cosines, and tangents:

$$\text{sih}\ t = \frac{e^t - e^{-t}}{2}; \quad \text{coh}\ t = \frac{e^t + e^{-t}}{2}; \quad \text{and tah}\ t = \frac{e^t - e^{-t}}{e^t + e^{-t}}$$

By the use of Euler's exponential formula, as is well known, and remembering that $n'^2 = -n^2$, eq. (51) may be put in the following form:

$$u = \Sigma e^{nr \cos\phi}\ [A_n \sin (nr \sin \phi) + A'_n \cos (nr \sin \phi)],$$

in which the sign of summation is to be extended to all possible values of A_n and A'_n. At the centre of any section for which r is zero, u must be zero also, for the axis of the piece is not shortened. This condition requires that $A'_n = 0$; u then becomes

$$u = \Sigma e^{nr \cos\phi}\ A_n \sin (nr \sin \phi).$$

The subsequent analysis will be simplified by introducing the form of the hyperbolic sine, and this may be done by adding and subtracting the same quantity to that already under the sign of summation, in such a manner that

$$u = \Sigma[A_n \sin (nr \sin \phi) . \text{sih} (nr \cos \phi) \\ + \tfrac{1}{2} A_n \sin (nr \sin \phi)\ e^{-nr \cos\phi}]. \quad (52)$$

Now if the product

$$\sin (nr \sin \phi)\ e^{-nr \cos\phi}$$

be developed in a series and multiplied by A_n, one term will consist of the quantity

$$-r^2 \sin\phi \cos\phi$$

multiplied by a constant, and if

$$\Sigma A_n \sin(nr\sin\phi)\, e^{-nr\cos\phi}$$

be replaced by simply,

$$-\alpha r^2 \sin\phi \cos\phi,$$

all the conditions of the problem will be found to be satisfied. This is equivalent to putting

$$-\alpha r^2 \sin\phi \cos\phi$$

for a general function of $r\sin\phi$ and $r\cos\phi$. This change will give the following form to u, first used by Saint-Venant:

$$u = \Sigma A_n \sin(nr\sin\phi)\,.\,\text{sih}\,(nr\cos\phi) - \alpha r^2 \sin\phi\cos\phi. \quad (53)$$

Fig. 4 represents the cross-section with C as the origin of co-ordinates and axis. The angle ϕ is measured positively

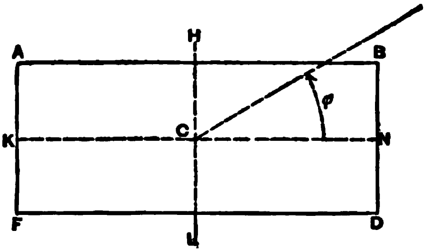

FIG. 4.

from CN toward CH. At the points N, H, K, and L, in the equation to the perimeter, dr_0 will be zero. Hence at those points, by eq. (21),

$$\frac{du}{dr}=\Sigma\lfloor A_n \sin(nr\sin\phi).n\cos\phi.\text{coh}(nr\cos\phi)$$
$$+A_n.n\sin\phi.\cos(nr\sin\phi).\text{sih}(nr\cos\phi)]$$
$$-2\alpha r\sin\phi\cos\phi=0.$$

At the points under consideration ϕ has the values 0°, 90°, 180°, 270°, and 360°. At the points N and K, $\phi=0°$ or 180°; hence $\sin\phi=0$, and both terms of the second member of $\frac{du}{dr}$ reduce to zero, whatever may be the value of n. But at H and L, $\phi=90°$ and 270°; hence $\sin\phi=+1$ or -1 and $\cos\phi=0$.

In order, then, that $\frac{du}{dr}=0$ at H and L, these must obtain:

$$\cos nr=\cos(-nr)=0.$$

If $HL=c$ and $KN=b$, then

$$\cos\frac{nc}{2}=\cos\left(-\frac{nc}{2}\right)=0. \quad . \quad . \quad . \quad . \quad (54)$$

If the signification of n be now somewhat changed so as to represent all possible whole numbers between 0 and ∞, eq. (54) will be satisfied by writing

$$\frac{2n-1}{c}\pi$$

for n in that equation. Eq. (53) will then become

$$u=\sum_1^\infty A_n \sin\left(\frac{2n-1}{c}\pi r\sin\phi\right).\operatorname{sih}\left(\frac{2n-1}{c}\pi r\cos\phi\right)$$
$$-\alpha r^2\sin\phi\cos\phi. \quad . \quad . \quad . \quad . \quad (55)$$

The quantity A_n yet remains to be determined by the aid of eq. (21), which expresses the condition existing at the perimeter of any section.

Now, for the portion BN of the perimeter,

$$r\cos\phi=\frac{b}{2},$$

and $\frac{dr_0}{r_0 d\phi}$ will be the tangent of $(-\phi)$, or

$$-\frac{dr_0}{r_0 d\phi}=-\tan(-\phi)=\tan\phi.$$

Hence eq. (21) becomes

$$\frac{\frac{du}{dr}}{\frac{du}{r\,d\phi}+\alpha r}=\tan\phi, \quad . \quad . \quad . \quad . \quad . \quad (56)$$

or

$$\alpha r\sin\phi=\frac{du}{dr}\cos\phi-\frac{du}{r\,d\phi}\sin\phi.$$

Substituting from eq. (55), then making

$$r\cos\phi=\frac{b}{2},$$

$$r\sin\phi=\sum_1^\infty A_n.\frac{2n-1}{2\alpha c}\pi.\operatorname{coh}\left(\frac{2n-1}{2c}\pi b\right).\sin\left(\frac{2n-1}{c}\pi r\sin\phi\right).$$

If $r \sin \phi$ be represented by the rectangular co-ordinate y, and another quantity by H, the above equation may be written

$$y = H_1 \sin \frac{\pi y}{c} + H_2 \sin \frac{3\pi y}{c} + H_3 \sin \frac{5\pi y}{c}$$

$$+ \ . \ . \ . \ - H_n \sin \left(\frac{2n-1}{c} \pi \right) y + \ . \ . \ .$$

If both sides of this equation be multiplied by

$$\sin \left(\frac{2n-1}{c} \pi y \right) . dy,$$

and if the integral then be taken between the limits 0 and $\frac{c}{2}$, it is known from the integral calculus that all terms except the n^{th} will disappear, and that

$$H_n = \int_0^{\frac{c}{2}} y . \sin \left(\frac{2n-1}{c} \pi y \right) . dy \div \int_0^{\frac{c}{2}} \sin^2 \left(\frac{2n-1}{c} \pi y \right) . dy.$$

Completing these simple integrations,

$$H_n = \left(\frac{c}{(2n-1)\pi} \right)^2 (-1)^{n-1} . \frac{4}{c}.$$

Hence

$$A_n = \frac{(-1)^{n-1} c^2}{(2n-1)^2 \pi^2} \cdot \frac{4}{c} \cdot \frac{2\alpha c}{(2n-1)\pi} \cdot \frac{1}{\operatorname{coh}\left(\frac{2n-1}{2c} \pi b \right)}.$$

If this value of A_n be put in eq. (55), and if rectangular co-ordinates

$$y = r\sin\phi \quad \text{and} \quad z = r\cos\phi$$

be introduced, that equation will become

$$u = -\alpha zy + \left(\frac{2}{\pi}\right)^3 . \alpha c^2 \sum_1^\infty \frac{(-1)^{n-1}\sin\left(\frac{2n-1}{c}\pi y\right).\operatorname{sih}\left(\frac{2n-1}{c}\pi z\right)}{(2n-1)^3\operatorname{coh}\left(\frac{2n-1}{2c}\pi b\right)}. \qquad (57)$$

This value of u placed in eq. (31) will enable the moment of torsion to be at once written.

The limits $+y_0$ and $-y_0$ are $+\frac{c}{2}$ and $-\frac{c}{2}$, and the limits $+z_0$ and $-z_0$ are $+\frac{b}{2}$ and $-\frac{b}{2}$. Hence

$$\Big[u\Big]_{-\frac{c}{2}}^{+\frac{c}{2}} = \alpha bc\left[-\frac{z}{b} + \left(\frac{2}{\pi}\right)^3\frac{c}{b}\sum_1^\infty \frac{2\operatorname{sih}\left(\frac{2n-1}{c}\pi z\right)}{(2n-1)^3\operatorname{coh}\left(\frac{2n-1}{2c}\pi b\right)}\right]$$

$= Q$, for brevity;

$$-\Big[u\Big]_{-\frac{b}{2}}^{+\frac{b}{2}} = \alpha bc\left[\frac{y}{c} - \left(\frac{2}{\pi}\right)^3\frac{c}{b}\sum_1^\infty \frac{(-1)^{n-1}.2\operatorname{sih}\left(\frac{2n-1}{2c}\pi b\right).\sin\left(\frac{2n-1}{c}\pi y\right)}{(2n-1)^3\operatorname{coh}\left(\frac{2n-1}{2c}\pi b\right)}\right] = R.$$

For the next integration

$$\int_{-\frac{b}{2}}^{+\frac{b}{2}} Qz\,dz = \alpha bc\left[-\frac{b^2}{12} \right.$$

$$\left. + \left(\frac{2}{\pi}\right)^3 \frac{c}{b} \sum_1^\infty \frac{\frac{2bc}{(2n-1)\pi}.\operatorname{coh}\frac{2n-1}{2c}\pi b - \frac{4c^2}{(2n-1)^2\pi^2}\operatorname{sih}\left(\frac{2n-1}{2c}\pi b\right)}{(2n-1)^3 \operatorname{coh}\left(\frac{2n-1}{2c}\pi b\right)} \right]$$

$$\int_{-\frac{c}{2}}^{+\frac{c}{2}} Ry\,dy = \alpha bc\left[\frac{c^2}{12} - \left(\frac{2}{\pi}\right)^3 \frac{c}{b} \sum_1^\infty \frac{\frac{4c^2}{(2n-1)^2\pi^2}\operatorname{sih}\left(\frac{2n-1}{2c}\pi b\right)}{(2n-1)^3 \operatorname{coh}\left(\frac{2n-1}{2c}\pi b\right)} \right].$$

Thus the integrations indicated in eq. (31) are completed. Hence

$$M = G\left\{ \int Qz\,dz + \int Ry\,dy + \alpha I_p \right\}.$$

Remembering that

$$I_p = bc\left(\frac{c^2+b^2}{12}\right),$$

$$M = G\alpha\left[\frac{bc^3}{6} + \frac{16bc^3}{\pi^4}\sum_1^\infty \frac{1}{(2n-1)^4} - \frac{64c^4}{\pi^5}\sum_1^\infty \frac{\operatorname{tah}\left(\frac{2n-1}{2c}\pi b\right)}{(2n-1)^5} \right]. \quad (58)$$

But it is known that

$$\sum_1^\infty \frac{1}{(2n-1)^4} = \frac{2}{1.2.3}.\frac{\pi^4}{2^5}.$$

Hence eq. (58) becomes

$$M = G\alpha bc^3\left[\frac{1}{3} - \frac{64}{\pi^5}\frac{c}{b}\sum_1^\infty \frac{\text{tah}\left(\frac{2n-1}{2c}\pi b\right)}{(2n-1)^5}\right]. \quad . \quad (59)$$

Since

$$\left(\frac{1}{1}+\frac{1}{3^5}+\frac{1}{5^5}+\ldots\right)$$
$$-\left(\frac{1-\text{tah}\,\pi}{1}+\frac{1-\text{tah}\,3\pi}{3^5}+\frac{1-\text{tah}\,5\pi}{5^5}+\ldots\right)$$
$$=\frac{\text{tah}\,\pi}{1}+\frac{\text{tah}\,3\pi}{3^5}+\frac{\text{tah}\,5\pi}{5^5}+\ldots,$$

and since

$$\frac{64}{\pi^5} = 0.209137,$$

and remembering that

$$\sum_1^\infty\left(\frac{1}{2n-1}\right)^5 = 1+\frac{1}{3^5}+\frac{1}{5^5}+\ldots = \left(1-\frac{1}{2^5}\right)\frac{\pi^5}{295.1215},$$

eq. (59) becomes

$$M = G\alpha bc^3\left[\frac{1}{3} - 0.210083\frac{c}{b}\right.$$
$$\left.+0.209137\frac{c}{b}\left(\frac{1-\text{tah}\,\frac{\pi b}{2c}}{1}+\frac{1-\text{tah}\,\frac{3\pi b}{2c}}{3^5}+\ldots\right)\right]. \quad (60)$$

Eq. (60) gives the value of the moment of torsion of a rectangular bar of material.

If z had been taken parallel to b, and y parallel to c, a moment of equal value would have been found, which can be at once written from eq. (60) by writing b for c and c for b.

That moment will be

$$M = G\alpha c b^3 \left[\frac{1}{3} - 0.210083 \frac{b}{c} + 0.209137 \frac{b}{c} \left(\frac{1 - \operatorname{tah} \frac{\pi c}{2b}}{1} + \frac{1 - \operatorname{tah} \frac{3\pi c}{2b}}{3^5} + \dots \right) \right]. \quad (61)$$

Eq. (60) *should be used when b is greater than c, and* eq. (61) *when c is greater than b,* because the series in the parentheses are then very rapidly converging, and not diverging. It will never be necessary to take more than three or four terms and one, only, will ordinarily be sufficient. The following are the values of

$$\left(1 - \operatorname{tah} \frac{n\pi}{2}\right)$$

for a few values of n:

$$\left(1 - \operatorname{tah} \frac{n\pi}{2}\right) = 0.083 : 0.00373 : 0.000162 : 0.000007;$$

$$n = 1 : 2 : 3 : 4$$

Square Section.

If $c = b$ either eq. (60) or eq. (61) gives

$$M = G\alpha b^4 \left[\frac{1}{3} - 0.2101 + 0.209 \left(1 - \operatorname{tah} \frac{\pi}{2}\right) \right];$$

$$\therefore M = 0.1406\, G\alpha b^4 = G\alpha \frac{A^4}{42.7\, I_p}, \quad . \quad . \quad (62)$$

in which A is the area $(=b_2)$ and I_p is the polar moment of inertia $\left(=\frac{b^4}{6}\right)$.

Rectangle in which $b=2c$.

If $b=2c$, eq. (60) gives

$$M=G\alpha.2c^4\left(\frac{1}{3}-0.105+0.1046\ (1-\text{tah}\ \pi)\right);$$

$$\therefore\ M=0.457\ G\alpha c^4=G\alpha\frac{A^4}{42\ I_p}, \quad . \quad . \quad . \quad (63)$$

in which A is the area $(=2c^2)$ and I_p = polar moment of inertia

$$=\frac{bc^3+b^3c}{12}=\frac{5c^4}{6}.$$

Rectangle in which $b=4c$.

If $b=4c$, eq. (60) then gives

$$M=G\alpha bc^3\left(\frac{1}{3}-0.0525\right)=1.123\ G\alpha c^4;$$

$$\therefore\ M=G\alpha\frac{A^4}{40.2\ I_p}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (64)$$

in which A = area = $4c^2$ and I_p = polar moment of inertia

$$=\frac{bc^3+b^3c}{12}=\frac{17c^4}{3}.$$

If b is greater than $2c$, it will be sufficiently near for all ordinary purposes to write

$$M = G\alpha\frac{bc^3}{3}\left(1 - 0.63\frac{c}{b}\right). \quad . \quad . \quad . \quad . \qquad (65)$$

Greatest Intensity of Shear.

There yet remains to be determined the greatest intensity of shear at any point in a section, and in searching for this quantity it will be convenient to use eqs. (28) and (29).

It will also be well to observe that by changing z to y, y to $-z$, c to b, and b to c, in eq. (57), there may be at once written

$$u = \alpha zy - \left(\frac{2}{\pi}\right)^3 .\alpha b^2 \sum_1^\infty \frac{(-1)^{n-1}\sin\left(\frac{2n-1}{b}\pi z\right).\text{sih}\left(\frac{2n-1}{b}\pi y\right)}{(2n-1)^3\text{coh}\left(\frac{2n-1}{2b}\pi c\right)}. \qquad (66)$$

This amounts to turning the co-ordinate axes 90°.

Since the resultant shear at any point is

$$T = \sqrt{T_2^2 + T_3^2},$$

it will be necessary to seek the maximum of

$$\left(\frac{du}{dy} + \alpha z\right)^2 + \left(\frac{du}{dz} - \alpha y\right)^2 = \frac{T^2}{G^2}.$$

The two following equations will then give the points desired:

$$\frac{d\left(\frac{T^2}{G^2}\right)}{dy}=\left(\frac{du}{dy}+\alpha z\right)\frac{d^2u}{dy^2}+\left(\frac{du}{dz}-\alpha y\right)\left(\frac{d^2u}{dz\,dy}-\alpha\right)=0; \quad (67)$$

$$\frac{d\left(\frac{T^2}{G^2}\right)}{dz}=\left(\frac{du}{dy}+\alpha z\right)\left(\frac{d^2u}{dz\,dy}+\alpha\right)+\left(\frac{du}{dz}-\alpha y\right)\frac{d^2u}{dz^2}=0. \quad (68)$$

It is unnecessary to reproduce the complete substitutions in these two equations, but such operations show that *the points of maximum values of T are at the middle points of the sides of the rectangular sections*, omitting the evident fact that $T=0$ at the centre. It will also be found that *the greatest intensity of shear will exist at the middle points of the greater sides.*

This result may be reached independent of any analytical test, by bearing in mind that an elongated ellipse closely approximates a rectangular section, and it has already been shown that the greatest intensity in an elliptical section is found at the extremities of the smaller axis.

By the aid of eqs. (28), (29), (57), and (66), it will also be found that $T_3=0$ at the extremities of the diameter c, and $T_2=0$ at the extremities of the diameter b. The maximum value of T will then be

$$T_m=-T_2=-G\left(\frac{du}{dz}-\alpha y\right)_{\substack{z=0\\ y=\frac{c}{2}}}. \quad . \quad . \quad . \quad (69)$$

By the use of eq. (57)

$$\frac{du}{dz}-\alpha y=$$

$$-2\alpha y+\left(\frac{2}{\pi}\right)^3.\alpha c^2\sum_1^\infty\frac{(-1)^{n-1}.\frac{\pi}{c}.\sin\left(\frac{2n-1}{c}\pi y\right).\operatorname{coh}\left(\frac{2n-1}{c}\pi z\right)}{(2n-1)^3\operatorname{coh}\left(\frac{2n-1}{2c}\pi b\right)}.$$

Putting $z=0$ and $y=\frac{c}{2}$ in this equation, there will result

$$T_m = G\alpha c\left[1-\frac{8}{\pi^2}\sum_1^\infty \frac{1}{(2n-1)^2 \operatorname{coh}\left(\frac{2n-1}{2c}\pi b\right)}\right] \quad . . \quad (70)$$

If b is greater than c the series appearing in this equation is very rapidly convergent, and it will never be necessary to use more than two or three terms if the section is square, and if b is four or five times c there may be written

$$T_m = G\alpha c. \quad . \ . \ . \ . \ . \ . \ . \quad (71)$$

Square Section.

Making $b=c$ in eq. (70), and making $n=1$, 2, and 3 (i.e., taking three terms of the series), there will result

$$T_m = 0.676\, G\alpha c; \quad \therefore\ G\alpha = 1.48\frac{T_m}{c}.$$

Inserting this value in eq. (62),

$$M = 0.21\, b^3 T_m = \frac{1.26 I T_m}{a}. \quad . \ . \ . \ . \ . \quad (72)$$

$$\therefore\ T_m = 0.8\frac{M}{I}a = 5\frac{M}{b^3}, \quad . \ . \ . \ . \quad (73)$$

in which

$$I = \frac{b^4}{12} \quad \text{and} \quad a = \frac{b}{2} = \frac{c}{2}.$$

Rectangular Section; $b=2c$.

Making $b=2c$ in eq. (70), and making $n=1$, only, there will result

$$T_m = 0.93\,G\alpha c; \quad \therefore\ G\alpha = 1.08\frac{T_m}{c}.$$

Inserting this value in eq. (63),

$$M = 0.49\,c^3T_m = 1.47\frac{IT_m}{a}. \quad . \quad . \quad . \quad . \quad (74)$$

$$\therefore\ T_m = 0.68\frac{M}{I}a = 2\frac{M}{c^3}, \quad . \quad . \quad . \quad . \quad (75)$$

in which

$$I = \frac{bc^3}{12} = \frac{c^4}{6} \quad \text{and} \quad a = \frac{c}{2}.$$

Rectangular Section; $b=4c$.

Making $b=4c$ in eq. (70), and making $n=1$, only,

$$T_m = 0.997\,G\alpha c; \quad \therefore\ G\alpha = 1.003\frac{T_m}{c}.$$

Inserting this value in eq. (64),

$$M = 1.126\,c^3T_m = 1.69\frac{IT_m}{a}. \quad . \quad . \quad . \quad (76)$$

$$\therefore\ T_m = 0.6\frac{M}{I}a = 0.9\frac{M}{c^3}, \quad . \quad . \quad . \quad . \quad . \quad (77)$$

in which

$$I = \frac{bc^3}{12} = \frac{c^4}{3} \quad \text{and} \quad a = \frac{c}{2}.$$

Circular Section about its Centre.

The torsion of a circular cylinder furnishes the simplest example of all.

If r_0 is the radius of the circular section, the polar equation of that section is

$$\frac{r_0^2}{2}=C \text{ (constant).}$$

Comparing this equation with eq. (33), it is seen that

$$c_1=c_2=c_3= \ldots \quad =c_1'=c_2'= \ldots \quad =0.$$

By eq. (32) this gives $u=0$. Hence *all sections remain plane during torsion.*

Eqs. (19) and (20) then give

$$T_{zr}=0 \quad \text{and} \quad T_{z\phi}=G\alpha r. \quad . \quad . \quad . \quad . \quad (78)$$

Eq. (23) gives for the moment of torsion

$$M=G\alpha I_p, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (79)$$

or

$$M=0.5\ \pi r_0^4 . G\alpha=\frac{A^4G}{4\pi^2 I_p}\alpha; \quad . \quad . \quad . \quad . \quad (80)$$

in which equation A is the area of the section and

$$I_p=\frac{\pi r_0^4}{2}.$$

The greatest intensity of shear in the section will be obtained by making $r=r_0$ in eq. (78), or

$$T_m = G\alpha r_0; \quad \therefore G\alpha = \frac{T_m}{r_0}. \quad . \quad . \quad . \quad . \quad . \quad (81)$$

Eq. (80) then becomes

$$M = 0.5\ \pi r_0^3 T_m = 2\frac{IT_m}{r_0}. \quad . \quad . \quad . \quad . \quad . \quad (82)$$

$$\therefore T_m = 0.64\frac{M}{r_0^3} = 0.5\frac{M}{I}r_0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (83)$$

in which $$I = \frac{\pi r_0^4}{4}.$$

It is thus seen that the circular section is the only one treated which remains plane during torsion.

General Observations.

The preceding examples will sufficiently exemplify the method to be followed in any case. Some general conclusions, however, may be drawn from a consideration of eq. (33).

If the perimeter is symmetrical about the line from which ϕ is measured, then r must be the same for $+\phi$ and $-\phi$; hence

$$c_1' = c_2' = c_3' = \ldots = 0.$$

If the perimeter is symmetrical about a line at right angles to the zero position of r, then r must be the same for

$$\phi = 90° + \phi' \quad \text{and} \quad 90° - \phi';$$

hence

$$c_1 = c_3 = c_5 \ldots \quad = c_2' = c_4' = c_6' = \ldots \quad = 0.$$

In connection with the first of these sets of results, eq. (32) shows that *every axis of symmetry of sections represented by* eq. (33) *will not be moved from its original position by torsion.*

If the section has two axes of symmetry passing through the origin of co-ordinates, then will all the above constants be zero, and its equation will become

$$\frac{r^2}{2} + c_2 r^2 \cos 2\phi + c_4 r^4 \cos 4\phi + c_6 r^6 \cos 6\phi + \ldots \quad = K.$$

Art. 7.—Torsional Oscillations of Circular Cylinders.

Two cases of torsional oscillations will be considered, in the first of which the cylindrical body twisted is supposed to be the only one in motion. In the second case, however, the mass of the twisted body will be neglected, and the motion of a heavy body, attached to its free end, will be considered. In both cases the section of the cylinder will be considered circular.

Since these cases are those of motion, the internal stresses are not, in general, in equilibrium; hence equations of motion must be used, and those of Art. 3 are most convenient. Of these last, the investigations of the preceding article show that eq. (4) is the only one which gives any conditions of motion in the problem under consideration.

Putting the value of

$$T = T_{\phi x} = G\frac{dw}{dx}$$

in eq. (4) of Art. 3, that equation may take the form

$$\frac{d^2w}{dt^2}=\frac{G}{m}\frac{d^2w}{dx^2}, \quad \text{or} \quad \frac{d^2w}{dt^2}-b^2\frac{d^2w}{dx^2}=0. \quad . \quad . \quad . \quad (1)$$

For brevity, b^2 is written for $\frac{G}{m}$.

That dimension of the cross-section of the body which lies in the direction of the radius will be assumed so small that w may be considered a function of x and t only. The results will then apply to small solid cylinders and all hollow ones with thin walls.

The general integral of eq. (1), on the assumption just made, is (Booles' "Differential Equations," Chap. XV, Ex. 1)

$$w=f(x+bt)+F(x-bt),$$

in which f and F signify *any arbitrary functions whatever.* Now it is evident that all oscillations are of a periodic character, i.e., at the end of certain equal intervals of time, w will have the same value. Hence since f and F are arbitrary forms, and since circular functions are periodic, there may be written

$$w=A_n\{\sin(\alpha_nx+\alpha_nbt)+\sin(\alpha_nx-\alpha_nbt)\}$$
$$-B_n\{\cos(\alpha_nx+\alpha_nbt)-\cos(\alpha_nx-\alpha_nbt)\}, \quad (2)$$

in which α_n, A_n, and B_n are coefficients to be determined.

Substituting for the sines and cosines of sums and differences of angles,

$$w=2\sin\alpha_nx(A_n\cos\alpha_nbt+B_n\sin\alpha_nbt). \quad . \quad . \quad (3)$$

Let the origin of co-ordinates be taken at the fixed end of the piece, w must then be equal to zero, as is shown by eq. (3). But there may be other points at which w is always equal to zero, whatever value the time t may have. These points, called *nodes*, are found by putting $w = 0$, or

$$\sin \alpha x = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

This equation is satisfied by taking

$$\alpha_n = \frac{\pi}{a}, \quad \frac{2\pi}{a}, \quad \frac{3\pi}{a}, \quad \dots, \quad \frac{n\pi}{a},$$

and $x = a$; in which a is the length of the piece.

Hence *at the distances*

$$a, \ \frac{a}{2}, \ \frac{a}{3}, \ \dots, \ \frac{a}{n}$$

from the fixed end of the piece, there will exist sections which are never distorted or moved from their positions of rest. These are called *nodes*, and one is assumed at the free end, although such an assumption is not necessary, since a is really the distance from the fixed end to the farthest *node* and not necessarily to the free end.

If, as is permissible, A_n and B_n be written for twice those quantities, the general value of w now becomes

$$\begin{aligned} w = {} & \sin \frac{\pi x}{a}\left(A_1 \cos \frac{\pi bt}{a} + B_1 \sin \frac{\pi bt}{a}\right) \\ & + \sin \frac{2\pi x}{a}\left(A_2 \cos \frac{2\pi bt}{a} + B_2 \sin \frac{2\pi bt}{a}\right) \\ & + \sin \frac{3\pi x}{a}\left(A_3 \cos \frac{3\pi bt}{a} + B_3 \sin \frac{3\pi bt}{a}\right) \\ & \quad . \quad . \quad . \quad . \quad . \\ & + \sin \frac{n\pi x}{a}\left(A_n \cos \frac{n\pi bt}{a} + B_n \sin \frac{n\pi bt}{a}\right). \quad . \quad . \qquad (5) \end{aligned}$$

The coefficients A and B are to be determined by the ordinary procedure for such cases. Let

$$w_1 = \phi(x)$$

be the expression for the initial or known strain at any point, for which the time t is zero. Then if A_n is any one of the coefficients A,

$$A_n = \frac{2}{a}\int_0^a \phi(x) \sin\frac{n\pi x}{a}dx. \quad . \quad . \quad . \quad . \quad (6)$$

The velocity at any point, or at any time, will be given by

$$\frac{dw}{dt} = -\sin\frac{\pi x}{a}\left(A_1 \sin\frac{\pi bt}{a} - B_1 \cos\frac{\pi bt}{a}\right)\frac{\pi b}{a}. \quad . \quad . \quad (7)$$

In the initial condition, when the time is zero, or $t=0$, it has the given, or known, value

$$\frac{dw_1}{dt} = \Phi(x) = \frac{\pi b}{a}\left(B_1 \sin\frac{\pi x}{a} + 2B_2 \sin\frac{2\pi x}{a} + 3B_3 \sin\frac{3\pi x}{a} + \dots\right).$$

Then, as before,

$$B_n = \frac{2}{n\pi b}\int_0^a \Phi(x) \sin\frac{n\pi x}{a}dx. \quad . \quad . \quad . \quad . \quad (8)$$

Thus the most general value of w is completely determined.

The intensity of shear at any place or time is given by

$$T = G\frac{dw}{dx}$$

w being taken from eq. (5).

The second case to be treated is that of the torsion pendulum, in which the mass of the twisted body is so inconsiderable in comparison with that of the heavy body, or bob, attached to its free end that it may be neglected.

Let M represent the mass of the pendulum bob, and k *its radius of gyration in reference to the axis about which it is to vibrate*, then will Mk^2 be its moment of inertia about the same axis.

The unbalanced moment of torsion, with the angle of torsion α, is, by eq. (9) of Art. 6,

$$G\alpha I_p.$$

The elementary quantity of work performed by this unbalanced couple, if β is the general expression for the angular velocity of the vibrating body, is

$$G\alpha I_p . \beta\, dt.$$

This quantity of energy is equal in amount but opposite in sign to the indefinitely small variation of actual energy in the bob; hence

$$G\alpha I_p \beta dt = -d\left(\frac{Mk_2\beta^2}{2}\right) = -Mk^2\beta d\beta.$$

But if a is the length of the piece twisted,

$$\beta = \frac{d(\alpha a)}{dt} \quad \text{and} \quad d\beta = \frac{d^2(\alpha a)}{dt};$$

$$\therefore \quad \left(\frac{GI_p}{a}\right)(\alpha a) = -Mk^2\frac{d^2(\alpha a)}{dt^2}.$$

Multiplying this equation by $2d(a\alpha)$, and for brevity putting

$$\left(\frac{GI_p}{a}\right)=H, \quad (Mk^2)=K;$$

then integrating and dropping the common factor a^2,

$$H\alpha^2=-K\left(\frac{d\alpha}{dt}\right)^2+C.$$

When $\alpha=\alpha_1$, the value of the angle of torsion at the extremity of an oscillation, the bob will come to rest and $\frac{d\alpha}{dt}$ will be zero. Hence

$$C=H\alpha_1^2$$

and

$$K\left(\frac{d\alpha}{dt}\right)^2=H\ (\alpha_1^2-\alpha^2).$$

$$\therefore \quad \frac{d\alpha}{\sqrt{\alpha_1^2-\alpha^2}}=\sqrt{\frac{H}{K}}.dt;$$

$$\therefore \quad \sin^{-1}\frac{\alpha}{\alpha_1}=t\sqrt{\frac{H}{K}}+(C'=0). \quad . \quad . \quad . \quad (9)$$

$C'=0$ because α and t can be put equal to zero together.

At the opposite extremities of a complete oscillation α will have the values

$$(+\alpha_1) \quad \text{and} \quad (-\alpha_1).$$

Putting these values in the expression

$$t=\sqrt{\frac{K}{H}}.\sin^{-1}\frac{\alpha}{\alpha_1}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

and taking the difference between the results thus obtained, the following interval of time for a complete oscillation will be found:

$$\tau = \pi\sqrt{\frac{K}{H}} = \pi\sqrt{\frac{Mk^2a}{GI_p}}. \quad . \quad . \quad . \quad . \quad (11)$$

The time required for an oscillation is thus seen to *vary directly as the square root of the moment of inertia of the bob and the length of the piece, and inversely as the square root of the coefficient of elasticity for shearing and the polar moment of inertia of the normal section of the piece twisted.*

The number of complete oscillations per second is $\frac{1}{\tau}$. If this number is the observed quantity, the following equation will give G:

$$G = \left(\frac{1}{\tau}\right)^2 \frac{\pi^2 Mk^2 a}{I_p}.$$

The formulas for this case should only be used when the mass of the cylindrical piece twisted is exceedingly small in comparison with M.

Art. 8.—Thick, Hollow Spheres.

In order to investigate the conditions of equilibrium of stress at any point within the material which forms a thick hollow sphere, it will be most convenient to use the equations of Art. 4. As in the case of a thick hollow cylinder, the interior and exterior surfaces of the sphere are supposed to be subjected to fluid pressure.

Let r' and r_1 be the interior and exterior radii, respectively.

Let $-p$ and $-p_1$ be the interior and exterior intensities, respectively.

Since each surface is subjected to normal pressure of uniform intensity *no tangential internal stress can exist*, but normal stresses in three rectangular co-ordinate directions may and do exist. Consequently, in the notation of Art. 4,

$$T_{\psi r} = T_{\phi r} = T_{\psi\phi} = 0.$$

With a given value of r, also, a uniform state of stress will exist. *Neither N_ψ nor N_ϕ can, then, vary with ϕ or ψ.* By the aid of these considerations, and after omitting R_0, Φ_0, Ψ_0, and the second members, the eqs. (1), (2), and (3) of Art. 4 reduce to

$$\frac{dN_r}{dr} + \frac{2N_r - N_\phi - N_\psi}{r} = 0; \quad . \quad . \quad . \quad . \quad (1)$$

$$-N_\psi + N_\phi = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

By eq. (2)

$$N_\psi = N_\phi.$$

Eq. (1) then becomes

$$\frac{dN_r}{dr} + 2\frac{N_r - N_\phi}{r} = 0. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

On account of the existing condition of stress which has just been indicated it at once results that

$$\eta = \omega = 0,$$

and that ρ is a function of r only.

Eqs. (4) to (10) of Art. 4 then reduce to

$$\theta = \frac{d\rho}{dr} + \frac{2\rho}{r}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$N_r = \frac{2G\mathfrak{r}}{1 - 2\mathfrak{r}}\theta + 2G\frac{d\rho}{dr}; \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$N_\psi = N_\phi = \frac{2G\mathfrak{r}}{1 - 2\mathfrak{r}}\theta + 2G\frac{\rho}{r}. \quad . \quad . \quad . \quad . \quad (6)$$

After substitution of these quantities, eq. (3) becomes

$$\frac{2G\mathfrak{r}}{1-2\mathfrak{r}}\left(\frac{d^2\rho}{dr^2}+\frac{2r\,d\rho-2\rho\,dr}{r^2\,dr}\right)+2G\frac{d^2\rho}{dr^2}+4G\frac{d\rho}{r\,dr}-4G\frac{\rho}{r^2}=0,$$

or

$$\frac{d^2\rho}{dr^2}+\frac{\left(d\frac{2\rho}{r}\right)}{dr}=0.$$

One integration gives

$$\frac{d\rho}{dr}+\frac{2\rho}{r}=c=\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (7)$$

Hence θ, the rate of variation of volume, is a constant quantity. Eq. (7) may take the form

$$r\,d\rho+2\rho\,dr=cr\,dr.$$

As it stands, this equation is not integrable, but, by inspecting its form, it is seen that r is an integrating factor. Multiplying both sides of the equation, then, by r,

$$r^2\,d\rho+2r\rho\,dr=d(r^2\rho)=cr^2\,dr;$$

$$\therefore\ r^2\rho=c\frac{r^3}{3}+b;\quad \therefore\ \rho=\frac{cr}{3}+\frac{b}{r^2}. \quad . \quad . \quad . \qquad (8)$$

Substituting from eqs. (7) and (8) in eq. (5),

$$N_r=\frac{2G\mathfrak{r}}{1-2\mathfrak{r}}c+\frac{2Gc}{3}-\frac{4bG}{r^3}; \quad . \quad . \quad . \quad . \quad . \qquad (9)$$

$$=A-\frac{4bG}{r^3}.$$

It is obvious what A represents.

When r' and r_1 are put for r, N_r becomes $-p$ and $-p_1$. Hence

$$A-\frac{4bG}{r^3}=-p$$

and

$$A-\frac{4bG}{r_1^{\,3}}=-p_1.$$

These equations express the conditions involved in eqs. (13), (14), and (15) of Art. 2.

The last equations give

$$4Gb=\frac{(p_1-p)\,r_1^{\,3}r'^3}{r'^3-r_1^{\,3}};$$

$$\therefore\ A=\frac{p_1r_1^{\,3}-pr'^3}{r'^3-r_1^{\,3}}.$$

These quantities make it possible to express N_r and N_ϕ independently of the constants of integration, c and b, for those intensities become

$$N_r=\frac{p_1r_1^{\,3}-pr'^3}{r'^3-r_1^{\,3}}-\frac{(p_1-p)\,r_1^{\,3}r'^3}{r'^3-r_1^{\,3}}\cdot\frac{1}{r^3};\quad .\quad (10)$$

$$N_\phi=N_\phi=\frac{p_1r_1^{\,3}-pr'^3}{r'^3-r_1^{\,3}}+\frac{(p_1-p)\,r_1^{\,3}r'^3}{2(r'^3-r_1^{\,3})}\cdot\frac{1}{r^2}.\quad .\quad (11)$$

Thus it is seen that $N_\phi=N_\phi$ has its greatest value for the interior surface; that intensity will be called h.

It is now required to find $r_1-r'=t$ in terms of h, p, and p_1.

If $r=r'$ in eq. (11),

$$2h(r'^3-r_1^{\,3})=3p_1r_1^{\,3}-p(2r'^3+r_1^{\,3}).$$

Dividing this equation by r'^3 and solving,

$$\frac{r_1^3}{r'^3} = \frac{2(h+p)}{2h-p+3p_1};$$

$$\therefore\ r_1 - r' = t = r'\sqrt[3]{\frac{2(h+p)}{2h-p+3p_1}} - r'. \quad . \quad . \quad (12)$$

If the intensities p and p_1 are given for any case, eq. (12) will give such a thickness that the greatest tension h (supposing p_1 considerably less than p) shall not exceed any assigned value. If the external pressure is very small compared with the internal, p_1 may be omitted.

The values of A and $4Gb$ allow the expressions for c and b to be at once written.

If p_1 is greater than p, nothing is changed except that $N_\phi = N_\phi$ becomes negative, or compression.

CHAPTER III.

THE ENERGY OF ELASTICITY.

Art. 9.—Work Expended in Producing Strains.

THE general expressions, in rectangular co-ordinates, for the unbalanced forces which act in the three co-ordinate directions upon any indefinitely small parallelopiped of material subjected to any state of stress whatever are given by multiplying each of eqs. (7), (8), and (9) of Art. 2 by $(dx\, dy\, dz)$. If an indefinitely small change in the state of stress takes place, that indefinitely small parallelopiped will suffer a displacement whose rectangular components are du, dv, dw; and the amount of work performed in moving it will be found by multiplying each of the three unbalanced forces, determined as above, by each of the three small strains belonging to the same direction with the force (as in Art. 2 u, v, and w are strains in the directions of x, y, and z). This differential quantity of work, integrated throughout the extent of the body, will give the elementary quantity of work required for the small deformation and motion of the whole body.

The resulting equations form the foundation of investigations in elastic vibrations and resilience; they also furnish the means of reaching some general conclusions in reference to suddenly applied loads.

Let dW represent the elementary quantity of work required for the motion only, then the operations which have just been indicated will give the following expression:

$$\begin{aligned}\iiint\Big[&\left(\frac{dN_1}{dx}dx\,dy\,dz+\frac{dT_3}{dy}dx\,dy\,dz+\frac{dT_2}{dz}dx\,dy\,dz\right)du\\ &+\left(\frac{dT_3}{dx}dx\,dy\,dz+\frac{dN_2}{dy}dx\,dy\,dz+\frac{dT_1}{dz}dx\,dy\,dz\right)dv\\ &+\left(\frac{dT_2}{dx}dx\,dy\,dz+\frac{dT_1}{dy}dx\,dy\,dz+\frac{dN_3}{dz}dx\,dy\,dz\right)dw\\ &+(X_0\,du+Y_0\,dv+Z_0\,dw)\;dx\,dy\,dz\Big]\\ &=\iiint m\left(du\frac{d^2u}{dt^2}+dv\frac{d^2v}{dt^2}+dw\frac{d^2w}{dt^2}\right)dx\,dy\,dz=dW. \qquad (1)\end{aligned}$$

This equation, however, can be put in a much simpler form and caused to take a shape which will show at a glance the true character of each part; dx, dy, and dz are differentials of independent variables, hence they are arbitrary and independent. Integrating by parts, therefore,

$$\begin{aligned}&\iiint\frac{dN_1}{dx}dx\,.\,dy\,dz\,.\,du\\ &\qquad=\iint(N_1'\,du'-N_1''\,du'')\;dy\,dz-\iiint N_1\,d\left(\frac{du}{dx}\right)\,dx\,dy\,dz;\end{aligned}$$

in which the *primes* indicate the values of N_1 and u at one point of the exterior surface of the body, and the *seconds* those values for another point of the exterior surface, these points being taken at opposite extremities of *a bar of the material whose normal section is* ($dy\,dz$) *and which extends entirely through the body in the direction of x.* Maintaining the same notation and proceeding,

$$\begin{aligned}&\iiint\frac{dT_3}{dy}dy\,.\,dx\,dz\,.\,du\\ &\qquad=\iint(T_3'\,du'-T_3''\,du'')\;dx\,dz-\iiint T_3 d\left(\frac{du}{dy}\right)\,dy\,dx\,dz;\end{aligned}$$

$$\iiint \frac{dT_2}{dz} dz \,.\, dx\, dy \,.\, du$$
$$= \iint (T_2{}' \, du' - T_2{}'' \, du') \, dx\, dy - \iiint T_2 d\left(\frac{du}{dz}\right) dz\, dx\, dy.$$

But by referring to the equations which immediately precede (13), (14), and (15) of Art. 2 it will be seen that the sum of these three *double* integrals will represent *the amount of work performed on the body by the external forces acting in the direction of the axis of* x. Precisely the same general results are obtained for the directions of y and z by treating in the same manner the remaining derivatives of the internal intensities in eq. (1). The preceding operations are typical; therefore they need not be repeated.

Again, by reference to the notation and demonstrations of Art. 1,

$$d\left(\frac{du}{dy}\right) + d\left(\frac{dv}{dx}\right) = d\phi_1;$$

$$d\left(\frac{dv}{dz}\right) + d\left(\frac{dw}{dy}\right) = d\phi_2;$$

$$d\left(\frac{dw}{dx}\right) + d\left(\frac{du}{dz}\right) = d\phi_3;$$

$$d\left(\frac{du}{dx}\right) = dl_1; \quad d\left(\frac{dv}{dy}\right) = dl_2; \quad d\left(\frac{dw}{dz}\right) = dl_3.$$

Finally

$$du\frac{d^2u}{dt^2} = \frac{1}{2}d\left(\frac{du}{dt}\right)^2; \quad dv\frac{d^2v}{dt^2} = \frac{1}{2}d\left(\frac{dv}{dt}\right)^2; \quad dw\frac{d^2w}{dt^2} = \frac{1}{2}d\left(\frac{dw}{dt}\right)^2.$$

Introducing these reductions and quantities, eq. (1) becomes

$$\begin{aligned}&\int P'da'(\cos\pi'\,du'+\cos\chi'\,dv'+\cos\rho'\,dw)\\
&-\int P''da''(\cos\pi''\,du''+\cos\chi''\,dv''+\cos\rho''\,dw'')\\
&-\iiint(N_1dl_1+N_2dl_2+N_3dl_3+T_3d\phi_1+T_1d\phi_2+T_2d\phi_3)dx\,dy\,dz\\
&+\iiint(X_0du+Y_0dv+Z_0dw)\,dx\,dy\,dz\\
&=\iiint m\frac{1}{2}d\left[\left(\frac{du}{dt}\right)^2+\left(\frac{dv}{dt}\right)^2+\left(\frac{dw}{dt}\right)^2\right]dx\,dy\,dz=dW.\quad.\quad.\quad(2)\end{aligned}$$

Eq. (2) shows clearly the distribution of the different portions of work expended. The first two (single) integrals evidently represent the total amount of work performed by forces acting on the exterior surface of the body; it will be indicated by dW_1. If the forces P' and P'' are of the same kind (i.e., both pulls or both pushes), the *algebraic* sum of any two terms of these integrals will be a *numerical* sum if they involve cosines of the same letter but of opposite signs.

The correct application of eq. (2) depends largely upon the proper observance of the signs which should affect P', P'', and the cosines.

The first triple integral in the first member of eq. (2), in which each intensity of stress is multiplied by the differential of its characteristic strain, and which will be indicated by dW_2, is evidently the amount of work required for the small distortion alone of the body. The quantity within the parentheses is called the *potential energy of the elasticity of a cubic unit of material*, since, *if it be multiplied by* ($dx\ dy\ dz$), *the product will express the amount of work that small portion of material can perform in returning to its original condition.*

This potential energy for a cubic unit is easily integrated by the aid of eqs. (11), (12), (13), (17), (18), and (19) of Art. 1. Making the substitutions from those equations and integrating,

$$H = \int (N_1 dl_1 + N_2 dl_2 + N_3 dl_3 + T_3 d\phi_1 + T_2 d\phi_3 + T_1 d\phi_2)$$

$$= 2G\left(\frac{l_1^2 + l_2^2 + l_3^2}{2}\right) + \frac{2Gr}{1 - 2r}\,\frac{(l_1 + l_2 + l_3)^2}{2} + G\left(\frac{\phi_1^2 + \phi_2^2 + \phi_3^2}{2}\right).$$

H is the potential energy of a cubic unit of material for a change of state extending from the limit 0 to the strains l_1, l_2, etc.

The last triple integral in the first member of eq. (2) expresses the work done by external forces which take hold of the mass of the body. Let it be represented by dW_3. This triple integral added to the first two single integrals, which belong to the surface of the body, will give the *total* work done by external forces.

The second member of the equation is the small variation of actual energy, which usually exists in consequence of vibrations.

Let V be the resultant velocity of the parallelopiped, then will

$$\frac{1}{2} dV^2 = V dV = \frac{1}{2} d\left[\left(\frac{du}{dt}\right)^2 + \left(\frac{dv}{dt}\right)^2 + \left(\frac{dw}{dt}\right)^2\right].$$

By transferring dW_2, the first two members of eq. (2) may take the form

$$dW_1 + dW_3 = dW_2 + \iiint mV\, dV\, dx\, dy\, dz;$$

$$\therefore\ W_1 + W_3 = W_2 + \iiiint mV\, dV\, dx\, dy\, dz. \quad . \quad (3)$$

Or, *the total external work performed is equal to the work done in distorting the body added to the change of actual energy.*

This result expresses the law of the conservation of energy for the elastic bodies considered.

If the external work is nothing, the first member of eq. (3) is zero. The actual energy will then exist in consequence of a state of vibration. Let its variable value be represented by U. Since dx, dy, and dz are arbitrary,

$$U = \iiint m\frac{V^2}{2}dx\,dy\,dz - C;$$

C representing a constant of integration. Under the circumstances assumed, then,

$$W_2 + U = C. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Hence, *the total energy of the vibrating body* (i.e., *the sum of the actual and potential*) *will be constant.*

Art. 10.—Resilience.

The term *resilience* is applied to the quantity of work which is required to be expended in order to produce a given state of strain in a body. The analytical expression for this amount of work is obtained directly from eq. (2) of the preceding article.

Let the simple case of a single straight bar be considered, and let all the external forces act parallel to the axis of the bar while they take hold of the end surfaces, which are normal sections. These external forces will be considered equal to the internal stresses developed; consequently no vibrations will exist. The action of the external forces X_0, Y_0, and Z_0 will also be omitted.

Now, if the axis of x be taken parallel to the axis of the bar, and if that end of the bar to which P'' is applied be fixed, there will result from the preceding conditions

$$\cos \pi' = \cos \pi'' = 1,$$
$$\cos \chi' = \cos \rho' = \cos \chi'' = \cos \rho'' = du'' = 0,$$
$$N_2 = N_3 = T_1 = T_2 = T_3 = 0.$$

Eq. (2) will then become

$$\int P' da\, du' = \int\int\int N_1\, dl_1\, dx\, dy\, dz. \quad . \quad . \quad . \quad (1)$$

But if the intensity P' is uniform and A the area of normal section, eq. (1) becomes

$$P'A\, du' = AN_1 x_1 dl_1, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

in which x_1 is the length of the bar.

From eq. (1) on page 3,

$$N_1 = El_1;$$

hence

$$\int P'A\, du' = Resilience = Ax_1 E\int_0^{l_1} l_1\, dl_1 = Ax_1 E\frac{l_1^2}{2}. \quad . \quad (3)$$

The quantity

$$El_1^2 = \frac{N_1^2}{E}$$

is called the "*Modulus of Resilience.*" This term is usually applied when N_1 is the greatest intensity allowed in the bar.

If one end of a bar placed in a vertical position is fixed, while a falling body whose weight is w acts upon the other end, the height of fall may be sufficient to produce rupture. Let h be the height of fall required and $N_1 = p$ the ultimate resistance of the material of the bar. In order that rupture may take place,

$$wh = \frac{Ax_1}{2} \cdot \frac{p^2}{E}; \quad \therefore \; h = x_1 \frac{A}{2w} \cdot \frac{p^2}{E} . \quad . \quad . \quad . \quad . \quad (4)$$

Eq. (4) shows that *the height of fall varies directly as the length of the piece.* It is virtually assumed, however, that the extension or compression is uniform throughout the length of the bar to the instant of rupture. This in reality is not true, and h will not vary as rapidly as x_1. The principle established in eq. (4) is equally true for torsion and bending.

Art. 11.—Suddenly Applied External Forces or Loads.

A very important deduction can be reached by an attentive consideration of eq. (2) of Art. 9, if it be assumed that the external forces P' and P'' are simple and direct functions of the external strains u, v, and w. In such a case the following relations will hold, in which a, b, and c are constants:

$$P' \cos \pi' = au'; \quad P' \cos \chi' = bv'; \quad P' \cos \rho' = cw';$$
$$P'' \cos \pi'' = au''; \quad P'' \cos \chi'' = bv''; \quad P'' \cos \rho'' = cw''.$$

Consequently the external work performed, omitting X_0, Y_0, and Z_0, in changing the body from a state of no stress to that indicated by the strains $\mathfrak{u}'$, $\mathfrak{v}'$, $\mathfrak{w}'$, $\mathfrak{u}''$, $\mathfrak{v}''$, $\mathfrak{w}''$, will be

$$\int dW_1 = \int da' \left(a\frac{\mathfrak{u}'^2}{2} + b\frac{\mathfrak{v}'^2}{2} + c\frac{\mathfrak{w}'^2}{2} \right)$$
$$- \int da'' \left(a\frac{\mathfrak{u}''^2}{2} + b\frac{\mathfrak{v}''^2}{2} + c\frac{\mathfrak{w}''^2}{2} \right) = W'';$$

in which equations the integrals are to be made to cover the whole extent of the surface.

If, instead of being variable, the forces P' and P'' are constant and equal to the *final* values of the preceding case (i.e., equal to $a\mathfrak{u}'$, $b\mathfrak{v}'$, $c\mathfrak{w}'$, $a\mathfrak{u}''$, etc.), the external work performed in bringing the body to the final state $\mathfrak{u}'$, $\mathfrak{v}'$, etc., will be

$$\int dW_1 = \int da' \ (a\mathfrak{u}'^2 + b\mathfrak{v}'^2 + c\mathfrak{w}'^2) - \int da'' \ (a\mathfrak{u}''^2 + b\mathfrak{v}''^2 + c\mathfrak{w}''^2) = 2W'.$$

This last case is that of "suddenly" applied external forces or loads, while the former is that of gradual application, in which the external forces, at each instant, are equal to the internal resistances. In the case of sudden application it is seen that the amount of work expended is twice as great as in the other case; consequently when the body arrives at the state of strain indicated by $\mathfrak{u}'$, $\mathfrak{v}'$, etc., *there remains to be expended just as much work as has already been performed*, and at the instant in question *it exists in the body in the shape of actual energy.*

But if an amount of energy equal to W' will produce the strains $\mathfrak{u}'$, $\mathfrak{v}'$, etc., and if, while the force acts which performed the work, an additional amount of energy equal to W' be expended on the body, additional strains equal to $\mathfrak{u}'$, $\mathfrak{v}'$, etc., will be produced in the body.

When the body comes to rest, therefore, the external strains will be $2\mathfrak{u}'$, $2\mathfrak{v}'$, $2\mathfrak{w}'$, etc. There is then no actual energy, all is potential.

Since the external strains are $2\mathfrak{u}'$, $2\mathfrak{v}'$, etc., the external work which has been performed up to this instant will be found by putting those quantities in the place of $\mathfrak{u}'$, $\mathfrak{v}'$, etc., in the expression for W', above. That expression will then become $4W'$.

For gradually applied loads eq. (2) of Art. 9 becomes simply

$$W' = \iiint H \, dx \, dy \, dz;$$

in which H is the potential energy per cubic unit for the state of strain corresponding to $\mathfrak{u}'$, $\mathfrak{v}'$, $\mathfrak{w}'$, etc. But if the loads be suddenly applied, in accordance with what has been given, eq. (2) of Art. 9 becomes

$$4W' = \iiint 4H \, dx \, dy \, dz.$$

Now the expression for H, given in Art. 9, shows that multiplying H by 4 *is the same thing as doubling the strains:*

$$l_1, \; l_2, \; l_3, \;\; \phi_1, \;\; \phi_2, \text{ and } \phi_3.$$

But by doubling the strains the intensities of stresses are doubled. Hence, *if the same loads are first applied gradually and then suddenly, the strains and stresses in the latter case will be double those in the former.* This is a very important principle in engineering practice, for it covers all cases of tension, compression, torsion, and bending. It also finds many important extensions in special cases of such structures as iron and steel bridges, particularly suspension bridges. For the considerations involved in this article show that in all cases of sudden application of loads actual energy will be stored and restored during different intervals of time, and consequently that vibrations will be initiated.

Eq. (2) of Art 9 furnishes a most convenient and elegant point of departure for investigations in such special cases, as will be exemplified in the next article.

Art. 12.—Longitudinal Oscillations of a Straight Bar of Uniform Section.

The complete solution of this problem will not be given, though it may be reached.

Let the bar be fixed at one end in a vertical position and let a heavy weight, W, act on the other. Also, let the axis of x be taken parallel to the axis of the bar, whose uniform normal section will be represented by A.

On account of the circumstances of application of the external forces and position of bar, the following equations of condition will exist:

$$\cos\chi' = \cos\rho' = \cos\chi'' = \cos\rho'' = du'' = N_2 = N_3$$
$$= T_1 = T_2 = T_3 = 0 = Y_0 = Z_0.$$

$$\frac{dv}{dt} = \frac{dw}{dt}$$

will be very small compared with $\frac{du}{dt}$, hence they will be omitted. P' is the heavy weight attached to the free end of the bar divided by A; consequently

$$\cos\pi' = 1.$$

Eq. (2) of Art. 9 now reduces to

$$\int P'\,da'\,du' - \iiint El_1\,dl_1\,dx\,dy\,dz + \iiint X_0\,du\,dx\,dy\,dz$$
$$= \iiint m\frac{1}{2}d\left(\frac{du}{dt}\right)^2 dx\,dy\,dz. \quad (1)$$

The integrals are to be extended throughout the whole of the bar. Since strains and stresses are uniform for any one cross-section of the bar, and because $X_0 = w =$ weight of a unit of volume of the bar (the force of gravity is the only external force which acts on the mass of the bar), eq. (1) becomes

$$Wdu' - AE\frac{l_1^2}{2}dx + Awu\,dx = \frac{m}{2}A\left(\frac{du}{dt}\right)dx + C\,dx. \quad . \quad (2)$$

This equation (*C* being a constant of integration) involves the complete problem of longitudinal oscillations. Two special cases, only, however, will be treated, in which the weight of the bar is so small compared with *W* that it may be neglected. This condition involves the omission of

$$Awu\,dx \quad \text{and} \quad \frac{m}{2}A\left(\frac{du}{dt}\right)^2 dx,$$

in eq. (2), and *makes* l_1 *constant throughout the length of the bar.*

Since the equation must be homogeneous, *C* will represent a quantity of actual energy; in fact, a part of *that quantity stored, at any instant, in W.*

If x_1 represents the length of the bar, *C* may be put equal to

$$\frac{W}{2gx_1}\left(\frac{du'}{dt}\right)^2.$$

Also, because l_1 is constant for the whole bar,

$$l_1 = \frac{u'}{x_1}.$$

Introducing all these changes in eq. (2) and integrating,

$$Wu' - AE\frac{u'^2}{2x_1} = \frac{W}{2g}\left(\frac{du'}{dt}\right)^2 + C'. \quad . \quad . \quad . \quad . \quad (3)$$

If *W* is suddenly applied to the bar while in a state of equilibrium or rest, for which

$$u' = \frac{du'}{dt} = 0,$$

C' will be zero, as the equation shows by such a substitution.

For this case eq. (3) becomes, after omitting the primes,

$$dt=\sqrt{\frac{Wx_1}{AEg}}\,\frac{du}{\sqrt{\frac{2Wx_1}{AE}u'-u^2}}.$$

$$\therefore\ t=\sqrt{\frac{Wx_1}{AEg}}\ \text{ver}\sin^{-1}\frac{AEu}{Wx_1}. \quad . \quad . \quad . \quad . \quad (4)$$

The limits of the amplitude are discovered by putting

$$\frac{du'}{dt}\ \text{(the velocity)}=0$$

in eq. (3), remembering that C' is also equal to zero. That operation will give

$$u=0 \quad \text{and} \quad u=\frac{2Wx_1}{AE}.$$

Putting these values in eq. (4) successively, and taking the difference of the results, the time occupied by one oscillation will be

$$T=\sqrt{\frac{Wx_1}{AEg}}\ \text{ver}\sin^{-1}2=\pi\sqrt{\frac{u_1}{g}}, \quad . \quad . \quad . \quad (5)$$

in which equation

$$u_1=\frac{Wx_1}{AE}$$

is the strain in the bar caused by a gradual application of W.

In the second case to be treated the bar is first supposed to take a vertical position, with the weight attached to its free end, in a state of equilibrium. An external force then depresses the free end a distance u_0, *measured from its position of equilibrium.* If the force F is now removed, the weight will make excursions on each side of its position of rest.

Let u_1 represent the value of u' corresponding to the weight W alone, as in the previous case; then let

$$u' = u_1 + u,$$

u being measured from the position of equilibrium of the weight W.

Eq. (3) will then take the form

$$W(u_1+u) - \frac{AE}{2x_1}(u_1+u)^2 = \frac{W}{2g}\left(\frac{d\,(u_1+u)}{dt}\right) + C. \quad . \quad (6)$$

When $u = u_0$ the body comes to rest. Hence

$$W(u_1+u_0) - \frac{AE}{2x_1}(u_1+u_0)^2 = C. \quad . \quad . \quad . \quad (7)$$

Subtracting eq. (7) from eq. (6),

$$W(u-u_0) - \frac{AE}{2x_1}[2u_1(u-u_0) + u^2 - u_0^2] = \frac{W}{2g}\left(\frac{du}{dt}\right)^2, \quad (8)$$

since

$$d(u_1+u) = du.$$

Remembering that

$$W(u-u_0) = \frac{AEu_1}{x_1}(u-u_0),$$

eq. (8) may take the form

$$dt = \sqrt{\frac{Wx_1}{AEg}}\,\frac{du}{\sqrt{u_0^2-u^2}}. \quad . \quad . \quad . \quad . \quad (9)$$

$$\therefore \quad t = \sqrt{\frac{u_1}{g}}\sin^{-1}\frac{u}{u_0}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Eq. (9) shows that the amplitude of a vibration is found by putting

$$u = +u_0 \quad \text{or} \quad -u_0.$$

Putting these values in eq. (10) and taking the difference of the results, the time of a single oscillation is found to be

$$T = \pi\sqrt{\frac{u_1}{g}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Eq. (11) is seen to be identical with eq. (5). In this case the amplitude is $2u_0$, and the body oscillates through its position of rest. Both oscillations are completely isochronous for the same weight W.

If n is the observed number of oscillations per second, either eq. (5) or (11) gives

$$E = \frac{1}{T^2} \cdot \frac{\pi^2 W x_1}{Ag} = n^2 \frac{\pi^2 W x_1}{Ag}, \quad . \quad . \quad . \quad . \quad . \quad (12)$$

from which E may be computed if W is very great compared with the weight of the bar or wire.

CHAPTER IV.

THEORY OF FLEXURE.

Art. 13.—General Formulæ.

If a prismatic portion of material is either supported at both ends, or fixed at one or both ends, and subjected to the action of external forces whose directions are normal to, *and cut*, the axis of the prismatic piece, that piece is said to be subjected to "flexure." If these external forces have lines of action which are oblique to the axis of the piece, it is subjected to combined flexure and direct stress.

Again, if the piece of material is acted upon by a couple having the same axis with itself, it will be subjected to "torsion."

The most general case possible is that which combines these three, and some general equations relating to it will first be established.

The co-ordinates axis of X will be taken to coincide with the axis of the prism, and *it will be assumed that all external forces act upon its ends only.* Since no external forces act upon its lateral surface, there will be taken

$$T_1 = N_2 = N_3 = 0,$$

retaining the notation of Art. 2. These conditions are not strictly true for the general case, but the errors are, at most, excessively small for the cases of direct stress or flexure, or for a combination of the two. By the use of eqs. (12), (21), and (22) of Art. 1 the conditions just given become

$$\frac{r}{1-2r}\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)+\frac{dv}{dy}=0; \quad . \quad . \quad . \quad (1)$$

$$\frac{r}{1-2r}\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)+\frac{dw}{dz}=0; \quad . \quad . \quad . \quad (2)$$

$$\frac{dv}{dz}+\frac{dw}{dy}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Eqs. (1) and (2) then give

$$\frac{dv}{dy}-\frac{dw}{dz}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

In consequence of eq. (4) eqs. (1) and (2) give

$$\frac{dv}{dy}=\frac{dw}{dz}=-r\frac{du}{dx}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

By the aid of eq. (5) and the use of eqs. (11), (13), and (20) of Art. 1, in eqs. (10), (11), and (12) of Art. 2 (in this case $X_0=Y_0=Z_0=0$), there will result

$$2\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz}=0; \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$\frac{d^2u}{dx\,dy}+\frac{d^2v}{dx^2}=0; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$\frac{d^2u}{dx\,dz}+\frac{d^2w}{dx^2}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Eqs. (3), (5), (6), (7), and (8) are five equations of condition by which the strains u, v, and w are to be determined.

Let eq. (6) be differentiated in respect to x:

$$2\frac{d^3u}{dx^3}+\frac{d^3u}{dy^2\,dx}+\frac{d^3u}{dz^2\,dx}=0.$$

From this equation let there be subtracted the sum of the results obtained by differentiating eq. (7) in respect to y and (8) in respect to z:

$$2\frac{d^3u}{dx^3}-\frac{d^3v}{dx^2\,dy}-\frac{d^3w}{dx^2\,dz}=0.$$

In this equation substitute the results obtained by differentiating eq. (5) twice in respect to x, there will result

$$\frac{d^3u}{dx^3}=\frac{d^2\left(\frac{du}{dx}\right)}{dx^2}=0. \quad . \quad . \quad . \quad . \quad . \quad (9)$$

This result, in the equation immediately preceding eq. (9), by the aid of eq. (5) will give

$$\frac{d^3v}{dx^2\,dy}=0.$$

After differentiating eq. (7) in respect to y, and substituting the value immediately above,

$$\frac{d^3u}{dy^2\,dx}=\frac{d^2\left(\frac{du}{dx}\right)}{dy^2}=0. \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Eqs. (9) and (10) enable the second equation preceding eq. (9) to give

$$\frac{d^3u}{dz^2\,dx}=\frac{d^2\left(\frac{du}{dx}\right)}{dz^2}=0. \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Let the results obtained by differentiating eq. (7) in respect to z and (8) in respect to y be added:

$$2\frac{d^3u}{dx\,dy\,dz}+\frac{d^3v}{dx^2\,dz}+\frac{d^3w}{dx^2\,dy}=0.$$

The sum of the second and third terms of the first member of this equation is zero, as is shown by twice differentiating eq. (3) in respect to x. Hence

$$\frac{d^3u}{dy\,dz\,dx}=\frac{d^2\left(\frac{du}{dx}\right)}{dy\,dz}=0. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (12)$$

Eqs. (9), (10), (11), and (12) are sufficient for the determination of the form of the function $\frac{du}{dx}$, if it be assumed to be algebraic, for

Eq. (9) shows that x^2 does not appear in it;
" (10) " " y^2 " " " "
" (11) " " z^2. " " " "
" (12) " " yz " " " "

The products xz and xy may, however, be found in the function. Hence if a, a_1, a_2, b, b_1, and b_2 are constants, there may be written

$$\frac{du}{dx}=a+a_1z+a_2y+x(b+b_1z+b_2y). \quad . \quad . \quad . \qquad (13)$$

Eq. (5) then gives

$$\frac{dv}{dy}=\frac{dw}{dz}=-r\{a+a_1z+a_2y+x(b+b_1z+b_2y)\}. \quad . \qquad (14)$$

Substituting from eq. (13) in eqs. (7) and (8),

$$\frac{d^2v}{dx^2}=-a_2-b_2x; \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

$$\frac{d^2w}{dx^2}=-a_1-b_1x. \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

The method of treatment of the various partial derivatives in the search for eqs. (13) and (14) is identical with that given by Clebsch in his "*Theorie der Elasticitat Fester Korper.*"

It is to be noticed that the preceding treatment has been entirely independent of the *form of cross-section* or *direction of external forces.*

It is evident from eqs. (13) and (14) that the constant a depends upon that component of the external force which acts parallel to the axis of the piece and produces tension or compression only. For (pages 9, 10) it is known that if a piece of material be subjected to direct stress only,

$$\frac{du}{dx}=a \quad \text{and} \quad \frac{dv}{dy}=\frac{dw}{dz}=-ra;$$

the negative sign showing that ra is opposite in kind to a, both being constant.

Again, if z and y are each equal to zero, eq. (13) shows that

$$\frac{du}{dx}=a+bx.$$

Hence bx is a part of the rate of strain in the direction of x which is *uniform over the whole of any normal section of the piece of material,* and it varies directly with x. But such a portion of the rate of strain can only be produced by an external force acting parallel to the axis of X, and whose intensity varies directly as x. But in the present case such a force does not exist. Hence b must equal zero.

The eqs. (13), (14), (15), and (16) show that a_1, b_1 and a_2, b_2 are symmetrical, so to speak, in reference to the coordinates z and y, while eqs. (13) and (14) show that the normal intensity N_1 is dependent on those, and no other, constants in pure flexure in which $a = 0$. It follows, therefore, that those two pairs of constants belong to the two cases of flexure about the two axes of Z and Y.

No direct stress N_1 can exist in torsion, which is simply a twisting or turning about the axis of X.

Since the generality of the deductions will be in no manner affected, pure flexure about the axis of Y will be considered. For this case

$$a = a_2 = b_2 = 0 = b.$$

Making these changes in (13) and (14),

$$\frac{du}{dx} = a_1 z + b_1 xz; \quad . \quad . \quad . \quad . \quad . \quad (17)$$

$$\frac{dv}{dy} = \frac{dw}{dz} = -r\frac{du}{dx} = -r(a_1 z + b_1 xz). \quad . \quad . \quad . \quad (18)$$

$$\therefore \ \theta = \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = z(a_1 + b_1 x)(1 - 2r). \ . \quad . \quad (19)$$

Also,

$$N_1 = \frac{2Gr}{1 - 2r}\theta + 2G\frac{du}{dx}.$$

$$\therefore \ N_1 = 2G(r + 1)(a_1 + b_1 x)z = E(a_1 + b_1 x)z, \quad . \quad (20)$$

since

$$2G(r+1)=E.$$

Taking the first derivative of N_1,

$$\frac{dN_1}{dz}=E(a_1+b_1x). \quad . \quad . \quad . \quad . \quad . \quad (21)$$

This important equation gives the law of variation of the intensity of stress acting parallel to the axis of a bent beam, in the case of pure flexure produced by forces exerted at its extremity. That equation proves that in a given normal section of the beam, whatever may be the form of the section, *the rate of variation of the normal intensity of stress is constant*; *the rate being taken along the direction of the external forces.*

It follows from this that N_1 must vary directly as the distance from some particular line in the normal section considered in which its value is zero. Since the external forces F are normal to the axis of the beam and direction of N_1, and because it is necessary for equilibrium that the sum of all the forces $N_1dy\ dz$, for a given section, must be equal to zero, it follows that on one side of this line tension must exist, and on the other compression.

Let N represent the normal intensity of stress at the distance unity from the line, b the variable width of the section parallel to y, and let $\Delta=bdz$. The sum of all the tensile stress in the section will be

$$\int_0^{z'} Nz\Delta = N\int_0^{z'} z\Delta.$$

The total compressive stress will be

$$N\int_{-z_1}^{0} z\Delta.$$

The integrals are taken between the limits o and the greatest value of z in each direction, so as to extend over the entire section. In order that equilibrium may exist, therefore,

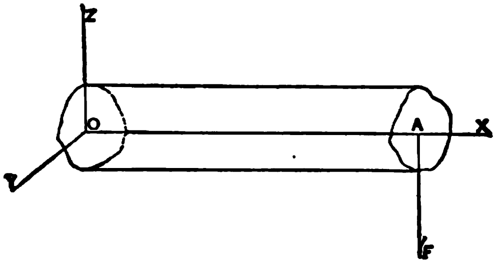

Fig. 1.

$$N\left\{\int_{0}^{z'} z\Delta + \int_{-z_1}^{0} z\Delta\right\} = 0.$$

$$\therefore \int_{-z_1}^{z'} z\Delta = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

Eq. (22) *shows that the line of no stress must pass through the centre of gravity of the normal section.*

This line of no stress is called the *neutral axis* of the section. Regarding the whole beam, there will be a surface which will contain all the neutral axes of the different sections, and it is called the *neutral surface* of the bent beam. The neutral axis of any section, therefore, is the line of intersection of the plane of section and neutral surface.

Hereafter the axis of X will be so taken as to traverse the centres of gravity of the different normal sections before flexure. The origin of co-ordinates will then be taken at the centre of gravity of the fixed end of the beam, as shown in Fig. 1.

The value of the expression (a_1+b_1x), in terms of the external bending moment, is yet to be determined. Consider any normal section of the beam located at the distance x from O, Fig. 1, and let $OA=l$. Also, let Fig. 2 represent the section considered, in which BC is the neutral axis and d' and d_1 the distances of the most remote fibres from BC. Let moments of all the forces acting upon the portion $(l-x)$ of the beam be taken about the neutral axis BC. If, again, b is the variable width of the beam, the internal resisting moment will be

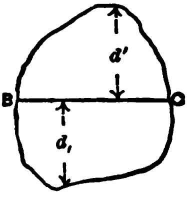

FIG. 2.

$$\int_{-d_1}^{d'} N_1bz\,dz=E(a_1+b_1x)\int_{-d_1}^{d'} z^2.bdz.$$

But the integral expression in this equation is *the moment of inertia of the normal section about the neutral axis*, which will hereafter be represented by I. The moment of the external force, or forces, F, will be $F(l-x)$, and it will be equal, but opposite in sign, to the internal resisting moment. Hence

$$F\ l-x)=M=-E(a_1+b_1x)I. \quad . \quad . \quad . \quad (23)$$

$$\therefore\ -(a_1+b_1x)=\frac{M}{EI}. \quad . \quad . \quad . \quad . \quad . \quad (24)$$

Substituting this quantity in eq. (16),

$$\frac{d^2w}{dx^2}=\frac{M}{EI}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

It has already been seen (page 38) that eq. (25) is one of the most important equations in the whole subject of the "Resistance of Materials."

An equation exactly similar to (25) may of course be written from eq. (15); but in such an expression M will represent the external bending moment about an axis parallel to the axis of Z.

No attempt has hitherto been made to determine the complete values of u, v, and w, for the mathematical operations involved are very extended. If, however, a beam be considered whose width, parallel to the axis of Y, is indefinitely small, u and w may be determined without difficulty. The conclusions reached in this manner will be applicable to any long rectangular beam without essential error.

If y is indefinitely small, all terms involving it as a factor will disappear in u and w; or, *the expressions for the strains u and w will be functions of z and x only.* But making u and w functions of z and x *only* is equivalent to a restriction of lateral strains to the direction of z *only*, or to the reduction of the direct strains one half, since direct strains and lateral strains in two directions accompany each other in the unrestricted case. Now as the lateral strain in one direction is supposed to retain the same amount as before, while the direct strain is considered only half as great, the value of their ratio for the present case will be twice as great as that used on pages 9 to 12. Hence $2r$ must be written for r, in order that that letter may represent the ratio for the unrestricted case, and this will be done in the following equations.

Since w and u are independent of y,

$$\frac{dw}{dy}=\frac{du}{dy}=0, \quad \text{and} \quad T_3=G\frac{dv}{dx}.$$

But, by eq. (14),

$$v=-2r(a_1+b_1x)zy+f(x, z).$$

By eq. (3), since

$$\frac{dw}{dy}=0,$$

$$\frac{dv}{dz}=-2r(a_1+b_1x)y+\frac{d}{dz}f(x,z)=0.$$

This equation, however, involves a contradiction, for it makes $f(x, z)$ equal to a function which involves y, which is impossible. Hence

$$f(x,z)=0.$$

Consequently

$$\frac{dv}{dz}=-2r(a_1+b_1x)y,$$

which is indefinitely small compared with

$$\frac{dv}{dy}=-2r(a_1+b_1x)z,$$

and is to be considered zero

Because $f(x, z)=0$,

$$\frac{dv}{dx}=-2rb_1zy.$$

This quantity is indefinitely small; hence

$$T_3=-2Grb_1zy$$

is of the same magnitude.

Under the assumption made in reference to y, there may be written, from eqs. (17) and (18),

$$u=a_1xz+b_1\frac{x^2}{2}z+f'(z); \quad . \quad . \quad . \quad . \quad (26)$$

$$w=-r(a_1z^2+b_1xz^2)+f(x). \quad . \quad . \quad . \quad (27)$$

Using eq. (26) in connection with eq. (6),

$$2b_1z = -\frac{d^2f'(z)}{dz^2}.$$

By two integrations,

$$f'(z) = -\frac{b_1z^3}{3} - c'z + c''. \quad . \quad . \quad . \quad . \quad (28)$$

Using eq. (27) in connection with eq. (8),

$$\frac{d^2f(x)}{dx^2} = -b_1x - a_1.$$

By two integrations,

$$f(x) = -b_1\frac{x^3}{6} - \frac{a_1x^2}{2} + c_1x + c_{11}.$$

The functions *u* and *w* now become

$$u = a_1xz + b_1\frac{x^2}{2}z - \frac{b_1z^3}{3} - c'z + c''; \quad . \quad . \quad . \quad (29)$$

$$w = -ra_1z^2 - rb_1xz^2 - b_1\frac{x^3}{6} - \frac{a_1x^2}{2} + c_1x + c_{11}. \quad . \quad (30)$$

The constants of integration *c'*, *c''*, etc., depend upon the values of *u* and *w*, and their derivatives, for certain reference values of the co-ordinates *x* and *z*, and also upon the manner of application of the external forces, *F*, at the end of the beam, Fig. 1. The last condition is involved in the application of eqs. (13), (14), and (15) of Art. 2.

In Fig. 1 let the beam be fixed at O. There will then result, for $x=0$ and $z=0$,

$$\left(\frac{du}{dz}=0\right)_{\substack{x=0\\z=0}};$$

$$(u=0, \quad \text{and} \quad w=0)_{\substack{z=0.\\x=0}}$$

In virtue of the last condition,

$$c'' = c_{11} = 0.$$

In consequence of the first,

$$c' = 0.$$

After inserting these values in eqs. (29) and (30),

$$\frac{du}{dz} = a_1x + b_1\frac{x^2}{2} - b_1z^2;$$

$$\frac{dw}{dx} = -rb_1z^2 - b_1\frac{x^2}{2} - a_1x + c_1.$$

$$\therefore \quad T_2 = G\left(\frac{du}{dz}+\frac{dw}{dx}\right) = -Gb_1(1+r)z^2 + Gc_1. \quad . \quad (31)$$

The surface of the end of the beam, on which F is applied, is at the distance l from the origin O and parallel to the plane ZY. Also, the force F has a direction parallel to the axis of Z. Using the notation of eqs. (13), (14), and (15) of Art. 2, these conditions give

$$\cos p = 1, \quad \cos q = 0, \quad \cos r = 0,$$

$$\cos \pi = 0, \quad \cos \chi = 0, \quad \cos \rho = 1.$$

Since, for $x=l$,

$$M=F(l-x)=0,$$

eqs. (24) and (20) give $N_1=0$ for all points of the end surface. Eq. (15) is, then, the only one of those equations which is available for the determination of c_1.

That equation becomes simply

$$T_2=P.$$

For a given value of z, therefore, any value may be assumed for T_2. For the upper and lower surfaces of the beam let the intensity of shear be zero; or for $z=\pm d$ let $T_2=0$. Hence, by eq. (31),

$$c_1=b_1(1+r)d^2.$$

$$\therefore\ T_2=Gb_1(1+r)(d^2-z^2);$$

$$\therefore\ T_2=\frac{Eb_1}{2}(d^2-z^2). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

The constants a_1 and b_1 still remain to be found. The only forces acting upon the portion $(l-x)$ of the beam are F and the sum of all the shears T_2 which act in the section x. Let Δy be the indefinitely small width of the beam, which, since z is finite, is thus really made constant. The principles of equilibrium require that

$$\int_{-d}^{+d}T_2.\Delta y.dz=Gb_1(1+r)\int_{-d}^{+d}(d^2.\Delta y.dz-z^2.\Delta y.dz)=F.$$

The first part of the integral will be $2\Delta yd^3$, and the second part will be the moment of inertia of the cross-section (made rectangular by taking Δy constant) about the neutral axis. Hence

$$2Gb_1(1+r)I = F, \quad \text{or} \quad b_1 = \frac{F}{2G(1+r)I} = \frac{F}{EI}. \quad . \quad (33)$$

$$\therefore \; T_2 = \frac{F}{2I}(d^2 - z^2). \quad . \quad . \quad . \quad . \quad . \quad (34)$$

If $x = 0$ in eq. (24),

$$a_1 = -\frac{Fl}{EI}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

Thus the two conditions of equilibrium are involved in the determination of a_1 and b_1. The complete values of the strains u and w are, finally,

$$u = \frac{F}{EI}\left(z\frac{x^2}{2} - \frac{z^3}{3} - xzl\right); \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

$$w = \frac{F}{EI}\left(lrz^2 - rxz^2 - \frac{x^3}{6} + \frac{lx^2}{2}\right) + \frac{Fd^2x}{2GI}. \quad . \quad . \quad (37)$$

These results are strictly true for rectangular beams of indefinitely small width, but they may be applied to any rectangular beam fixed at one end and loaded at the other, with sufficient accuracy for the ordinary purposes of the civil engineer. It is to be remembered that the load at the end is supposed to be applied according to the law given by eq. (34), a condition which is never realized. Hence these formulæ are better applicable to long than short beams.

The greatest value of T_2, in eq. (34), is found at the neutral axis by making $z=0$; for which it becomes

$$T_2=\frac{Fd^2}{2I}=\frac{3}{2}\cdot\frac{F}{2d}. \quad . \quad . \quad . \quad . \quad . \quad (38)$$

$\frac{F}{2d}$ is the *mean intensity* of shear in the cross-section; hence *the greatest intensity of shear is once and a half as great as the mean.*

In eq. (36), if $z=0$, $u=0$. Hence no point of the neutral surface suffers longitudinal displacement.

In eq. (37) the last term of the second member is that part of the vertical deflection due to the shear at the neutral surface, as is shown by eq. (38). The first term of the second member, being independent of x, is that part of the deflection which arises wholly from the deformation of the normal cross-section.

The usual modification of this treatment, designed to supply formulæ for the ordinary experience of the engineer, has already been given in preceding articles.

APPENDIX II.

ANOTHER METHOD FOR FINDING PITCH OF RIVETS.

A SIMPLE method of finding the pitch of rivets piercing the vertical legs of the flange angles and the web plate of a plate girder at any section of the beam may readily be found by using the general but elementary expression for the bending moment,

$$\Sigma Px = M.$$

By differentiating this equation,

$$\Sigma P . dx = Sdx = dM;$$

S representing the total transverse shear.

If dM is the change of bending moment for the distance along the flange represented by the pitch of rivets, p, the change of flange stress for the same distance will be found by dividing dM by the effective depth of the girder, h. If the pitch of rivets, p, be placed in the preceding equation in place of dx, the corresponding change of flange stress will represent the amount of stress transferred to the flange by one rivet. Representing that variation of flange stress by v, the last of the preceding equations may be written

$$Sp = hv; \quad \therefore \; p = \frac{hv}{S}.$$

In this equation v represents either the bearing capacity of one rivet against the web plate or against the two flange angles, or the double shearing value of the same rivet, i.e., the least of those three values. Ordinarily the bearing of the rivet against the web plate will be less than either of the two other quantities; hence that bearing value would then be substituted for v. In general the least of the three preceding values for one rivet is to be substituted for v in an actual computation. The total transverse shear S is always known at any section or may readily be determined. The preceding formula for the pitch, therefore, is a very simple one and is much employed at the present time in plate-girder design.

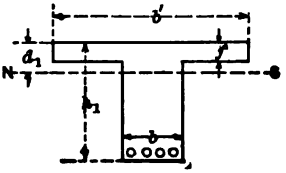

FIG. 1

APPENDIX III.

ART. 76*a*.—T BEAMS OF REINFORCED CONCRETE.

THE general formulæ of Art. 74 belong to beams of any shape of cross section whatever; it is only necessary, therefore, in this case, to apply them to the T-shaped section. Two conditions may arise, in one of which the neutral axis may lie in the flange of the beam whose cross section is shown in Fig. 1, or, as shown in that figure, it may lie below the flange. As is usually the case in actual work, the tensile resistance of the concrete will finally be neglected. This latter condition makes it necessary to consider only the case shown by Fig. 1.

Position of Neutral Axis.

Using the notation of Art. 74 under the conditions outlined above, but first recognizing the tensile resistance of the concrete,

$$\int_0^{d_1} z dA_1 = \int_{d_1-f}^{d_1} z \,.\, b^1 dz + \int_0^{d_1-f} z \,.\, bdz$$

$$= b^1 f\left(d_1 - \frac{f}{2}\right) + b\,\frac{(d_1-f)^2}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 1$$

Introducing this value in Eq. 3 of Art. 74, and using Eq. 2 of Art. 75,

$$b^1fd_1 - b^1\frac{f^2}{2} + \frac{bd_1^2}{2} - bd_1f + \frac{bf^2}{2} - \frac{d_1^2bn}{2} + nbh_1d_1 - n\frac{bh_1^2}{2}$$

$$+ \frac{E_2}{E_1}A_2(d_2 - rh_2) = 0.$$

$$\therefore \quad d_1^2 + d_1\frac{2\left(f\left(\frac{b^1}{b} - 1\right) + nh_1 + \frac{E_2}{E_1}\frac{A_2}{b}\right)}{1 - n} = \frac{\left(\frac{b^1}{b} - 1\right)f^2 + nh_1^2}{1 - n}$$

$$+ 2\frac{E_2}{E_1}\frac{A_2}{b}\frac{(a + rh_2)}{1 - n} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad 2$$

The solution of this quadratic equation will give

$$d_1 = \frac{\left(f\left(\frac{b^1}{b} - 1\right) + nh_1 + \frac{E_2}{E_1}\frac{A_2}{b}\right)}{1 - n}$$

$$\pm\sqrt{\frac{\left(\frac{b_1}{b} - 1\right)f^2 + nh_1^2}{1 - n} + 2\frac{E_2}{E_1}\frac{A_2}{b}\frac{(a + rh_2)}{1 - n} + \frac{\left(f\left(\frac{b_1}{b} - 1\right) + nh_1 + \frac{E_2}{E_1}\frac{A_2}{b}\right)^2}{(1 - n)^2}} \quad 3$$

If the two coefficients of elasticity for concrete in tension and compression are the same, as is always assumed in actual work, $n = 1$. This value gives indetermination in Eq. 3, but it is only necessary to multiply both members of Eq. 2 by $(1 - n)$ and then make $n = 1$. These operations give

$$d_1 = \frac{\frac{1}{2}\left(\frac{b^1}{b} - 1\right)f^2 + \frac{h_1^2}{2} + \frac{E_2}{E_1}\frac{A_2}{b}(a + rh_2)}{f\left(\frac{b^1}{b} - 1\right) + h_1 + \frac{E_2}{E_1}\frac{A_2}{b}} \quad . \; . \; . \; . \; . \quad 4$$

If the entire steel reinforcement is on the tension side of the beam, $r = 1$, and in Eqs. 3 and 4, $a + rh_2 = a + h_2$.

The tensile capacity of the concrete is practically always neglected; hence $n = 0$ in Eq. 3, and

$$d_1 = -f\left(\frac{b^1}{b} - 1\right) - \frac{E_2}{E_1}\frac{A_2}{b} \pm \sqrt{\left(\frac{b^1}{b} - 1\right)f^2 + 2\frac{E_2}{E_1}\frac{A_2}{b}\left(a + rh_2\right) + \left(f\left(\frac{b^1}{b} - 1\right) + \frac{E_2}{E_1}\frac{A_2}{b}\right)^2}. \quad 5$$

These formulæ locate the neutral axis by giving the distance d_1 for all cases.

Bending Moment.

Proceeding in precisely the same manner indicated in Art. 76, but adapting the general formulæ to the shape under consideration, the equation for the bending moment M corresponding to Eq. 5 of Art. 76 becomes

$$\begin{aligned} M = {} & b^1 d_1 . \frac{k_1}{2} . \frac{2}{3} d_1 - (b^1 - b)(d_1 - f) . \frac{d_1 - f}{d_1} k_1 . \frac{1}{2} . \frac{2}{3} (d_1 - f) \\ & + b(h_1 - d_1)\frac{(h_1 - d_1)}{d_1} k_1 . \frac{1}{2} . \frac{2}{3}(h_1 - d_1) \\ & + \frac{E_2}{E_1}\frac{d_2}{d_1} k_1 . d_2 (1 - r) A_2 \\ & + \frac{E_2}{E_1}\frac{h_2 - d_2}{d_1} k_1 . rA_2 . (h_2 - d_2) \quad . \; . \; . \; . \; . \; 6 \end{aligned}$$

Hence,

$$\begin{aligned} M = {} & \frac{b^1 d_1^2}{3} k_1 - (b^1 - b)\frac{(d_1 - f)^3}{3\, d_1} k_1 + b\frac{(h_1 - d_1)^3}{3\, d_1} k_1 \\ & + \frac{E_2 d_2^2}{E_1 d_1}(1 - r) A_2 k_1 + \frac{E_2}{E_1}\frac{(h_2 - d_2)^2}{d_1} rA_2 k_1 \quad . \; . \; 7 \end{aligned}$$

If concrete in tension be ignored and if all the steel reinforcement be on the tension side ($r = 1$), the third and fourth terms of the second member of Eq. 7 disappear and

$$M = \frac{b'd_1^2}{3}k_1 - (b' - b)\frac{(d_1 - f)^2}{3\,d_1}k_1 + \frac{E_2}{E_1}\frac{(h_2 - d_2)^2}{d_1}A_2k_1. \qquad 8$$

Equation 8 is the form of M usually required in actual work.

Problem. A T beam 14 feet 6 inches long with an effective span of 13.3 feet, has a flange 5 feet 6 inches wide and 4 inches thick, the total depth of the beam being 26 inches. The external loading produces a bending moment of 1,915,000 inch-pounds; it is required to design the beam, *i.e.*, to find the width b of the vertical part of the beam and the steel reinforcement wholly on the tension side of the cross section. The greatest allowed compression in the concrete is 500 pounds per square inch, and 16,000 pounds per square inch tension in the steel. A trial value of 16.5 inches will be taken for b, and the steel reinforcement will be taken as five 1¼ inch round bars placed with centers 2.5 inches from the bottom of the beam. Hence,

$b' = 66''$; $b = 16.''5$; $f = 4''$; $a + h_2 = 23.''5$; $A_2 = 6$ square inches; $\frac{b'}{b} = 4$; $\frac{A_2}{b} = \frac{6}{16.5} = \frac{2}{5.5}$; $h_1 = 26''$; $r = 1$.

These quantities inserted in Eq. 5 give, with $\frac{E_2}{E_1}$ taken as 12,

$$d_1 = -16.36 \pm 22.82 = 6.46 \text{ inches.}$$

Hence, $\qquad h_2 - d_2 = a + h_2 - d_1 = 17$ inches.

With d_1 and $h_2 - d_2$ now known, Eq. 8 gives,

$$M = 2{,}045{,}500 \text{ inch-pounds.}$$

As this is but little larger than the external bending moment given above, the dimensions used will be considered satisfactory. The tension in the steel is $12 . \frac{17}{6.46} . 500 = 15800$ pounds per square inch.

This beam as here designed was constructed for an office building in the city of New York.

The width, b', of flange which may be taken as a proper part of the beam is a matter of judgment, as it cannot be determined by analysis. It has been stated in some important building regulations that b' may be taken as large as $10b$, but that value is too high in some cases at least. The ratio of $b' \div b$ should vary with f, being smaller when f is small and *vice versa.*

If $b' = b$ in the preceding values of d_1 and M, those values at once take the forms for a rectangular section, as they should.

In the case of T-beams as they ordinarily occur in practice, it will be found that as $d_1 - f$ is small the second term of the second member of Eq. 8 may be omitted without much error. That omission will make the resulting equation identical with Eq. 11 of Art. 76. Hence some engineers, when the tensile resistance of the concrete is neglected, regard T-beams as of rectangular section with the width b'. This simplifies design work in all ordinary cases without incurring more than a small error.

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TABLES AND PLATES

OF

Structural Steel Shapes

MANUFACTURED BY

CAMBRIA STEEL CO.

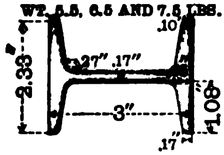
EXPLANATORY NOTE

There will be found on some of the succeeding pages a number of cross references to matters printed on other pages of the Cambria Steel Company's Handbook than those found in the following tables. These references do not in any way affect the use of the tables, and may be disregarded in every case. They simply indicate other matters which may be found in the Cambria Steel Company's Handbook. Indeed, every reader of this book will find much of practical use and convenience in that handbook.

STANDARD BEAMS.

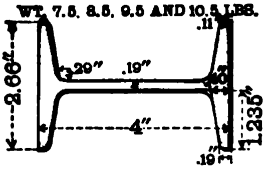
B. 5

WT. 5.5, 6.5 AND 7.5 LBS.



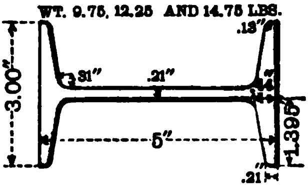
B. 9

WT. 7.5, 8.5, 9.5 AND 10.5 LBS.



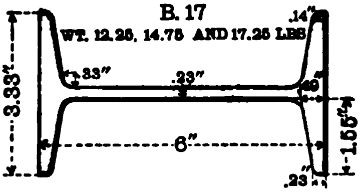
B. 13

WT. 9.75, 12.25 AND 14.75 LBS.



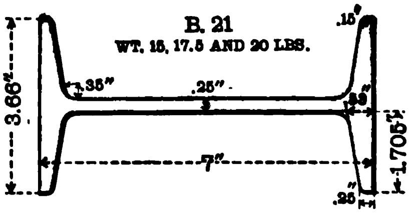
B. 17

WT. 12.25, 14.75 AND 17.25 LBS.

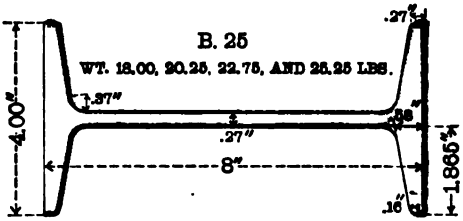


B. 21

WT. 15, 17.5 AND 20 LBS.

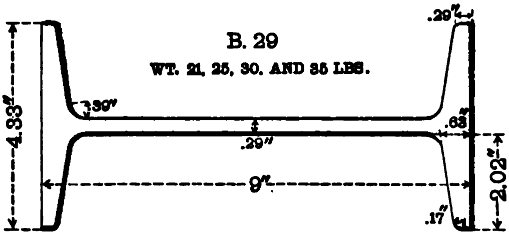


STANDARD BEAMS.



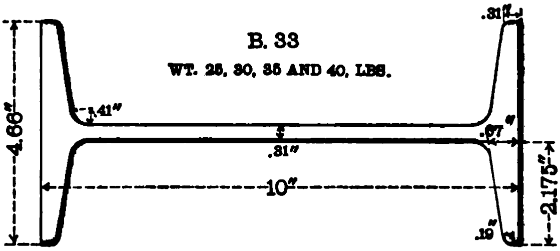
B. 25

WT. 18.00, 20.25, 22.75, AND 25.25 LBS.



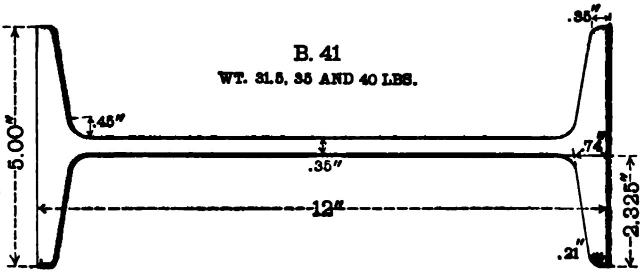
B. 29

WT. 21, 25, 30, AND 35 LBS.



B. 33

WT. 25, 30, 35 AND 40, LBS.

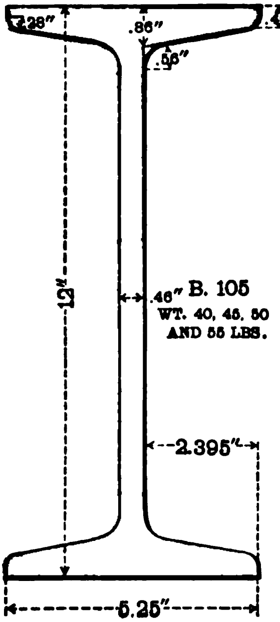


B. 41

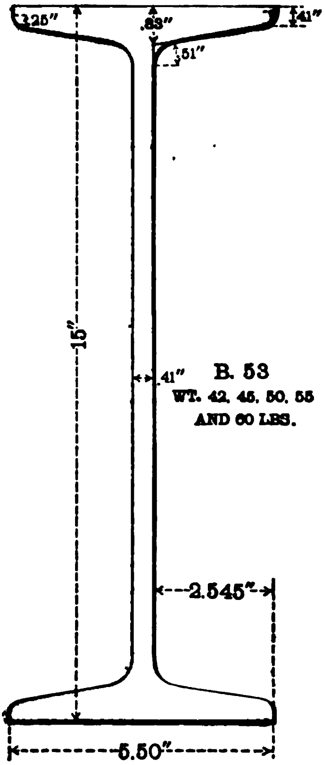
WT. 31.5, 35 AND 40 LBS.

BEAMS.

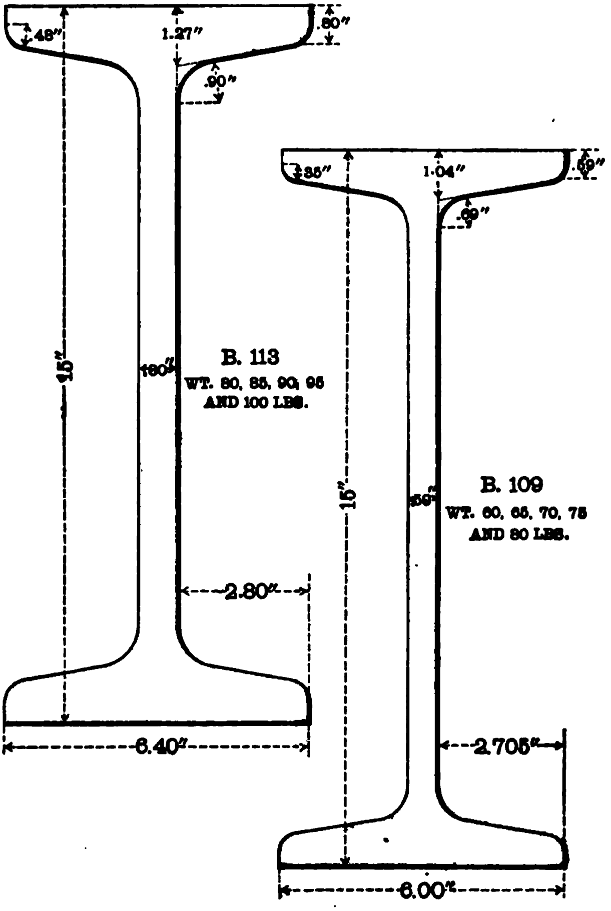
SPECIAL 12" BEAM.



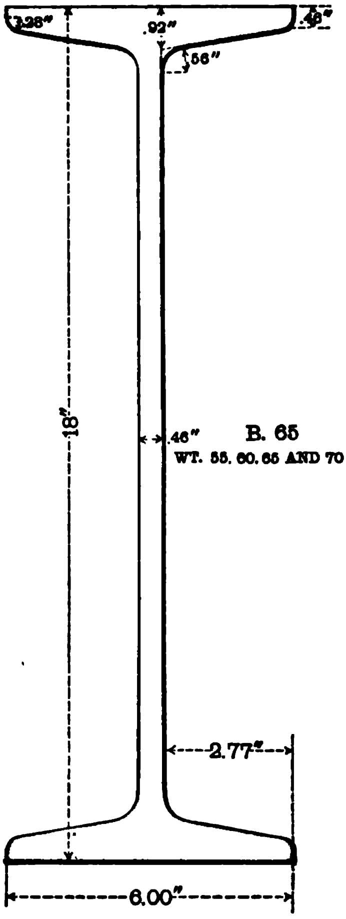
STANDARD 15" BEAM.



SPECIAL BEAMS.



STANDARD BEAMS.

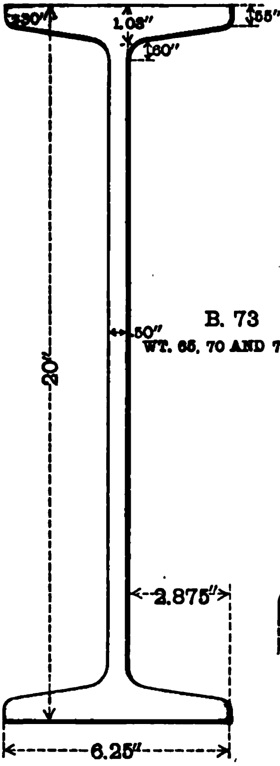


B. 65

WT. 55, 60, 65 AND 70 LBS.

BEAMS.

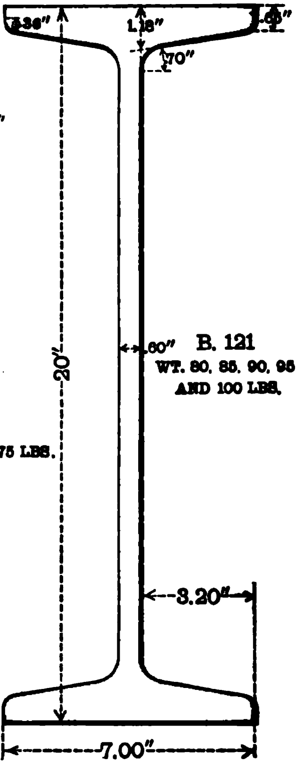
STANDARD 20" BEAM



B. 73

WT. 65, 70 AND 75 LBS.

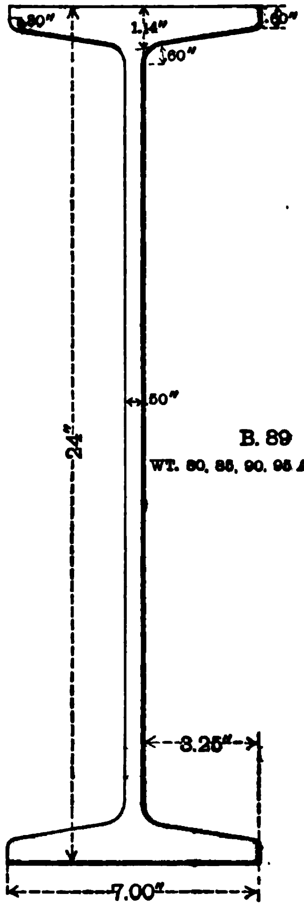
SPECIAL 20" BEAM



B. 121

WT. 80, 85, 90, 95 AND 100 LBS.

STANDARD BEAMS.



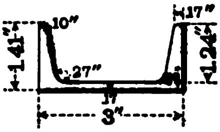
B. 89

WT. 80, 85, 90, 95 AND 100 LBS.

STANDARD CHANNELS.

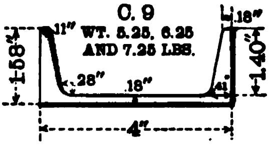
C. 5

WT. 4, 5 AND 6 LBS.



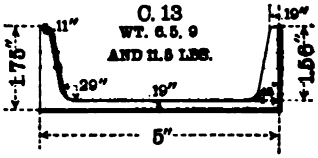
C. 9

WT. 5.25, 6.25 AND 7.25 LBS.



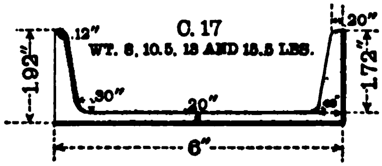
C. 13

WT. 6.5, 9 AND 11.5 LBS.



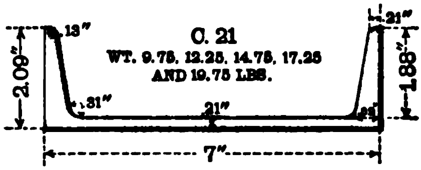
C. 17

WT. 8, 10.5, 13 AND 15.5 LBS.



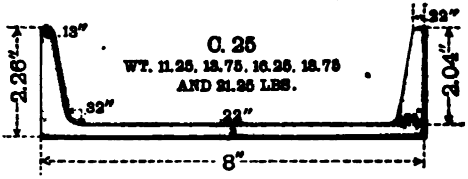
C. 21

WT. 9.75, 12.25, 14.75, 17.25 AND 19.75 LBS.

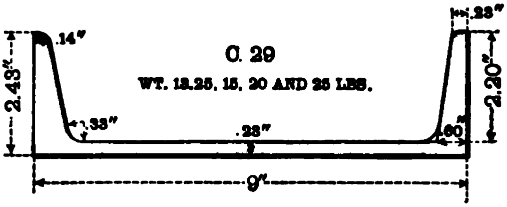


C. 25

WT. 11.25, 13.75, 16.25, 18.75 AND 21.25 LBS.

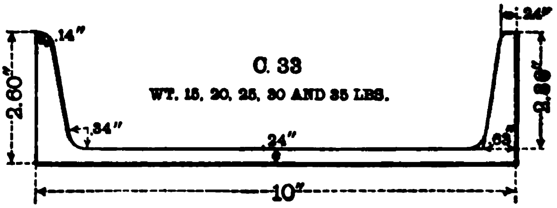


STANDARD CHANNELS.



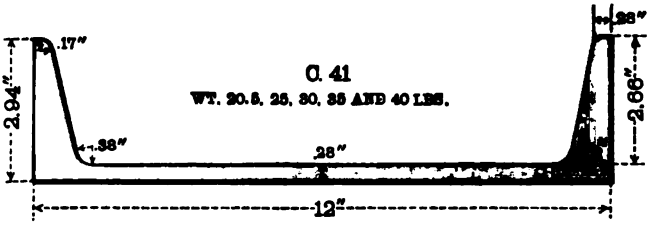
C. 29

WT. 13.25, 15, 20 AND 25 LBS.



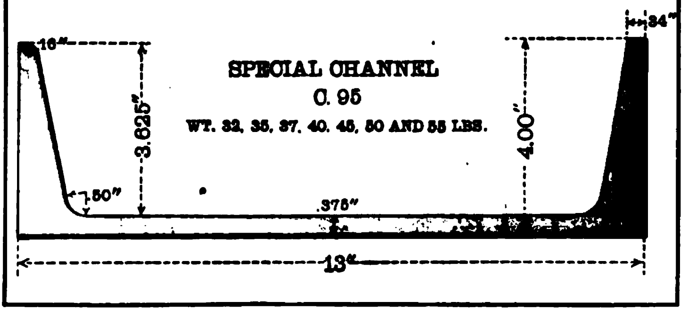
C. 33

WT. 15, 20, 25, 30 AND 35 LBS.



C. 41

WT. 20.5, 25, 30, 35 AND 40 LBS.

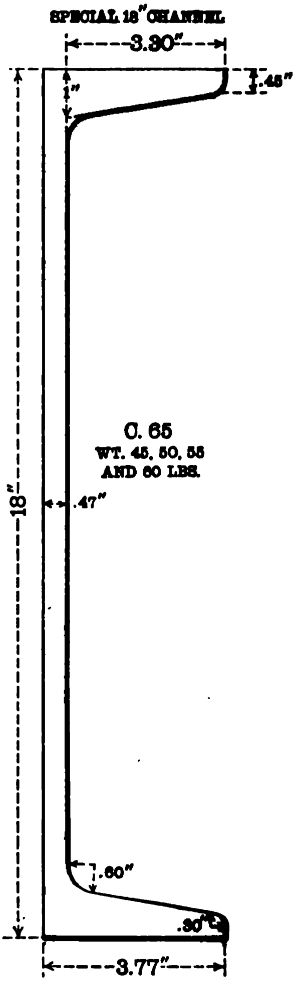


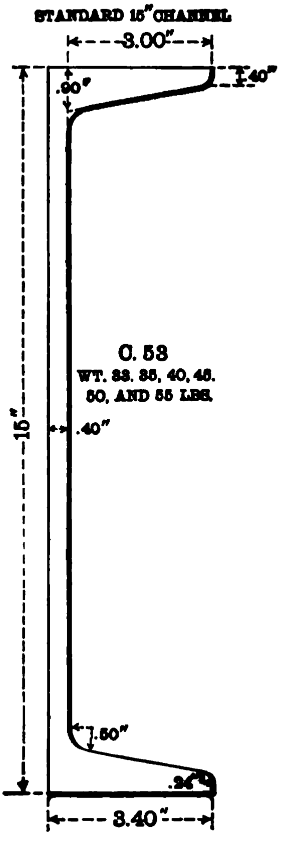
SPECIAL CHANNEL

C. 95

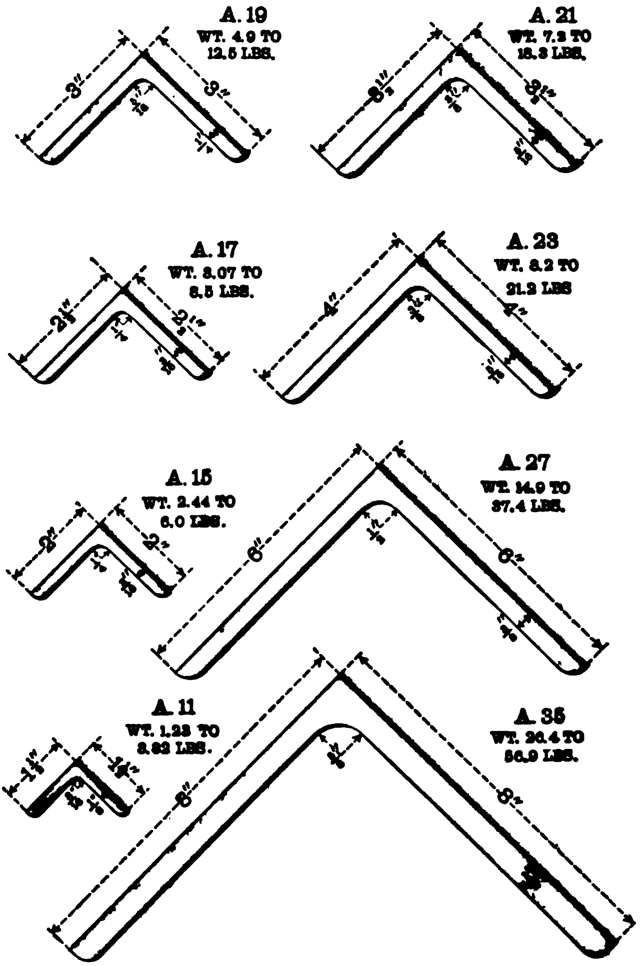
WT. 32, 35, 37, 40, 45, 50 AND 55 LBS.

CHANNELS.


SPECIAL 18" CHANNEL
3.30"
.45"
1"
C. 65
WT. 45, 50, 55 AND 60 LBS.
18"
.47"
.60"
.30"
3.77"


STANDARD 15" CHANNEL
3.00"
.40"
.90"
C. 53
WT. 33, 35, 40, 45, 50, AND 55 LBS.
15"
.40"
.50"
.25"
3.40"

STANDARD ANGLES WITH EQUAL LEGS.



A. 19
WT. 4.9 TO
12.5 LBS.

A. 21
WT. 7.2 TO
18.3 LBS.

A. 17
WT. 3.07 TO
8.5 LBS.

A. 23
WT. 8.2 TO
21.2 LBS

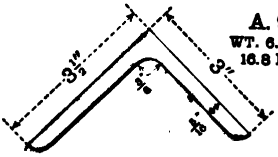
A. 15
WT. 2.44 TO
6.0 LBS.

A. 27
WT. 14.9 TO
37.4 LBS.

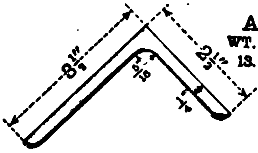
A. 11
WT. 1.23 TO
3.82 LBS.

A. 35
WT. 26.4 TO
56.9 LBS.

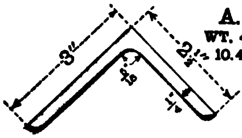
STANDARD ANGLES WITH UNEQUAL LEGS.



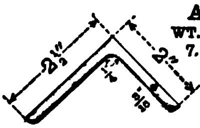
A. 97
WT. 6.6 TO
16.8 LBS.



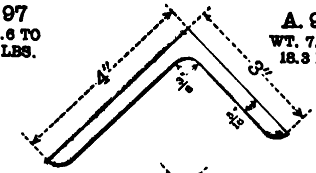
A. 95
WT. 4.9 TO
13.4 LBS.



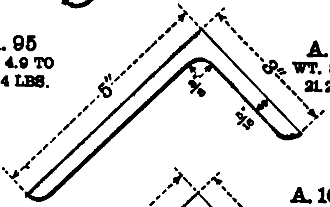
A. 93
WT. 4.5 TO
10.4 LBS.



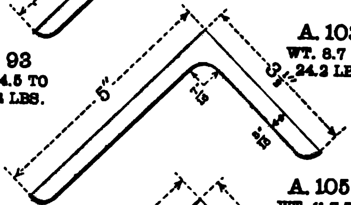
A. 91
WT. 2.75 TO
7.6 LBS.



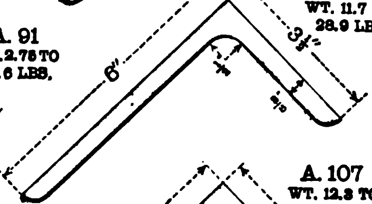
A. 99
WT. 7.2 TO
18.3 LBS.



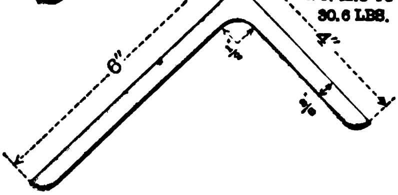
A. 101
WT. 8.2 TO
21.2 LBS.



A. 103
WT. 8.7 TO
24.2 LBS.

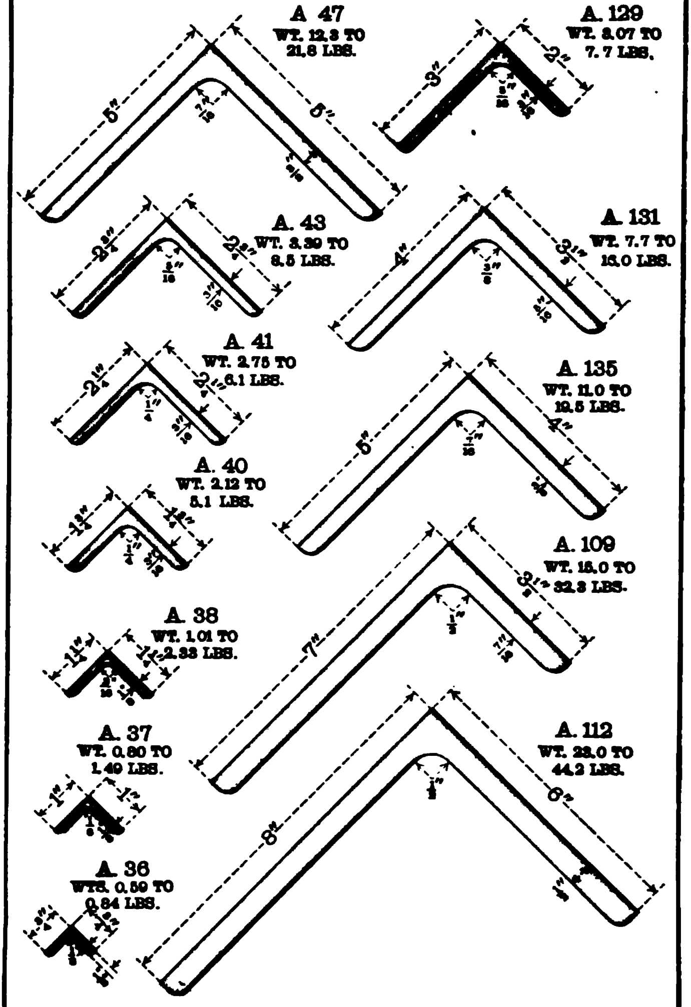


A. 105
WT. 11.7 TO
28.9 LBS.



A. 107
WT. 12.3 TO
30.6 LBS.

SPECIAL ANGLES.



A 47
WT. 12.3 TO 21.8 LBS.

A. 129
WT. 3.07 TO 7.7 LBS.

A. 43
WT. 3.39 TO 8.5 LBS.

A. 131
WT. 7.7 TO 16.0 LBS.

A. 41
WT. 2.75 TO 6.1 LBS.

A. 135
WT. 11.0 TO 19.5 LBS.

A. 40
WT. 2.12 TO 5.1 LBS.

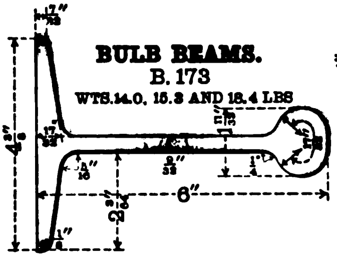
A. 109
WT. 15.0 TO 32.3 LBS.

A. 38
WT. 1.01 TO 2.33 LBS.

A. 37
WT. 0.80 TO 1.49 LBS.

A. 112
WT. 28.0 TO 44.2 LBS.

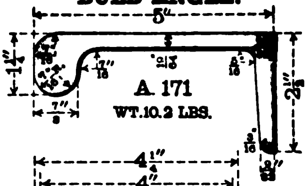
A. 36
WTS. 0.59 TO 0.84 LBS.



BULB BEAMS.

B. 173

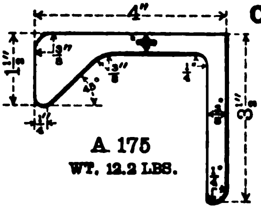
WTS. 14.0, 15.3 AND 18.4 LBS



BULB ANGLE.

A. 171

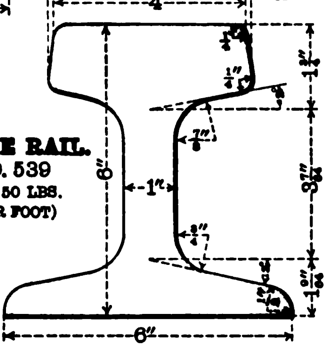
WT. 10.2 LBS.



TOP. GUARD ANGLE.

A. 175

WT. 12.2 LBS.

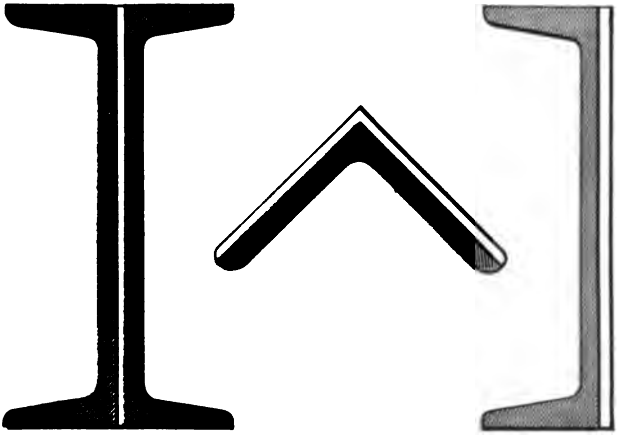


CRANE RAIL.

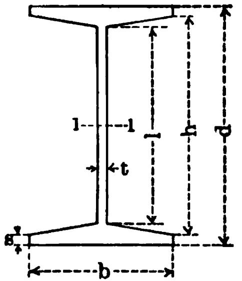
NO. 539

WT. 50 LBS. (PER FOOT)

METHOD OF INCREASING SECTIONAL AREA.



STANDARD BEAMS.



The following Formulas and Diagram relate to the Properties of I-Beams:

Weight per foot = Area × 3.4.

$$\text{Area} = td + 2s(b-t) + \frac{(b-t)^2}{12}.$$

$$\text{Section Modulus} = s = \frac{2I}{d}.$$

$$\text{Slope of Flange} = g = \frac{h-l}{b-t} = \tfrac{1}{6} \text{ for Standard Beams.}$$

I = Moment of Inertia, Neutral Axis (1–1) parallel to flange.

$$I = \tfrac{1}{12}\left[bd^3 - \frac{1}{4g}(h^4-l^4)\right] \text{ or } \frac{bd^3}{12} - \tfrac{1}{8}(h^4-l^4) \text{ for Standard Beams.}$$

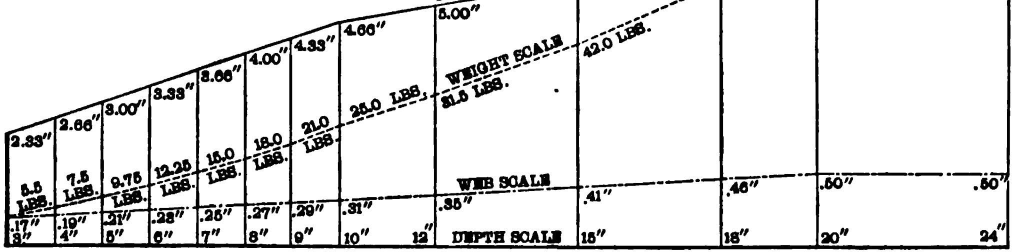
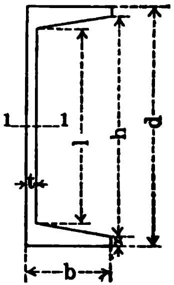


DIAGRAM FOR MINIMUM STANDARD BEAMS.

STANDARD CHANNELS



The following Formulas and Diagram relate to the Properties of Channels:

Weight per foot = Area × 3.4.

$$\text{Area} = td + 2s\,(b-t) + \frac{(b-t)^2}{6}.$$

$$\text{Section Modulus} = s = \frac{2I}{d},$$

$$\text{Slope of Flange} = g = \frac{h-l}{2(b-t)}, \text{ or } \tfrac{1}{6} \text{ for Standard Channels.}$$

I = Moment of Inertia, Neutral Axis (1–1) parallel to flange.

$$I = \tfrac{1}{12}\left[bd^3 - \frac{1}{8g}\,(h^4-l^4)\right] \text{ or } \frac{bd^3}{12} - \frac{h^4-l^4}{16} \text{ for Standard Channels.}$$

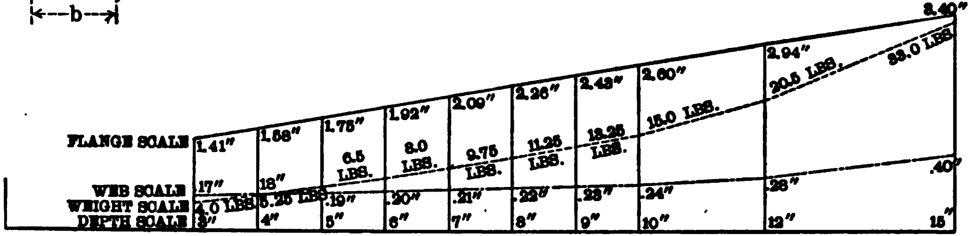
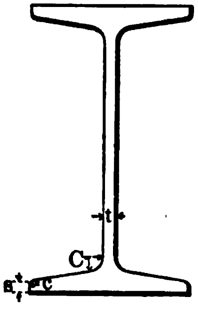
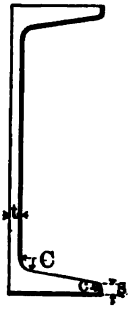


DIAGRAM FOR MINIMUM STARDARD CHANNELS.

STANDARD BEAMS AND CHANNELS.





The following data are common to all Standard I-Beams and Channels, with the exceptions stated:

$c = \frac{6}{10}$ Minimum Web.
$C =$ Minimum Web $+ \frac{1}{10}$ inch.

$s =$ Minimum Thickness of Web $=$ t Minimum for all Channels and Beams, except 20″ I and 24″ I.

For 20″ Standard I, $s = .55''$, t Minimum $= .50''$.
For 24″ Standard I, $s = .60''$, t Minimum $= .50''$.

The Slope of Flange of all Standard Beams and Channels is $16\frac{2}{3}\%$ $= 9^\circ - 27' - 44'' = 2''$ per foot.

TABLES
OF
STEEL PLATES

SHEARED PLATES.

Width in Inches.	THICKNESS IN INCHES.								
	3/16	1/4	5/16	3/8	7/16	1/2	9/16	5/8	11/16
	MAXIMUM LENGTH IN INCHES.								
24– 29	400	525	575	600	600	600	600	600	575
30– 35	375	525	550	600	600	625	625	600	575
36– 41	375	475	525	550	550	575	575	575	575
42– 47	400	525	550	575	600	600	600	575	575
48– 53	400	525	575	600	600	600	600	600	575
54– 59	400	525	550	600	600	625	625	600	575
60– 65	375	525	550	600	600	625	625	600	575
66– 71	350	475	500	575	575	600	600	600	575
72– 77	325	425	450	525	550	575	575	575	575
78– 83		400	425	475	500	525	525	525	525
84– 89		375	400	425	450	475	475	475	475
90– 95		325	350	375	400	425	425	425	425
96–101		300	325	350	375	400	400	400	400
102–107		275	300	325	350	375	375	375	375
108–113		250	275	300	325	350	350	350	350
114–119		175	200	225	250	275	275	275	275
120–125			175	200	225	250	250	250	250
126								175	175
Diam. of Heads.	72	115	117	124	124	127	127	127	127

Minimum Diameter of Heads = 30 inches.

SHEARED PLATES.

THICKNESS IN INCHES.										Width in Inches.
¾	13/16	⅞	15/16	1	1⅛	1¼	1½	1¾	2	
MAXIMUM LENGTH IN INCHES.										
575	550	550	525	525	500	450	425	375	350	24– 29
575	550	500	475	475	450	450	400	375	350	30– 35
550	525	500	475	475	450	425	400	375	350	36– 41
575	525	500	500	500	475	425	400	375	350	42– 47
575	550	550	525	525	500	450	400	375	350	48– 53
575	550	550	525	525	500	450	400	375	350	54– 59
575	550	550	525	525	475	425	400	350	325	60– 65
575	550	550	525	525	475	425	375	350	325	66– 71
575	550	525	500	500	475	425	375	350	300	72– 77
525	500	475	450	450	425	375	325	300	275	78– 83
475	450	450	425	425	375	350	300	275	250	84– 89
425	400	400	375	375	350	325	280	260	250	90– 95
400	375	375	350	325	300	275	260	250	225	96–101
375	350	350	325	300	275	250	240	220	220	102–107
350	325	325	300	275	250	250	225	200	175	108–113
300	275	275	250	250	225	200	175	160	150	114–119
275	250	250	225	225	200	200	175	160	150	120–125
200	200	200	175	175	160	160	150	144	144	126
127	127	127	127	127	127	127	127	127	127	Diam. of Heads.

Minimum Diameter of Heads = 30 inches.

EDGED PLATES.

Width in Inches.	THICKNESS IN INCHES.														
	$\frac{3}{16}$	$\frac{1}{4}$	$\frac{5}{16}$	$\frac{3}{8}$	$\frac{7}{16}$	$\frac{1}{2}$	$\frac{9}{16}$	$\frac{5}{8}$	$\frac{3}{4}$	$\frac{7}{8}$	1	$1\frac{1}{4}$	$1\frac{1}{2}$	$1\frac{3}{4}$	2
	LENGTH IN FEET.														
8–27	75	85	85	85	85	85	85	85	85	85	85	68	56	48	42
28		85	85	85	85	85	85	85	85	85	84	67	56	48	42
29		85	85	85	85	85	85	85	85	85	81	64	54	46	40
30		85	85	85	85	85	85	85	85	85	78	62	52	44	39
31		85	85	85	85	85	85	85	85	85	75	60	50	43	37
32		85	85	85	85	85	85	85	85	84	73	58	49	42	36
33		85	85	85	85	85	85	85	85	81	71	57	47	40	35
34		85	85	85	85	85	85	85	85	79	69	55	46	39	34
35		85	85	85	85	85	85	85	85	76	67	53	44	38	33
36		85	85	85	85	85	85	85	85	74	65	52	43	37	32

THIN SHEARED SHEETS.

Width in Inches.	THICKNESS IN INCHES.												
	.065	.070	.075	.080	.085	.090	.095	.100	.110	.125	.135	.150	.165
	LENGTH IN FEET.												
8–13	20	20	20	24	24	26	26	26	26	26	26	26	26
14–16	20	20	20	20	20	24	26	26	26	26	26	26	26
17–19	18	18	18	20	20	24	26	26	26	26	26	26	26
20–23	16	16	16	18	18	22	24	24	26	26	26	26	26
24–26	14	14	14	16	16	20	22	22	24	24	26	26	26
27–28	14	14	14	16	16	18	20	20	24	24	24	26	26
29–30	12	12	12	14	16	18	18	18	20	20	24	26	26
31–34	10	10	10	14	16	18	18	18	20	20	22	24	24

EXPLANATIONS OF THE TABLES OF PROPERTIES OF STANDARD AND SPECIAL I-BEAMS, STANDARD AND SPECIAL CHANNELS, AND STANDARD AND SPECIAL ANGLES WITH EQUAL AND UNEQUAL LEGS.

PROPERTIES OF I-BEAMS.

PAGES 158 TO 161 INCLUSIVE.

The figures or values in the various columns give the section numbers, dimensions, weights, areas and properties of the sections as noted in the different headings.

The columns which require special explanation are as follows:

SECTION MODULUS—Column 8.

This is obtained from the moment of inertia in column 7 by dividing it by the distance from the neutral axis to the most remote fibre, which in this case is one-half the depth of the beam.

COEFFICIENTS OF STRENGTH—Columns 13 and 14.

The coefficients of strength F and F′ have been computed for fibre stresses of 16 000 and 12 500 pounds per square inch respectively, as stated in the headings of the columns, and are the safe loads in pounds uniformly distributed, including its own weight, for a beam one foot long. Thus the safe load for any span may be obtained by dividing the proper coefficient by the length of the span in feet.

The coefficients of strength were obtained from the following formulæ:

$$F = \tfrac{2}{3} \times 16\,000 \times S$$
$$F' = \tfrac{2}{3} \times 12\,500 \times S$$

in which S is the section modulus.

COEFFICIENTS OF DEFLECTION—Columns 15 and 16.

The Coefficients of Deflection N and N′ for uniform and center loads, respectively, were obtained from the following formulæ:

$$N = \frac{Wl^3}{76.8EI} \qquad N' = \frac{Pl^3}{48EI}$$

in which

P and W = 1 000 pounds.
l = 12 inches.
E = 29 000 000.
I = moment of inertia about axis 1-1.

These coefficients are, therefore, the deflections in inches of a beam one foot long with a load of 1 000 pounds, hence, the deflection of a beam for any load and span may be obtained by multiplying the proper coefficient by the cube of the span in feet, and by the number of 1 000-pound units in the given load.

PROPERTIES OF STANDARD AND SPECIAL CHANNELS.

PAGES 162 TO 165 INCLUSIVE.

The various columns in the Tables of Properties of Standard Channels are similar to those in the Tables of Properties of I-Beams, as explained above, with the addition of column 11, which gives the Section Modulus about an axis through the center of gravity parallel to the web, and column 13, which gives the distance of the center of gravity from the outside of the web.

In this case the Section Modulus $S' = \frac{I'}{b - x}$ the notation being as given at the heads of the columns.

PROPERTIES OF ANGLES.

The values in the Tables of Properties of Standard and Special Angles, with Equal Legs, pages 166 to 171, are these stated in the headings, and those in the Tables of Properties of Standard and Special Angles, with Unequal Legs, on pages 172 to 177, are similar, but with the addition of values for I″, S″ and r″ about the inclined axis 3-3, the position of which, in order to give the minimum values, was determined by the formula on page 142 or the value of the tangent of $2a$. After determining the position of the inclined axis, the properties corresponding thereto were obtained by the formula on page 142.

MOMENTS OF INERTIA OF RECTANGLES.

A Table of Moments of Inertia of Rectangles, about a transverse axis through the center of gravity, is added on pages 178 and 179 for convenience in calculating the Moments of Inertia, Section Moduli, and Radii of Gyration for compound shapes in which plates are used.

GENERAL FORMULÆ FOR PROPERTIES AND FLEXURE.

Formulæ for obtaining the Properties of Standard Sections are given on pages 142 and 143, and for various usual sections on pages 144 to 151 inclusive.

General formulæ for Flexure of Beams, Bending Moments, and Deflections for various cases of loading are given on pages 136 to 141 inclusive.

EXAMPLES OF APPLICATION OF THE TABLES OF PROPERTIES.

EXAMPLE I.

What is the proper size of I-Beam to carry a load of 35 000 pounds concentrated at the center of a span of 25 feet, the fibre stress not to exceed 16 000 pounds per square inch?

In the Tables of Properties of Standard I-Beams, the column headed F gives the coefficient of strength for a uniform load corresponding to a fibre stress of 16 000 pounds per square inch.

The coefficient of strength for a concentrated load at the center is twice that for the same load uniformly distributed, hence the coefficient necessary to meet the conditions is 35 000 × 25 × 2 = 1 750 000. From the Table of Properties of Standard I-Beams, page 161, column 13, the coefficient F for a 24-inch 80-pound beam is found to be 1 855 310. The weight of the beam itself is 80 × 25 = 2000 pounds, which corresponds to a coefficient of 2000 × 25 = 50 000, which deducted from 1 855 310 gives a net coefficient of 1 805 310. A 24-inch beam weighing 80 pounds per foot is, therefore, the proper size.

EXAMPLE II.

What is the deflection of the beam in the preceding example under the given load?

In the Table of Properties of Standard I-Beams, pages 158 to 161 inclusive, the coefficient of deflection for beams with center loads is given in column 16. To obtain the required deflection it is only necessary to multiply the coefficient by the cube of the span and the number of 1 000 pounds units contained in the load.

Thus for the given example the deflection in inches =

$$.0000006 \times 25^3 \times \frac{35\,000}{1\,000} = .328 \text{ inch.}$$

EXAMPLE III.

What is the safe load uniformly distributed that can be placed on an 8-inch standard channel weighing 11.25 pounds per foot, with a clear span of 15 feet for a maximum fibre stress of 12 500 pounds per square inch, the web to be placed vertically?

From the Table of Properties of Standard Channels, page 163, column 16, the coefficient of strength F′ for the given channel under the conditions named, is found to be 67 300. Hence, the total load may be 67 300 ÷ 15 = 4487 pounds, and, as the channel itself weighs 169 pounds, the net superimposed load which it can safely carry under the given conditions is 4318 pounds.

EXAMPLE IV.

What is the fibre stress in a 5″ x 3″ angle weighing 8.2 pounds per foot if loaded at the center with a weight of 1500 pounds, used as a beam with a span of 6 feet, the 5-inch leg to be placed vertically?

The bending moment at the center will be

$$\frac{W_1 l}{4} + \frac{W_2 l}{8} = \frac{1\,500 \times 72}{4} + \frac{8.2 \times 6 \times 72}{8} = 27\,443 \text{ inch pounds.}$$

Referring to the Table of Properties of Standard Angles, Unequal Legs, on page 175, the Section Modulus for this angle, corresponding to the axis 2—2, is found to be 1.89.

The maximum fibre stress is obtained by dividing the bending moment by the section modulus, thus: $\frac{27\,443}{1.89} = 14\,520$, which is the maximum fibre stress in pounds per square inch at the point most remote from the neutral axis, which in this case is the extremity of the longer leg of the angle.

The second term in the above expression for the bending moment is that due to the weight of the angle itself and is inconsiderable, so that in practice it might be neglected for short spans, but should be taken into consideration for the longer ones.

PROPERTIES OF COMPOUND SHAPES.

The moments of inertia, section moduli, and radii of gyration of compound sections used as beams or columns, composed of plates and angles, channels, beams, or any combination of these, may be obtained with the aid of the Tables of Properties as follows:

The first step is to find the center of gravity of the proposed section, which in the case of symmetrical sections is at the center of the figure.

For unsymmetrical sections the position of the center of gravity may be determined by multiplying the areas of the component parts by the distances of their centers of gravity from any convenient line, taken as an axis, and dividing the sum of these products by the sum of the areas, which will give the distance of the center of gravity of the compound section from the assumed axis.

The position of the center of gravity for all sizes of angles and channels, is given in the Tables of Properties for these shapes, and is given for various geometrical sections on pages 144 to 151 inclusive, in connection with their other properties.

After determining the position of the center of gravity of a compound section, as explained above, the moment of inertia about an axis through its center of gravity may be found by taking the sum of the moments of inertia of each component part about an axis through its own center of gravity, parallel to the axis of the compound section, and adding thereto the sum of products obtained by multiplying the area of each component part by the square of the distance of its center of gravity from the axis of the compound section.

Having thus obtained the moment of inertia of the compound section, the section modulus may be obtained by dividing this moment of inertia by the distance from the neutral axis to the most remote extremity of the section.

The square of the radius of gyration for the compound section may be obtained by dividing the moment of inertia by the total area.

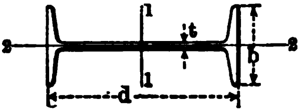
The moment of inertia of a compound section about any axis other than that through its center of gravity may be found in a manner similar to that above described.

TABLES

OF

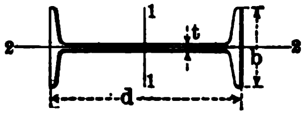
PROPERTIES OF ROLLED AND COMPOUND SHAPES

PROPERTIES OF STANDARD I-BEAMS.



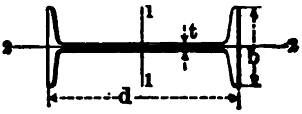
1	2	3	4	5	6	7	8	9	10	11
Section Number.	Depth of Beam.	Weight per Foot.	Area of Section.	Thickness of Web.	Width of Flange.	Moment of Inertia Axis 1-1.	Section Modulus Axis 1-1.	Radius of Gyration Axis 1-1.	Moment of Inertia Axis 2-2.	Radius of Gyration Axis 2-2.
	d		A	t	b	I	S	r	I′	r′
	Inches.	Pounds.	Sq. Ins.	Inch.	Inches.	Inches.⁴	Inches.³	Inches.	Inches.⁴	Inch.
B 5	3	5.50	1.63	.17	2.33	2.5	1.7	1.23	.46	.53
"	"	6.50	1.91	.26	2.42	2.7	1.8	1.19	.53	.52
"	"	7.50	2.21	.36	2.52	2.9	1.9	1.15	.60	.52
B 9	4	7.50	2.21	.19	2.66	6.0	3.0	1.64	.77	.59
"	"	8.50	2.50	.26	2.73	6.4	3.2	1.59	.85	.58
"	"	9.50	2.79	.34	2.81	6.7	3.4	1.54	.93	.58
"	"	10.50	3.09	.41	2.88	7.1	3.6	1.52	1.01	.57
B13	5	9.75	2.87	.21	3.00	12.1	4.8	2.05	1.23	.65
"	"	12.25	3.60	.36	3.15	13.6	5.4	1.94	1.45	.63
"	"	14.75	4.34	.50	3.29	15.1	6.1	1.87	1.70	.63
B17	6	12.25	3.61	.23	3.33	21.8	7.3	2.46	1.85	.72
"	"	14.75	4.34	.35	3.45	24.0	8.0	2.35	2.09	.69
"	"	17.25	5.07	.47	3.57	26.2	8.7	2.27	2.36	.68
B21	7	15.00	4.42	.25	3.66	36.2	10.4	2.86	2.67	.78
"	"	17.50	5.15	.35	3.76	39.2	11.2	2.76	2.94	.76
"	"	20.00	5.88	.46	3.87	42.2	12.1	2.68	3.24	.74
B25	8	18.00	5.33	.27	4.00	56.9	14.2	3.27	3.78	.84
"	"	20.25	5.96	.35	4.08	60.2	15.0	3.18	4.04	.82
"	"	22.75	6.69	.44	4.17	64.1	16.0	3.10	4.36	.81
"	"	25.25	7.43	.53	4.26	68.0	17.0	3.03	4.71	.80
B29	9	21.00	6.31	.29	4.33	84.9	18.9	3.67	5.16	.90
"	"	25.00	7.35	.41	4.45	91.9	20.4	3.54	5.65	.88
"	"	30.00	8.82	.57	4.61	101.9	22.6	3.40	6.42	.85
"	"	35.00	10.29	.73	4.77	111.8	24.8	3.30	7.31	.84
B33	10	25.00	7.37	.31	4.66	122.1	24.4	4.07	6.89	.97
"	"	30.00	8.82	.45	4.80	134.2	26.8	3.90	7.65	.93
"	"	35.00	10.29	.60	4.95	146.4	29.3	3.77	8.52	.91
"	"	40.00	11.76	.75	5.10	158.7	31.7	3.67	9.50	.90
B41	12	31.50	9.26	.35	5.00	215.8	36.0	4.83	9.50	1.01
"	"	35.00	10.29	.44	5.09	228.3	38.0	4.71	10.07	.99
"	"	40.00	11.76	.56	5.21	245.9	41.0	4.57	10.95	.96
B53	15	42.00	12.48	.41	5.50	441.8	58.9	5.95	14.62	1.08
"	"	45.00	13.24	.46	5.55	455.8	60.8	5.87	15.09	1.07
"	"	50.00	14.71	.56	5.65	483.4	64.5	5.73	16.04	1.04
"	"	55.00	16.18	.66	5.75	511.0	68.1	5.62	17.06	1.03
"	"	60.00	17.65	.76	5.84	538.6	71.8	5.52	18.17	1.01

PROPERTIES OF STANDARD I-BEAMS.



12	13	14	15	16	1
Increase of Thickness of Web for each Pound Increase in Weight.	Coefficient of Strength.		Coefficient of Deflection.		Section Number.
	For Fibre Stress of 16 000 Pounds per Square Inch for Buildings.	For Fibre Stress of 12500 Pounds per Square Inch for Bridges.	Uniform Load.	Center Load.	
f	F	F'	N	N'	
.098	17650	13790	.00031253	.00050006	B 5
	19140	14950	.00028827	.00046124	"
	20710	16180	.00026644	.00042630	"
.074	31810	24850	.00013009	.00020815	B 9
	33890	26480	.00012209	.00019535	"
	35980	28110	.00011500	.00018400	"
	38070	29750	.00010868	.00017389	"
.059	51590	40300	.00006417	.00010267	B13
	58100	45390	.00005698	.00009117	"
	64630	50490	.00005122	.00008195	"
.049	77460	60520	.00003561	.00005698	B17
	85270	66610	.00003235	.00005177	"
	93110	72740	.00002963	.00004741	"
.042	110410	86260	.00002142	.00003427	B21
	119400	93290	.00001980	.00003168	"
	128560	100430	.00001839	.00002943	"
.037	151660	118490	.00001364	.00002183	B25
	160510	125400	.00001289	.00002062	"
	170970	133570	.00001210	.00001936	"
	181430	141740	.00001140	.00001825	"
.033	201300	157260	.00000914	.00001462	B29
	217930	170260	.00000844	.00001350	"
	241460	188640	.00000762	.00001219	"
	264990	207020	.00000694	.00001110	"
.029	260470	203500	.00000635	.00001017	B33
	286250	223630	.00000578	.00000925	"
	312390	244050	.00000530	.00000848	"
	338530	264480	.00000489	.00000782	"
.025	383670	299740	.00000360	.00000575	B41
	405800	317030	.00000340	.00000544	"
	437170	341540	.00000316	.00000505	"
.020	628270	490840	.00000176	.00000281	B53
	648310	506490	.00000170	.00000272	"
	687530	537130	.00000161	.00000257	"
	726740	567770	.00000152	.00000243	"
	765960	598410	.00000144	.00000231	"

PROPERTIES OF STANDARD I-BEAMS.

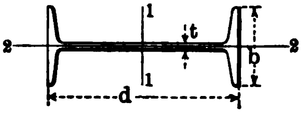


1	2	3	4	5	6	7	8	9	10	11
Section Number.	Depth of Beam.	Weight per Foot.	Area of Section.	Thickness of Web.	Width of Flange.	Moment of Inertia Axis 1-1.	Section Modulus Axis 1-1.	Radius of Gyration Axis 1-1.	Moment of Inertia Axis 2-2.	Radius of Gyration Axis 2-2.
	d		A	t	b	I	S	r	I′	r′
	Inches.	Pounds.	Sq. Ins.	Inch.	Inches.	Inches.⁴	Inches.³	Inches.	Inches.⁴	Inch.
B 65	18	55.0	15.93	.46	6.00	795.6	88.4	7.07	21.19	1.15
"	"	60.0	17.65	.56	6.10	841.8	93.5	6.91	22.38	1.13
"	"	65.0	19.12	.64	6.18	881.5	97.9	6.79	23.47	1.11
"	"	70.0	20.59	.72	6.26	921.2	102.4	6.69	24.62	1.09
B 73	20	65.0	19.08	.50	6.25	1169.5	117.0	7.83	27.86	1.21
"	"	70.0	20.59	.58	6.33	1219.8	122.0	7.70	29.04	1.19
"	"	75.0	22.06	.65	6.40	1268.8	126.9	7.58	30.25	1.17
B 89	24	80.0	23.32	.50	7.00	2087.2	173.9	9.46	42.86	1.36
"	"	85.0	25.00	.57	7.07	2167.8	180.7	9.31	44.35	1.33
"	"	90.0	26.47	.63	7.13	2238.4	186.5	9.20	45.70	1.31
"	"	95.0	27.94	.69	7.19	2309.0	192.4	9.09	47.10	1.30
"	"	100.0	29.41	.75	7.25	2379.6	198.3	8.99	48.55	1.28

PROPERTIES OF SPECIAL I-BEAMS.

B105	12	40.0	11.84	.46	5.25	268.9	44.8	4.77	13.81	1.08
"	"	45.0	13.24	.58	5.37	285.7	47.6	4.65	14.89	1.06
"	"	50.0	14.71	.70	5.49	303.4	50.6	4.54	16.12	1.05
"	"	55.0	16.18	.82	5.61	321.0	53.5	4.45	17.46	1.04
B109	15	60.0	17.67	.59	6.00	609.0	81.2	5.87	25.96	1.21
"	"	65.0	19.12	.69	6.10	636.1	84.8	5.77	27.42	1.20
"	"	70.0	20.59	.78	6.19	663.7	88.5	5.68	29.00	1.19
"	"	75.0	22.06	.88	6.29	691.2	92.2	5.60	30.68	1.18
"	"	80.0	23.53	.98	6.39	718.8	95.8	5.53	32.46	1.17
B113	15	80.0	23.57	.80	6.40	789.1	105.2	5.79	41.31	1.32
"	"	85.0	25.00	.90	6.50	815.9	108.8	5.71	43.46	1.32
"	"	90.0	26.47	.99	6.59	843.4	112.5	5.64	45.79	1.32
"	"	95.0	27.94	1.09	6.69	871.0	116.1	5.58	48.25	1.31
"	"	100.0	29.41	1.19	6.79	898.6	119.8	5.53	50.84	1.31
B121	20	80.0	23.73	.60	7.00	1466.3	146.6	7.86	45.81	1.39
"	"	85.0	25.00	.66	7.06	1508.5	150.9	7.77	47.25	1.37
"	"	90.0	26.47	.74	7.14	1557.5	155.8	7.67	48.98	1.36
"	"	95.0	27.94	.81	7.21	1606.6	160.7	7.58	50.78	1.35
"	"	100.0	29.41	.88	7.28	1655.6	165.6	7.50	52.65	1.34

PROPERTIES OF STANDARD I-BEAMS.

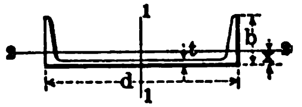


12	13	14	15	16	1
Increase of Thickness of Web for each Pound Increase in Weight.	Coefficient of Strength. For Fibre Stress of 16 000 Pounds per Square Inch for Buildings.	Coefficient of Strength. For Fibre Stress of 12 500 Pounds per Square Inch for Bridges.	Coefficient of Deflection. Uniform Load.	Coefficient of Deflection. Center Load.	Section Number.
f	F	F′	N	N′	
.016	942880	736620	.00000098	.00000156	B 65
	997680	779440	.00000092	.00000148	“
	1044740	816200	.00000088	.00000141	“
	1091800	852970	.00000084	.00000135	“
.015	1247490	974600	.00000066	.00000106	B 78
	1301110	1016490	.00000064	.00000102	“
	1353400	1057340	.00000061	.00000098	“
.0123	1855310	1449460	.00000037	.00000060	B 89
	1926950	1505430	.00000036	.00000057	“
	1989700	1554450	.00000035	.00000056	“
	2052440	1603470	.00000034	.00000054	“
	2115190	1652490	.00000033	.00000052	“

PROPERTIES OF SPECIAL I-BEAMS.

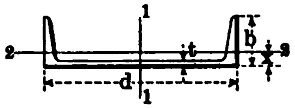
.025	478130	373540	.00000288	.00000462	B105
	507930	396820	.00000272	.00000435	“
	539300	421320	.00000256	.00000409	“
	570670	445830	.00000242	.00000387	“
.020	866130	676670	.00000127	.00000204	B109
	904660	706770	.00000122	.00000195	“
	943870	737400	.00000117	.00000187	“
	983090	768040	.00000112	.00000180	“
	1022300	798670	.00000108	.00000173	“
.020	1122290	876790	.00000098	.00000157	B113
	1160340	906520	.00000095	.00000152	“
	1199550	937150	.00000092	.00000147	“
	1238770	967790	.00000089	.00000143	“
	1277980	998420	.00000086	.00000138	“
.015	1564060	1221920	.00000053	.00000085	B121
	1609100	1257110	.00000051	.00000082	“
	1661390	1297960	.00000050	.00000080	“
	1713670	1338810	.00000048	.00000077	“
	1765960	1379660	.00000047	.00000075	“

PROPERTIES OF STANDARD CHANNELS.



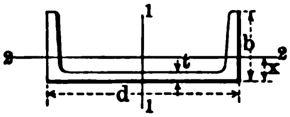
1	2	3	4	5	6	7	8	9	10	11	12
Section Number.	Depth of Channel.	Weight per Foot.	Area of Section.	Thickness of Web.	Width of Flange.	Moment of Inertia Axis 1-1.	Section Modulus Axis 1-1.	Radius of Gyration Axis 1-1.	Moment of Inertia Axis 2-2.	Section Modulus Axis 2-2.	Radius of Gyration Axis 2-2.
	d		A	t	b	I	S	r	I′	S′	r′
	Inches.	Pounds.	Sq. Ins.	Inch.	Inches.	Inches.⁴	Ins.³	Inches.	Inches.⁴	Ins.³	Inch.
C 5	3	4.00	1.19	.17	1.41	1.6	1.1	1.17	.20	.21	.41
“	“	5.00	1.47	.26	1.50	1.8	1.2	1.12	.25	.24	.41
“	“	6.00	1.76	.36	1.60	2.1	1.4	1.08	.31	.27	.42
C 9	4	5.25	1.55	.18	1.58	3.8	1.9	1.56	.32	.29	.45
“	“	6.25	1.84	.25	1.65	4.2	2.1	1.51	.38	.32	.45
“	“	7.25	2.13	.33	1.73	4.6	2.3	1.46	.44	.35	.46
C13	5	6.50	1.95	.19	1.75	7.4	3.0	1.95	.48	.38	.50
“	“	9.00	2.65	.33	1.89	8.9	3.5	1.83	.64	.45	.49
“	“	11.50	3.38	.48	2.04	10.4	4.2	1.75	.82	.54	.49
C17	6	8.00	2.38	.20	1.92	13.0	4.3	2.34	.70	.50	.54
“	“	10.50	3.09	.32	2.04	15.1	5.0	2.21	.88	.57	.53
“	“	13.00	3.82	.44	2.16	17.3	5.8	2.13	1.07	.65	.53
“	“	15.50	4.56	.56	2.28	19.5	6.5	2.07	1.28	.74	.53
C21	7	9.75	2.85	.21	2.09	21.1	6.0	2.72	.98	.63	.59
“	“	12.25	3.60	.32	2.20	24.2	6.9	2.59	1.19	.71	.57
“	“	14.75	4.34	.42	2.30	27.2	7.8	2.50	1.40	.79	.57
“	“	17.25	5.07	.53	2.41	30.2	8.6	2.44	1.62	.87	.56
“	“	19.75	5.81	.63	2.51	33.2	9.5	2.39	1.85	.96	.56
C25	8	11.25	3.35	.22	2.26	32.3	8.1	3.10	1.33	.79	.63
“	“	13.75	4.04	.31	2.35	36.0	9.0	2.98	1.55	.87	.62
“	“	16.25	4.78	.40	2.44	39.9	10.0	2.89	1.78	.95	.61
“	“	18.75	5.51	.49	2.53	43.8	11.0	2.82	2.01	1.02	.60
“	“	21.25	6.25	.58	2.62	47.8	11.9	2.76	2.25	1.11	.60
C29	9	13.25	3.89	.23	2.43	47.3	10.5	3.49	1.77	.97	.67
“	“	15.00	4.41	.29	2.49	50.9	11.3	3.40	1.95	1.03	.66
“	“	20.00	5.88	.45	2.65	60.8	13.5	3.21	2.45	1.19	.65
“	“	25.00	7.35	.61	2.81	70.7	15.7	3.10	2.98	1.36	.64
C33	10	15.00	4.46	.24	2.60	66.9	13.4	3.87	2.30	1.17	.72
“	“	20.00	5.88	.38	2.74	78.7	15.7	3.66	2.85	1.34	.70
“	“	25.00	7.35	.53	2.89	91.0	18.2	3.52	3.40	1.50	.68
“	“	30.00	8.82	.68	3.04	103.2	20.6	3.42	3.99	1.67	.67
“	“	35.00	10.29	.82	3.18	115.5	23.1	3.35	4.66	1.87	.67
C41	12	20.50	6.03	.28	2.94	128.1	21.4	4.61	3.91	1.75	.81
“	“	25.00	7.35	.39	3.05	144.0	24.0	4.43	4.53	1.91	.78
“	“	30.00	8.82	.51	3.17	161.6	26.9	4.28	5.21	2.09	.77
“	“	35.00	10.29	.64	3.30	179.3	29.9	4.17	5.90	2.27	.76
“	“	40.00	11.76	.76	3.42	196.9	32.8	4.09	6.63	2.46	.75
C53	15	33.00	9.90	.40	3.40	312.6	41.7	5.62	8.23	3.16	.91
“	“	35.00	10.29	.43	3.43	319.9	42.7	5.57	8.48	3.22	.91
“	“	40.00	11.76	.52	3.52	347.5	46.3	5.44	9.39	3.43	.89
“	“	45.00	13.24	.62	3.62	375.1	50.0	5.32	10.29	3.63	.88
“	“	50.00	14.71	.72	3.72	402.7	53.7	5.23	11.22	3.85	.87
“	“	55.00	16.18	.82	3.82	430.2	57.4	5.16	12.19	4.07	.87

PROPERTIES OF STANDARD CHANNELS.



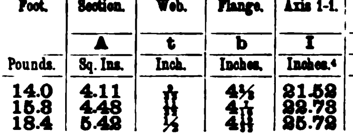
13	14	15	16	17	18	1
Distance of Center of Gravity from Outside of Web.	Increase of Thickness of Web for each Pound Increase in Weight.	Coef. of Strength.		Coef. of Deflection.		Section Number.
		Fibre Stress 16000 Pounds per Sq. Inch for Buildings.	Fibre Stress 12500 Pounds per Sq. Inch for Bridges.	Uniform Load.	Center Load.	
x	f	F	F'	N	N'	
Inch.	Inches.					
.44	.098	11630	9090	.0004743	.0007589	C 5
.44		13140	10270	.0004199	.0006718	"
.46		14710	11490	.0003751	.0006001	"
.46	.074	20230	15800	.0002046	.0003273	C 9
.46		22270	17400	.0001858	.0002973	"
.46		24360	19030	.0001698	.0002717	"
.49	.059	31640	24720	.0001046	.0001674	C13
.48		37860	29570	.0000875	.0001399	"
.51		44390	34680	.0000746	.0001193	"
.52	.049	46210	36100	.0000597	.0000955	C17
.50		53750	42000	.0000513	.0000821	"
.52		61600	48120	.0000448	.0000717	"
.55		69440	54250	.0000397	.0000636	"
.55	.042	64270	50210	.0000368	.0000588	C21
.53		73650	57540	.0000321	.0000514	"
.53		82740	64690	.0000286	.0000457	"
.55		91950	71840	.0000257	.0000411	"
.58		101100	78990	.0000234	.0000374	"
.58	.037	86140	67300	.0000240	.0000384	C25
.56		95990	75000	.0000216	.0000345	"
.56		106450	83170	.0000194	.0000311	"
.57		116910	91340	.0000177	.0000283	"
.59		127370	99510	.0000162	.0000260	"
.61	.033	112170	87630	.0000164	.0000262	C29
.59		120540	94170	.0000153	.0000244	"
.58		144070	112550	.0000128	.0000204	"
.62		167590	130930	.0000110	.0000176	"
.64	.029	142680	111470	.0000116	.0000186	C33
.61		167940	131210	.0000099	.0000158	"
.62		194090	151630	.0000085	.0000136	"
.65		220230	172060	.0000075	.0000120	"
.69		246380	192480	.0000067	.0000107	"
.70	.025	227750	177930	.0000061	.0000097	C41
.68		256000	200000	.0000054	.0000086	"
.68		287370	224510	.0000048	.0000077	"
.69		318750	249020	.0000043	.0000069	"
.72		350120	273530	.0000039	.0000063	"
.79	.020	444520	347280	.0000025	.0000040	C53
.79		455080	355500	.0000024	.0000039	"
.78		494250	386130	.0000022	.0000036	"
.79		533470	416770	.0000021	.0000033	"
.80		572680	447410	.0000019	.0000031	"
.82		611900	478050	.0000018	.0000029	"

PROPERTIES OF SPECIAL CHANNELS.



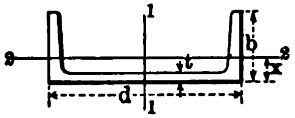
1	2	3	4	5	6	7	8	9	10	11	12	13
Section Number.	Depth of Chan'l.	W'ght per Foot.	Area of Section.	Thickness of Web.	Width of Flange.	Thickness of Flange.	Slope of Flange.	Moment of Inertia Axis 1-1.	Section Modulus Axis 1-1.	Radius of Gyration Axis 1-1.	Moment of Inertia Axis 2-2.	Section Modulus Axis 2-2.
	d		A	t	b	s	g	I	S	r	I'	S'
	Ins.	Lbs.	Sq. Ins.	Inch.	Inches.	Inch.		Inches.⁴	Ins.³	Inches.	Inches.⁴	Ins.³
C 86	6	15.2	4.46	.35	3.50	.34	.02	25.0	8.3	2.37	5.19	2.14
C 88	6	19.0	5.58	.41	3.56	.46	.02	31.1	10.4	2.36	6.79	2.85
"	6	21.6	6.36	.54	3.69	"	"	33.4	11.1	2.29	7.85	3.10
C 89	7	20.9	6.15	.45	3.45	.48	.02	44.6	12.7	2.69	6.74	2.81
C101	8	21.5	6.30	.40	3.50	.48	.02	60.7	15.2	3.01	7.20	2.94
C103	8	23.8	7.00	.50	3.50	.48	.02	63.6	15.7	3.01	7.42	2.96
C 90	10	21.7	6.38	.38	3.38	.41	.02	91.3	18.3	3.78	6.22	2.48
C 92	10	27.2	8.00	.54	3.50	.41	.02	104.6	20.9	3.62	7.20	2.70
C 95	13	32.0	9.30	.38	4.00	.34	.15	237.5	36.5	5.05	11.54	3.86
"	"	35.0	10.29	.45	4.08	"	"	251.5	38.7	4.94	12.54	4.06
"	"	37.0	10.88	.50	4.12	"	"	259.8	40.0	4.89	13.10	4.17
"	"	40.0	11.76	.56	4.19	"	"	272.2	41.9	4.81	13.94	4.33
"	"	45.0	13.24	.68	4.30	"	"	292.9	45.1	4.70	15.32	4.59
"	"	50.0	14.71	.79	4.42	"	"	313.7	48.3	4.62	16.71	4.86
"	"	55.0	16.18	.90	4.53	"	"	334.4	51.4	4.55	18.14	5.14
C 65	18	45.0	13.25	.47	3.77	.45	.17	584.3	64.9	6.64	12.89	4.40
"	"	50.0	14.71	.55	3.85	"	"	623.1	69.2	6.51	13.90	4.61
"	"	55.0	16.18	.63	3.93	"	"	662.0	73.6	6.40	14.93	4.82
"	"	60.0	17.65	.72	4.02	"	"	703.3	78.1	6.31	15.96	5.03

PROPERTIES OF BULB BEAMS.



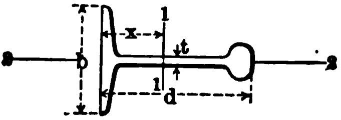
1	2	3	4	5	6	7	8	9
Section Number.	Depth of Beam.	Weight per Foot.	Area of Section.	Thickness of Web.	Width of Flange.	Moment of Inertia Axis 1-1.	Section Modulus Axis 1-1.	Radius of Gyration Axis 1-1.
	d		A	t	b	I	S	r
	Inches.	Pounds.	Sq. Ins.	Inch.	Inches.	Inches.⁴	Inches.³	Inches.
B178	6	14.0	4.11	$\frac{5}{16}$	$4\frac{1}{8}$	21.52	6.12	2.29
"	"	15.8	4.48	$\frac{3}{8}$	$4\frac{3}{16}$	22.78	6.55	2.25
"	"	18.4	5.42	$\frac{1}{2}$	$4\frac{5}{16}$	25.72	7.59	2.18

PROPERTIES OF SPECIAL CHANNELS.



14	15	16	17	18	19	20	1
Radius of Gyration Axis 2-2.	Distance of Center of Gravity from Outside of Web.	Increase of Thickness of Web for each Lb. Increase in Weight.	Coef. of Strength. Fibre Stress 16 000 Lbs. per Sq. Inch for Buildings.	Coef. of Strength. Fibre Stress 12 500 Lbs. per Sq. Inch for Bridges.	Coef. of Deflection. Uniform Load.	Coef. of Deflection. Center Load.	Section Number.
r′ Inch.	x Inch.	f Inch.	F	F′	N	N′	
1.03	1.03	.049	88920	69470	.0000310	.0000496	C 86
1.10	1.18	.049	110450	86290	.0000250	.0000400	C 88
1.11	1.16	"	118770	92790	.0000232	.0000372	"
1.05	1.05	.042	135950	106210	.0000174	.0000278	C 89
1.07	1.05	.037	161930	126510	.0000128	.0000204	C101
1.03	.99	.037	167470	130830	.0000122	.0000195	C103
.99	.87	.029	194750	152150	.0000085	.0000136	C 90
.95	.83	.029	222930	174170	.0000074	.0000119	C 92
1.11	1.01	.023	389710	304460	.0000033	.0000052	C 95
1.10	.99	"	412750	322460	.0000031	.0000049	"
1.10	.98	"	426340	333080	.0000030	.0000048	"
1.09	.97	"	446740	349010	.0000029	.0000046	"
1.08	.97	"	480720	375560	.0000027	.0000043	"
1.07	.98	"	514710	402120	.0000025	.0000040	"
1.06	1.00	"	548700	428670	.0000023	.0000037	"
.99	.84	.016	692270	540830	.0000014	.0000022	C 65
.97	.83	"	738520	576970	.0000012	.0000020	"
.96	.83	"	784600	612970	.0000012	.0000019	"
.95	.85	"	833560	651220	.0000011	.0000018	"

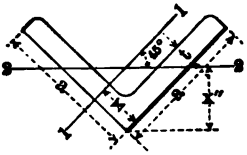
PROPERTIES OF BULB BEAMS.



10	11	12	13	14	15	1
Distance of Center of Gravity from Outside of Flange.	Increase of Thickness of Web for each Lb. Increase in Weight.	Coef. of Strength. Fibre Stress 16 000 Pounds per Sq. Inch for Buildings.	Coef. of Strength. Fibre Stress 12 500 Pounds per Sq. Inch for Bridges.	Coef. of Deflection. Uniform Load.	Coef. of Deflection. Center Load.	Section Number.
x Inches.	f	F	F′	N	N′	
2.49	.049	65380	51080	.0000361	.0000577	B178
2.58	"	69860	54580	.0000341	.0000546	"
2.61	"	80930	63230	.0000302	.0000483	"

PROPERTIES OF STANDARD ANGLES.

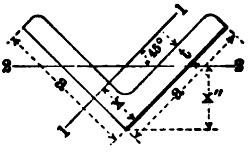
EQUAL LEGS.



1	2	3	4	5	6	7	8
Section Number.	Dimensions.	Thickness.	Weight per Foot.	Area of Section.	Distance of Center of Gravity from Back of Leg.	Moment of Inertia Axis 1-1.	Section Modulus Axis 1-1.
	a x a	t		A	x	I	S
	Inches.	Inch.	Pounds.	Sq. Ins.	Inch.	Inches.⁴	Inches.³
A11	1½ x 1½	⅛	1.23	.36	.42	.08	.072
"	"	3/16	1.80	.53	.44	.11	.104
"	"	¼	2.34	.69	.47	.14	.134
"	"	5/16	2.86	.84	.49	.16	.162
"	"	⅜	3.35	.98	.51	.19	.188
"	"	7/16	3.82	1.12	.53	.21	.214
A15	2 x 2	3/16	2.44	.72	.57	.27	.19
"	"	¼	3.19	.94	.59	.35	.25
"	"	5/16	3.92	1.15	.61	.42	.30
"	"	⅜	4.7	1.36	.64	.48	.35
"	"	7/16	5.3	1.56	.66	.54	.40
"	"	½	6.0	1.75	.68	.59	.45
A17	2½ x 2½	3/16	3.07	.90	.69	.55	.30
"	"	¼	4.1	1.19	.72	.70	.39
"	"	5/16	5.0	1.47	.74	.85	.48
"	"	⅜	5.9	1.73	.76	.98	.57
"	"	7/16	6.8	2.00	.78	1.11	.65
"	"	½	7.7	2.25	.81	1.23	.73
"	"	9/16	8.5	2.50	.83	1.34	.80
A19	3 x 3	¼	4.9	1.44	.84	1.24	.58
"	"	5/16	6.1	1.78	.87	1.51	.71
"	"	⅜	7.2	2.11	.89	1.76	.83
"	"	7/16	8.3	2.43	.91	1.99	.95
"	"	½	9.4	2.75	.93	2.22	1.07
"	"	9/16	10.4	3.06	.95	2.43	1.19
"	"	⅝	11.5	3.36	.98	2.62	1.30
"	"	11/16	12.5	3.65	1.00	2.81	1.40

PROPERTIES OF STANDARD ANGLES.

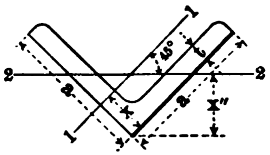
EQUAL LEGS.



9	10	11	12	13	1
Radius of Gyration Axis 1-1.	Distance of Center of Gravity from External Apex.	Least Moment of Inertia Axis 2-2.	Section Modulus Axis 2-2.	Least Radius of Gyration Axis 2-2.	Section Number.
r	x''	I''	S''	r''	
Inch.	Inches.	Inches.^{4}	Inches.^{3}	Inch.	
.47	.60	.031	.053	.30	A11
.46	.63	.045	.072	.29	"
.45	.66	.058	.088	.29	"
.44	.69	.070	.101	.29	"
.44	.72	.082	.114	.29	"
.43	.75	.094	.126	.29	"
.62	.80	.11	.14	.39	A15
.61	.84	.14	.17	.39	"
.60	.87	.17	.20	.39	"
.59	.90	.20	.22	.39	"
.59	.93	.23	.25	.38	"
.58	.96	.26	.27	.38	"
.78	.98	.22	.22	.49	A17
.77	1.01	.29	.28	.49	"
.76	1.05	.35	.33	.49	"
.75	1.08	.41	.38	.48	"
.75	1.11	.46	.42	.48	"
.74	1.14	.52	.46	.48	"
.73	1.17	.58	.49	.48	"
.93	1.19	.50	.42	.59	A19
.92	1.22	.61	.50	.59	"
.91	1.26	.72	.57	.58	"
.91	1.29	.82	.64	.58	"
.90	1.32	.92	.70	.58	"
.89	1.35	1.02	.76	.58	"
.88	1.38	1.12	.81	.58	"
.88	1.41	1.22	.86	.58	"

PROPERTIES OF STANDARD ANGLES.

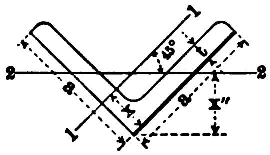
EQUAL LEGS.



1	2	3	4	5	6	7	8
Section Number.	Dimensions.	Thickness.	Weight per Foot.	Area of Section.	Distance of Center of Gravity from Back of Leg.	Moment of Inertia Axis 1-1.	Section Modulus Axis 1-1.
	a x a	t		A	x	I	S
	Inches.	Inch.	Pounds.	Sq. Ins.	Inches.	Inches.⁴	Inches.³
A21	3½ x 3½	5/16	7.2	2.09	.99	2.45	.98
"	"	3/8	8.5	2.48	1.01	2.87	1.15
"	"	7/16	9.8	2.87	1.04	3.26	1.32
"	"	1/2	11.1	3.25	1.06	3.64	1.49
"	"	9/16	12.4	3.62	1.08	3.99	1.65
"	"	5/8	13.6	3.98	1.10	4.33	1.81
"	"	11/16	14.8	4.34	1.12	4.65	1.96
"	"	3/4	16.0	4.69	1.15	4.96	2.11
"	"	13/16	17.1	5.03	1.17	5.25	2.25
"	"	7/8	18.3	5.36	1.19	5.53	2.39
A23	4 x 4	5/16	8.2	2.40	1.12	3.71	1.29
"	"	3/8	9.8	2.86	1.14	4.36	1.52
"	"	7/16	11.3	3.31	1.16	4.97	1.75
"	"	1/2	12.8	3.75	1.18	5.56	1.97
"	"	9/16	14.3	4.18	1.21	6.12	2.19
"	"	5/8	15.7	4.61	1.23	6.66	2.40
"	"	11/16	17.1	5.03	1.25	7.17	2.61
"	"	3/4	18.5	5.44	1.27	7.66	2.81
"	"	13/16	19.9	5.84	1.29	8.14	3.01
"	"	7/8	21.2	6.23	1.31	8.59	3.20
A27	6 x 6	3/8	14.9	4.36	1.64	15.39	3.53
"	"	7/16	17.2	5.06	1.66	17.68	4.07
"	"	1/2	19.6	5.75	1.68	19.91	4.61
"	"	9/16	21.9	6.43	1.71	22.07	5.14
"	"	5/8	24.2	7.11	1.73	24.16	5.66
"	"	11/16	26.5	7.78	1.75	26.19	6.17
"	"	3/4	28.7	8.44	1.78	28.15	6.66
"	"	13/16	31.0	9.09	1.80	30.06	7.15
"	"	7/8	33.1	9.73	1.82	31.92	7.63
"	"	15/16	35.3	10.37	1.84	33.72	8.11
"	"	1	37.4	11.00	1.86	35.46	8.57
A35	8 x 8	1/2	26.4	7.75	2.19	48.65	8.37
"	"	9/16	29.6	8.68	2.21	54.09	9.34
"	"	5/8	32.7	9.61	2.23	59.43	10.30
"	"	11/16	35.8	10.53	2.25	64.64	11.25
"	"	3/4	38.9	11.44	2.28	69.74	12.18
"	"	13/16	42.0	12.34	2.30	74.72	13.11
"	"	7/8	45.0	13.23	2.32	79.58	14.02
"	"	15/16	48.1	14.12	2.34	84.34	14.91
"	"	1	51.0	15.00	2.37	88.98	15.80
"	"	1 1/16	54.0	15.87	2.39	93.53	16.67
"	"	1 1/8	56.9	16.73	2.41	97.97	17.53

PROPERTIES OF STANDARD ANGLES.

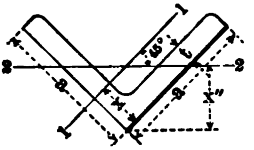
EQUAL LEGS.



9	10	11	12	13	1
Radius of Gyration Axis 1-1.	Distance of Center of Gravity from External Apex.	Least Moment of Inertia Axis 2-2.	Section Modulus Axis 2-2.	Least Radius of Gyration Axis 2-2.	Section Number.
r	x''	I''	S''	r''	
Inches.	Inches.	Inches.⁴	Inches.³	Inch.	
1.08	1.40	.99	.71	.69	A21
1.07	1.43	1.16	.81	.68	"
1.07	1.46	1.33	.91	.68	"
1.06	1.50	1.50	1.00	.68	"
1.05	1.53	1.66	1.09	.68	"
1.04	1.56	1.82	1.17	.68	"
1.04	1.59	1.97	1.24	.67	"
1.03	1.62	2.13	1.31	.67	"
1.02	1.65	2.28	1.38	.67	"
1.02	1.68	2.43	1.45	.67	"
1.24	1.58	1.50	.95	.79	A23
1.23	1.61	1.77	1.10	.79	"
1.23	1.64	2.02	1.23	.78	"
1.22	1.67	2.28	1.36	.78	"
1.21	1.71	2.52	1.48	.78	"
1.20	1.74	2.76	1.59	.77	"
1.19	1.77	3.00	1.70	.77	"
1.19	1.80	3.23	1.80	.77	"
1.18	1.83	3.46	1.89	.77	"
1.17	1.86	3.69	1.99	.77	"
1.88	2.32	6.19	2.67	1.19	A27
1.87	2.34	7.13	3.04	1.19	"
1.86	2.38	8.04	3.37	1.18	"
1.85	2.41	8.94	3.70	1.18	"
1.84	2.45	9.81	4.01	1.17	"
1.83	2.48	10.67	4.31	1.17	"
1.83	2.51	11.52	4.59	1.17	"
1.82	2.54	12.35	4.86	1.17	"
1.81	2.57	13.17	5.12	1.16	"
1.80	2.60	13.98	5.37	1.16	"
1.80	2.64	14.78	5.61	1.16	"
2.51	3.09	19.56	6.33	1.59	A35
2.50	3.12	21.79	6.98	1.58	"
2.49	3.16	23.97	7.60	1.58	"
2.48	3.19	26.13	8.20	1.58	"
2.47	3.22	28.24	8.77	1.57	"
2.46	3.25	30.33	9.33	1.57	"
2.45	3.28	32.38	9.86	1.56	"
2.44	3.32	34.40	10.38	1.56	"
2.44	3.35	36.40	10.88	1.56	"
2.43	3.38	38.38	11.36	1.56	"
2.42	3.41	40.33	11.83	1.55	"

PROPERTIES OF SPECIAL ANGLES.

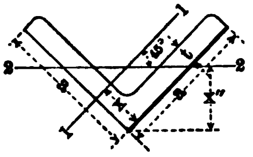
EQUAL LEGS.



1	2	3	4	5	6	7	8
Section Number.	Dimensions.	Thickness	Weight per Foot.	Area of Section.	Distance of Center of Gravity from Back of Leg.	Moment of Inertia Axis 1-1.	Section Modulus Axis 1-1.
	a x a	t		A	x	I	S
	Inches.	Inch.	Pounds.	Sq. Ins.	Inch.	Inches.⁴	Inches.³
A36	¾ x ¾	⅛	.59	.17	.23	.009	.017
"	"	3/16	.84	.25	.25	.012	.024
A37	1 x 1	⅛	.80	.23	.30	.022	.031
"	"	3/16	1.16	.34	.32	.030	.044
"	"	¼	1.49	.44	.34	.037	.056
A38	1¼ x 1¼	⅛	1.01	.30	.36	.044	.049
"	"	3/16	1.48	.43	.38	.061	.071
"	"	¼	1.92	.56	.40	.077	.091
"	"	5/16	2.33	.68	.42	.090	.109
A40	1¾ x 1¾	3/16	2.12	.62	.51	.18	.14
"	"	¼	2.77	.81	.53	.23	.19
"	"	5/16	3.39	1.00	.55	.27	.23
"	"	⅜	3.99	1.17	.57	.31	.26
"	"	7/16	4.6	1.34	.59	.35	.30
"	"	½	5.1	1.50	.61	.38	.33
A41	2¼ x 2¼	3/16	2.75	.81	.63	.39	.24
"	"	¼	3.62	1.06	.65	.50	.32
"	"	5/16	4.5	1.31	.68	.61	.39
"	"	⅜	5.3	1.55	.70	.70	.45
"	"	7/16	6.1	1.78	.72	.79	.52
A42	2¾ x 2¾	3/16	3.39	1.00	.76	.73	.37
"	"	¼	4.5	1.31	.78	.95	.48
"	"	5/16	5.6	1.62	.80	1.15	.59
"	"	⅜	6.6	1.92	.82	1.33	.69
"	"	7/16	7.6	2.22	.85	1.51	.79
"	"	½	8.5	2.50	.87	1.67	.89
A47	5 x 5	⅜	12.3	3.61	1.39	8.74	2.42
"	"	7/16	14.3	4.18	1.41	10.02	2.79
"	"	½	16.2	4.75	1.43	11.25	3.16
"	"	9/16	18.1	5.31	1.46	12.44	3.51
"	"	⅝	20.0	5.86	1.48	13.58	3.86
"	"	11/16	21.8	6.40	1.50	14.68	4.20

PROPERTIES OF SPECIAL ANGLES.

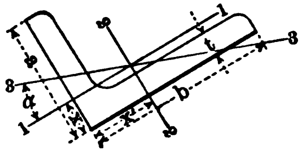
EQUAL LEGS.



9	10	11	12	13	1
Radius of Gyration Axis 1-1.	Distance of Center of Gravity from External Apex.	Least Moment of Inertia Axis 2-2.	Section Modulus Axis 2-2.	Least Radius of Gyration Axis 2-2.	Section Number.
r	x''	I''	S''	r''	
Inch.	Inch.	Inches.⁴	Inches.³	Inch.	
.22	.33	.004	.011	.14	A36
.22	.36	.005	.014	.14	"
.30	.42	.009	.021	.19	A37
.30	.45	.013	.028	.19	"
.29	.48	.016	.034	.19	"
.38	.51	.018	.035	.24	A38
.38	.54	.025	.047	.24	"
.37	.57	.033	.057	.24	"
.36	.60	.040	.066	.24	"
.54	.72	.073	.10	.34	A40
.53	.75	.094	.13	.34	"
.52	.78	.113	.15	.34	"
.51	.81	.133	.16	.34	"
.51	.84	.152	.18	.34	"
.50	.87	.171	.20	.34	"
.70	.89	.16	.18	.44	A41
.69	.92	.21	.22	.44	"
.68	.96	.25	.26	.44	"
.67	.99	.29	.30	.43	"
.67	1.02	.33	.33	.43	"
.86	1.07	.30	.28	.54	A43
.85	1.10	.38	.35	.54	"
.84	1.13	.47	.41	.54	"
.83	1.17	.55	.47	.53	"
.83	1.20	.63	.52	.53	"
.82	1.23	.70	.57	.53	"
1.56	1.96	3.53	1.79	.99	A47
1.55	2.00	4.05	2.03	.98	"
1.54	2.03	4.56	2.25	.98	"
1.53	2.06	5.06	2.46	.98	"
1.52	2.09	5.55	2.66	.97	"
1.51	2.12	6.03	2.84	.97	"

PROPERTIES OF STANDARD ANGLES.

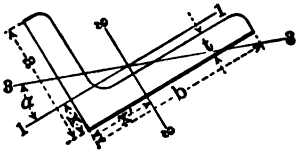
UNEQUAL LEGS.



1	2	3	4	5	6	7	8
Section Number.	Dimensions.	Thickness.	Weight per Foot.	Area of Section.	Distance of Center of Gravity from Back of Longer Leg.	Moment of Inertia Axis 1-1.	Section Modulus Axis 1-1.
	b x a	t		A	x	I	S
	Inches.	Inch.	Pounds.	Sq. Ins.	Inch.	Inches.⁴	Inches.³
A91	2½ x 2	3/16	2.75	.81	.51	.29	.20
"	"	¼	3.62	1.06	.54	.37	.25
"	"	5/16	4.5	1.31	.56	.45	.31
"	"	⅜	5.3	1.55	.58	.51	.36
"	"	7/16	6.1	1.78	.60	.58	.41
"	"	½	6.8	2.00	.63	.64	.46
"	"	9/16	7.6	2.22	.65	.69	.51
A93	3 x 2½	¼	4.5	1.31	.66	.74	.40
"	"	5/16	5.6	1.62	.68	.90	.49
"	"	⅜	6.6	1.92	.71	1.04	.58
"	"	7/16	7.6	2.22	.73	1.18	.66
"	"	½	8.5	2.50	.75	1.30	.74
"	"	9/16	9.5	2.78	.77	1.42	.82
"	"	⅝	10.4	3.05	.79	1.53	.90
A95	3½ x 2½	¼	4.9	1.44	.61	.78	.41
"	"	5/16	6.1	1.78	.64	.94	.50
"	"	⅜	7.2	2.11	.66	1.09	.59
"	"	7/16	8.3	2.43	.68	1.23	.68
"	"	½	9.4	2.75	.70	1.36	.76
"	"	9/16	10.4	3.06	.73	1.49	.84
"	"	⅝	11.5	3.36	.75	1.61	.92
"	"	11/16	12.5	3.65	.77	1.72	.99
"	"	¾	13.4	3.94	.79	1.83	1.07
A97	3½ x 3	5/16	6.6	1.93	.81	1.58	.72
"	"	⅜	7.9	2.30	.83	1.85	.85
"	"	7/16	9.1	2.65	.85	2.09	.98
"	"	½	10.2	3.00	.88	2.33	1.10
"	"	9/16	11.4	3.34	.90	2.55	1.21
"	"	⅝	12.5	3.67	.92	2.76	1.33
"	"	11/16	13.6	4.00	.94	2.96	1.44
"	"	¾	14.7	4.31	.96	3.15	1.54
"	"	13/16	15.8	4.62	.98	3.33	1.65
"	"	⅞	16.8	4.92	1.00	3.50	1.75
A99	4 x 3	5/16	7.2	2.09	.76	1.65	.73
"	"	⅜	8.5	2.48	.78	1.92	.87
"	"	7/16	9.8	2.87	.80	2.18	.99
"	"	½	11.1	3.25	.83	2.42	1.12
"	"	9/16	12.4	3.62	.85	2.66	1.23
"	"	⅝	13.6	3.98	.87	2.87	1.35
"	"	11/16	14.8	4.34	.89	3.08	1.46
"	"	¾	16.0	4.69	.92	3.28	1.57
"	"	13/16	17.1	5.03	.94	3.47	1.68
"	"	⅞	18.3	5.36	.96	3.66	1.79

PROPERTIES OF STANDARD ANGLES.

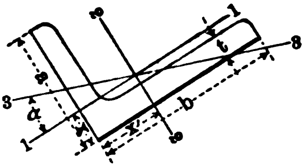
UNEQUAL LEGS.



9	10	11	12	13	14	15	1
Radius of Gyration Axis 1-1.	Distance of Center of Gravity from Back of Shorter Leg.	Moment of Inertia Axis 2-2.	Section Modulus Axis 2-2.	Radius of Gyration Axis 2-2.	Tangent of Angle.	Least Radius of Gyration Axis 3-3.	Section Number.
r	x'	I'	S'	r'	α	r''	
Inch.	Inch.	Inches.^{4}	Inches.^{3}	Inches.		Inch.	
.60	.76	.51	.29	.79	.632	.43	A91
.59	.79	.65	.38	.78	.626	.42	"
.58	.81	.79	.47	.78	.620	.42	"
.58	.83	.91	.55	.77	.614	.42	"
.57	.85	1.03	.62	.76	.607	.42	"
.56	.88	1.14	.70	.75	.600	.42	"
.56	.90	1.24	.77	.75	.592	.42	"
.75	.91	1.17	.56	.95	.684	.53	A93
.74	.93	1.42	.69	.94	.680	.53	"
.74	.96	1.66	.81	.93	.676	.52	"
.73	.98	1.88	.93	.92	.672	.52	"
.72	1.00	2.08	1.04	.91	.666	.52	"
.72	1.02	2.28	1.15	.91	.661	.52	"
.71	1.04	2.46	1.26	.90	.655	.52	"
.74	1.11	1.80	.75	1.12	.506	.54	A95
.73	1.14	2.19	.93	1.11	.501	.54	"
.72	1.16	2.56	1.09	1.10	.496	.54	"
.71	1.18	2.91	1.26	1.09	.491	.54	"
.70	1.20	3.24	1.41	1.09	.486	.53	"
.70	1.23	3.55	1.56	1.08	.480	.53	"
.69	1.25	3.85	1.71	1.07	.472	.53	"
.69	1.27	4.13	1.85	1.06	.468	.53	"
.68	1.29	4.40	1.99	1.06	.461	.54	"
.90	1.06	2.33	.95	1.10	.724	.63	A97
.90	1.08	2.72	1.13	1.09	.721	.62	"
.89	1.10	3.10	1.29	1.08	.718	.62	"
.88	1.13	3.45	1.45	1.07	.714	.62	"
.87	1.15	3.79	1.61	1.07	.711	.62	"
.87	1.17	4.11	1.76	1.06	.707	.62	"
.86	1.19	4.41	1.91	1.05	.703	.62	"
.85	1.21	4.70	2.05	1.04	.698	.62	"
.85	1.23	4.98	2.20	1.04	.694	.62	"
.84	1.25	5.24	2.33	1.03	.689	.63	"
.89	1.26	3.38	1.23	1.27	.554	.65	A99
.88	1.28	3.96	1.46	1.26	.551	.64	"
.87	1.30	4.52	1.68	1.25	.547	.64	"
.86	1.33	5.05	1.89	1.25	.543	.64	"
.86	1.35	5.55	2.09	1.24	.538	.64	"
.85	1.37	6.03	2.30	1.23	.534	.64	"
.84	1.39	6.49	2.49	1.22	.529	.64	"
.84	1.42	6.93	2.68	1.22	.524	.64	"
.83	1.44	7.35	2.87	1.21	.518	.64	"
.83	1.46	7.75	3.05	1.20	.512	.64	"

PROPERTIES OF STANDARD ANGLES.

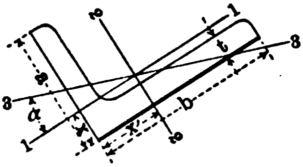
UNEQUAL LEGS.



1	2	3	4	5	6	7	8
Section Number.	Dimensions.	Thickness.	Weight per Foot.	Area of Section.	Distance of Center of Gravity from Back of Longer Leg.	Moment of Inertia Axis 1-1.	Section Modulus Axis 1-1.
	b x a	t		A	x	I	S
	Inches.	Inch.	Pounds.	Sq. Ins.	Inch.	Inches.⁴	Inches.³
A101	5 x 3	5/16	8.2	2.40	.68	1.75	.75
"	"	3/8	9.8	2.86	.70	2.04	.89
"	"	7/16	11.3	3.31	.73	2.32	1.02
"	"	1/2	12.8	3.75	.75	2.58	1.15
"	"	9/16	14.3	4.18	.77	2.83	1.27
"	"	5/8	15.7	4.61	.80	3.06	1.39
"	"	11/16	17.1	5.03	.82	3.29	1.51
"	"	3/4	18.5	5.44	.84	3.51	1.62
"	"	13/16	19.9	5.84	.86	3.71	1.74
"	"	7/8	21.2	6.23	.88	3.91	1.85
A103	5 x 3½	5/16	8.7	2.56	.84	2.72	1.02
"	"	3/8	10.4	3.05	.86	3.18	1.21
"	"	7/16	12.0	3.53	.88	3.63	1.39
"	"	1/2	13.6	4.00	.91	4.05	1.56
"	"	9/16	15.2	4.47	.93	4.45	1.73
"	"	5/8	16.8	4.92	.95	4.83	1.90
"	"	11/16	18.3	5.37	.97	5.20	2.06
"	"	3/4	19.8	5.81	1.00	5.55	2.22
"	"	13/16	21.3	6.25	1.02	5.89	2.37
"	"	7/8	22.7	6.67	1.04	6.21	2.52
"	"	15/16	24.2	7.09	1.06	6.52	2.67
A105	6 x 3½	3/8	11.7	3.42	.79	3.34	1.23
"	"	7/16	13.5	3.97	.81	3.81	1.41
"	"	1/2	15.3	4.50	.83	4.25	1.59
"	"	9/16	17.1	5.03	.86	4.67	1.77
"	"	5/8	18.9	5.55	.88	5.08	1.94
"	"	11/16	20.6	6.06	.90	5.47	2.11
"	"	3/4	22.4	6.56	.93	5.84	2.27
"	"	13/16	24.0	7.06	.95	6.20	2.43
"	"	7/8	25.7	7.55	.97	6.55	2.59
"	"	15/16	27.3	8.03	.99	6.88	2.74
"	"	1	28.9	8.50	1.01	7.21	2.90
A107	6 x 4	3/8	12.3	3.61	.94	4.90	1.60
"	"	7/16	14.3	4.18	.96	5.60	1.85
"	"	1/2	16.2	4.75	.99	6.27	2.08
"	"	9/16	18.1	5.31	1.01	6.91	2.31
"	"	5/8	20.0	5.86	1.03	7.52	2.54
"	"	11/16	21.8	6.40	1.06	8.11	2.76
"	"	3/4	23.6	6.94	1.08	8.68	2.97
"	"	13/16	25.4	7.47	1.10	9.23	3.18
"	"	7/8	27.2	7.98	1.12	9.75	3.39
"	"	15/16	28.9	8.50	1.14	10.26	3.59
"	"	1	30.6	9.00	1.17	10.75	3.79

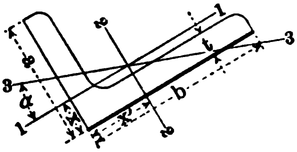
PROPERTIES OF STANDARD ANGLES.

UNEQUAL LEGS.



9	10	11	12	13	14	15	1
Radius of Gyration Axis 1-1.	Distance of Center of Gravity from Back of Shorter Leg.	Moment of Inertia Axis 2-2.	Section Modulus Axis 2-2.	Radius of Gyration Axis 2-2.	Tangent of Angle.	Least Radius of Gyration Axis 3-3.	Section Number.
r	x'	I'	S'	r'	α	r''	
Inch.	Inches.	Inches.⁴	Inches.³	Inch.		Inch.	
.85	1.68	6.26	1.89	1.61	.368	.66	A101
.84	1.70	7.37	2.24	1.61	.364	.65	“
.84	1.73	8.43	2.58	1.60	.361	.65	“
.83	1.75	9.45	2.91	1.59	.357	.65	“
.82	1.77	10.43	3.23	1.58	.353	.65	“
.82	1.80	11.37	3.55	1.57	.349	.64	“
.81	1.82	12.28	3.86	1.56	.345	.64	“
.80	1.84	13.15	4.16	1.55	.340	.64	“
.80	1.86	13.98	4.46	1.55	.336	.64	“
.79	1.88	14.78	4.75	1.54	.331	.64	“
1.03	1.59	6.60	1.94	1.61	.489	.77	A103
1.02	1.61	7.78	2.29	1.60	.485	.76	“
1.01	1.63	8.90	2.64	1.59	.482	.76	“
1.01	1.66	9.99	2.99	1.58	.479	.75	“
1.00	1.68	11.03	3.32	1.57	.476	.75	“
.99	1.70	12.03	3.65	1.56	.472	.75	“
.98	1.72	12.99	3.97	1.56	.468	.75	“
.98	1.75	13.92	4.28	1.55	.464	.75	“
.97	1.77	14.81	4.58	1.54	.460	.75	“
.96	1.79	15.67	4.88	1.53	.455	.75	“
.96	1.81	16.49	5.17	1.53	.451	.75	“
.99	2.04	12.86	3.24	1.94	.350	.77	A105
.98	2.06	14.76	3.75	1.93	.347	.76	“
.97	2.08	16.59	4.24	1.92	.344	.76	“
.96	2.11	18.37	4.72	1.91	.341	.75	“
.96	2.13	20.08	5.19	1.90	.338	.75	“
.95	2.15	21.74	5.65	1.89	.334	.75	“
.94	2.18	23.34	6.10	1.89	.331	.75	“
.94	2.20	24.89	6.55	1.88	.327	.75	“
.93	2.22	26.39	6.98	1.87	.323	.75	“
.93	2.24	27.84	7.41	1.86	.320	.75	“
.92	2.26	29.15	7.80	1.85	.317	.75	“
1.17	1.94	13.47	3.32	1.93	.446	.88	A107
1.16	1.96	15.46	3.83	1.92	.443	.87	“
1.15	1.99	17.40	4.33	1.91	.440	.87	“
1.14	2.01	19.26	4.83	1.90	.438	.87	“
1.13	2.03	21.07	5.31	1.90	.434	.86	“
1.13	2.06	22.82	5.78	1.89	.431	.86	“
1.12	2.08	24.51	6.25	1.88	.428	.86	“
1.11	2.10	26.15	6.70	1.87	.425	.86	“
1.11	2.12	27.73	7.15	1.86	.421	.86	“
1.10	2.14	29.26	7.59	1.86	.418	.86	“
1.09	2.17	30.75	8.02	1.85	.414	.86	“

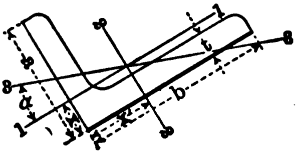
PROPERTIES OF SPECIAL ANGLES.
UNEQUAL LEGS.



1	2	3	4	5	6	7	8
Section Number.	Dimensions.	Thickness.	Weight per Foot.	Area of Section.	Distance of Center of Gravity from Back of Longer Leg.	Moment of Inertia Axis 1-1.	Section Modulus Axis 1-1.
	b x a	t		A	x	I	S
	Inches.	Inch.	Pounds.	Sq. Ins.	Inch.	Inches.⁴	Inches.³
A129	3 x 2	3/16	3.07	.90	.47	.31	.20
"	"	1/4	4.1	1.19	.49	.39	.26
"	"	5/16	5.0	1.47	.51	.47	.32
"	"	3/8	5.9	1.73	.54	.54	.37
"	"	7/16	6.8	2.00	.56	.61	.42
"	"	1/2	7.7	2.25	.58	.67	.47
A131	4 x 3½	5/16	7.7	2.25	.93	2.55	.99
"	"	3/8	9.1	2.67	.96	2.99	1.17
"	"	7/16	10.6	3.09	.98	3.40	1.35
"	"	1/2	11.9	3.50	1.00	3.79	1.52
"	"	9/16	13.3	3.90	1.02	4.17	1.68
"	"	5/8	14.7	4.30	1.04	4.49	1.83
"	"	11/16	16.0	4.68	1.07	4.86	2.00
A135	5 x 4	3/8	11.0	3.23	1.03	4.66	1.57
"	"	7/16	12.8	3.75	1.05	5.32	1.81
"	"	1/2	14.5	4.25	1.07	5.96	2.04
"	"	9/16	16.2	4.75	1.10	6.56	2.26
"	"	5/8	17.8	5.23	1.12	7.14	2.48
"	"	11/16	19.5	5.72	1.14	7.70	2.69
A109	7 x 3½	7/16	15.0	4.40	.75	3.95	1.44
"	"	1/2	17.0	5.00	.78	4.41	1.62
"	"	9/16	19.1	5.59	.80	4.86	1.80
"	"	5/8	21.0	6.17	.82	5.28	1.97
"	"	11/16	23.0	6.75	.85	5.69	2.14
"	"	3/4	24.9	7.31	.87	6.08	2.31
"	"	13/16	26.8	7.87	.89	6.46	2.48
"	"	7/8	28.7	8.42	.91	6.83	2.64
"	"	15/16	30.5	8.97	.94	7.18	2.80
"	"	1	32.3	9.50	.96	7.53	2.96
A112	8 x 6	1/2	23.0	6.75	1.47	21.68	4.79
"	"	9/16	25.7	7.56	1.50	24.04	5.34
"	"	5/8	28.5	8.36	1.52	26.33	5.88
"	"	11/16	31.2	9.15	1.54	28.56	6.40
"	"	3/4	33.8	9.94	1.56	30.72	6.92
"	"	13/16	36.5	10.72	1.59	32.82	7.44
"	"	7/8	39.1	11.48	1.61	34.86	7.94
"	"	15/16	41.7	12.25	1.63	36.85	8.43
"	"	1	44.2	13.00	1.65	38.78	8.92

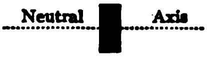
PROPERTIES OF SPECIAL ANGLES.

UNEQUAL LEGS.



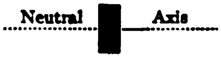
9	10	11	12	13	14	15	1
Radius of Gyration Axis 1-1.	Distance of Center of Gravity from Back of Shorter Leg.	Moment of Inertia Axis 2-2.	Section Modulus Axis 2-2.	Radius of Gyration Axis 2-2.	Tangent of Angle.	Radius of Gyration Axis 3-3.	Section Number.
r	x'	I'	S'	r'	α	r"	
Inch.	Inches.	Inches.⁴	Inches.³	Inches.		Inch.	
.58	.97	.84	.41	.97	.446	.44	A129
.57	.99	1.09	.54	.96	.440	.43	"
.57	1.02	1.32	.66	.95	.434	.43	"
.56	1.04	1.53	.78	.94	.428	.43	"
.55	1.06	1.73	.89	.93	.421	.43	"
.55	1.08	1.92	1.00	.92	.414	.43	"
1.07	1.18	3.56	1.26	1.26	.757	.73	A131
1.06	1.21	4.18	1.49	1.25	.755	.73	"
1.05	1.23	4.76	1.72	1.24	.753	.72	"
1.04	1.25	5.32	1.94	1.23	.750	.72	"
1.03	1.27	5.86	2.15	1.23	.747	.72	"
1.02	1.29	6.37	2.35	1.22	.742	.72	"
1.02	1.32	6.86	2.56	1.21	.742	.72	"
1.20	1.53	8.14	2.34	1.59	.631	.85	A135
1.19	1.55	9.32	2.70	1.58	.629	.85	"
1.18	1.57	10.46	3.05	1.57	.626	.85	"
1.18	1.60	11.55	3.39	1.56	.623	.85	"
1.17	1.62	12.61	3.73	1.55	.620	.84	"
1.16	1.64	13.62	4.05	1.54	.617	.84	"
.95	2.50	22.56	5.01	2.26	.267	.76	A109
.94	2.53	25.41	5.68	2.25	.264	.75	"
.93	2.55	28.18	6.34	2.25	.262	.75	"
.93	2.57	30.86	6.96	2.24	.259	.75	"
.92	2.60	33.47	7.60	2.23	.257	.74	"
.91	2.62	35.99	8.22	2.22	.253	.74	"
.91	2.64	38.45	8.83	2.21	.250	.74	"
.90	2.66	40.82	9.42	2.20	.247	.74	"
.89	2.69	43.13	10.00	2.19	.244	.74	"
.89	2.71	45.37	10.58	2.19	.241	.74	"
1.79	2.47	44.31	8.02	2.56	.558	1.30	A112
1.78	2.50	49.26	8.95	2.55	.556	1.30	"
1.77	2.52	54.10	9.87	2.54	.554	1.29	"
1.77	2.54	58.82	10.77	2.54	.554	1.29	"
1.76	2.56	63.42	11.67	2.53	.553	1.28	"
1.75	2.59	67.92	12.55	2.52	.549	1.28	"
1.74	2.61	72.32	13.41	2.51	.546	1.28	"
1.73	2.63	76.59	14.27	2.50	.545	1.28	"
1.73	2.65	80.78	15.11	2.49	.543	1.28	"

MOMENTS OF INERTIA OF RECTANGLES.



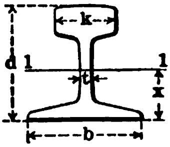
Depth in Inches.	Width of Rectangle in Inches.						
	1/4	5/16	3/8	7/16	1/2	9/16	5/8
2	.17	.21	.25	.29	.33	.38	.42
3	.56	.70	.84	.98	1.13	1.27	1.41
4	1.33	1.67	2.00	2.33	2.67	3.00	3.33
5	2.60	3.26	3.91	4.56	5.21	5.86	6.51
6	4.50	5.63	6.75	7.88	9.00	10.13	11.25
7	7.15	8.93	10.72	12.51	14.29	16.08	17.86
8	10.67	13.33	16.00	18.67	21.33	24.00	26.67
9	15.19	18.98	22.78	26.58	30.38	34.17	37.97
10	20.83	26.04	31.25	36.46	41.67	46.87	52.08
11	27.73	34.66	41.59	48.53	55.46	62.39	69.32
12	36.00	45.00	54.00	63.00	72.00	81.00	90.00
13	45.77	57.21	68.66	80.10	91.54	102.98	114.43
14	57.17	71.46	85.75	100.04	114.33	128.63	142.92
15	70.31	87.89	105.47	123.05	140.63	158.20	175.78
16	85.33	106.67	128.00	149.33	170.67	192.00	213.33
17	102.35	127.94	153.53	179.12	204.71	230.30	255.89
18	121.50	151.88	182.25	212.63	243.00	273.38	303.75
19	142.90	178.62	214.34	250.07	285.79	321.52	357.24
20	166.67	208.33	250.00	291.67	333.33	375.00	416.67
21	192.94	241.17	289.41	337.64	385.88	434.11	482.34
22	221.83	277.29	332.75	388.21	443.67	499.13	554.58
23	253.48	316.85	380.22	443.59	506.96	570.33	633.70
24	288.00	360.00	432.00	504.00	576.00	648.00	720.00
25	325.52	406.90	488.28	569.66	651.04	732.42	813.80
26	366.17	457.71	549.25	640.79	732.33	823.88	915.42
27	410.06	512.58	615.09	717.61	820.13	922.64	1025.16
28	457.33	571.67	686.00	800.33	914.67	1029.00	1143.33
29	508.10	635.13	762.16	889.18	1016.21	1143.23	1270.26
30	562.50	703.13	843.75	984.38	1125.00	1265.63	1406.25
32	682.67	853.33	1024.00	1194.67	1365.33	1536.00	1706.67
34	818.83	1023.54	1228.25	1432.96	1637.67	1842.38	2047.08
36	972.00	1215.00	1458.00	1701.00	1944.00	2187.00	2430.00
38	1143.17	1428.96	1714.75	2000.54	2286.33	2572.13	2857.92
40	1333.33	1666.67	2000.00	2333.33	2666.67	3000.00	3333.33
42	1543.50	1929.38	2315.25	2701.13	3087.00	3472.88	3858.75
44	1774.67	2218.33	2662.00	3105.67	3549.33	3993.00	4436.67
46	2027.83	2534.79	3041.75	3548.71	4055.67	4562.63	5069.58
48	2304.00	2880.00	3456.00	4032.00	4608.00	5184.00	5760.00
50	2604.17	3255.21	3906.25	4557.29	5208.33	5859.38	6510.42
52	2929.33	3661.67	4394.00	5126.33	5858.67	6591.00	7323.33
54	3280.50	4100.63	4920.75	5740.88	6561.00	7381.13	8201.25
56	3658.67	4573.33	5488.00	6402.67	7317.33	8232.00	9146.67
58	4064.83	5081.04	6097.25	7113.46	8129.67	9145.87	10163.08
60	4500.00	5625.00	6750.00	7875.00	9000.00	10125.00	11250.00

MOMENTS OF INERTIA OF RECTANGLES.



Width of Rectangle in Inches.						Depth in Inches.
11/16	3/4	13/16	7/8	15/16	1	
.46	.50	.54	.58	.63	.67	2
1.55	1.69	1.83	1.97	2.11	2.25	3
3.67	4.00	4.33	4.67	5.00	5.33	4
7.16	7.81	8.46	9.11	9.77	10.42	5
12.38	13.50	14.63	15.75	16.88	18.00	6
19.65	21.44	23.22	25.01	26.80	28.58	7
29.33	32.00	34.67	37.33	40.00	42.67	8
41.77	45.56	49.36	53.16	56.95	60.75	9
57.29	62.50	67.71	72.92	78.13	83.33	10
76.26	83.19	90.12	97.05	103.98	110.92	11
99.00	108.00	117.00	126.00	135.00	144.00	12
125.87	137.31	148.75	160.20	171.64	183.08	13
157.21	171.50	185.79	200.08	214.38	228.67	14
193.36	210.94	228.52	246.09	263.67	281.25	15
234.67	256.00	277.33	298.67	320.00	341.33	16
281.47	307.06	332.65	358.24	383.83	409.42	17
334.13	364.50	394.88	425.25	455.63	486.00	18
392.96	428.69	464.41	500.14	535.86	571.58	19
458.33	500.00	541.67	583.33	625.00	666.67	20
530.58	578.81	627.05	675.28	723.52	771.75	21
610.04	665.50	720.96	776.42	831.87	887.33	22
697.07	760.44	823.81	887.18	950.55	1013.92	23
792.00	864.00	936.00	1008.00	1080.00	1152.00	24
895.18	976.56	1057.94	1139.32	1220.70	1302.08	25
1006.96	1098.50	1190.04	1281.58	1373.13	1464.67	26
1127.67	1230.19	1332.70	1435.22	1537.73	1640.25	27
1257.67	1372.00	1486.33	1600.67	1715.00	1829.33	28
1397.29	1524.31	1651.34	1778.36	1905.39	2032.42	29
1546.88	1687.50	1828.13	1968.75	2109.38	2250.00	30
1877.33	2048.00	2218.67	2389.33	2560.00	2730.67	32
2251.79	2456.50	2661.21	2865.92	3070.63	3275.33	34
2673.00	2916.00	3159.00	3402.00	3645.00	3888.00	36
3143.71	3429.50	3715.29	4001.08	4286.88	4572.67	38
3666.67	4000.00	4333.33	4666.67	5000.00	5333.33	40
4244.63	4630.50	5016.38	5402.25	5788.13	6174.00	42
4880.33	5324.00	5767.67	6211.33	6655.00	7098.67	44
5576.54	6083.50	6590.46	7097.42	7604.38	8111.33	46
6336.00	6912.00	7488.00	8064.00	8640.00	9216.00	48
7161.46	7812.50	8463.54	9114.58	9765.63	10416.67	50
8055.67	8788.00	9520.33	10252.67	10985.00	11717.33	52
9021.88	9841.50	10661.63	11481.75	12301.88	13122.00	54
10061.33	10976.00	11890.67	12805.33	13720.00	14634.67	56
11178.29	12194.50	13210.71	14226.92	15243.12	16259.33	58
12375.00	13500.00	14625.00	15750.00	16875.00	18000.00	60

PROPERTIES AND PRINCIPAL DIMENSIONS OF STANDARD T-RAILS.

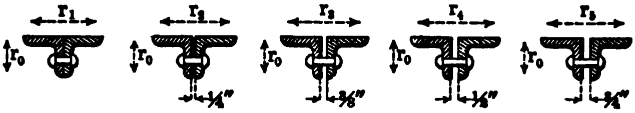


Section Number.	Weight per Yard.	Area.	b	d	k	t	x	Axis 1-1. Moment of Inertia.	Axis 1-1. Section Modulus.
	Pounds.	Sq. Ins.	Inches.	Inches.	Inches.	Inch.	Inches.	I	S
580	12	1.18	2	2	1	3/16	0.92	0.55	0.53
579	16	1.57	2 3/8	2 3/8	1 11/64	7/32	1.1	1.1	0.95
578	20	2.00	2 5/8	2 5/8	1 11/32	1/4	1.2	1.7	1.3
577	25	2.5	2 3/4	2 3/4	1 1/2	19/64	1.3	2.6	1.8
576	30	2.9	3 1/8	3 1/8	1 11/16	21/64	1.4	3.6	2.3
575	35	3.4	3 5/16	3 5/16	1 3/4	23/64	1.6	4.9	2.9
545	40	3.9	3 1/2	3 1/2	1 7/8	25/64	1.7	6.6	3.6
549	45	4.4	3 11/16	3 11/16	2	27/64	1.8	8.1	4.2
542	50	4.9	3 7/8	3 7/8	2 1/8	7/16	1.9	9.8	4.9
537	55	5.4	4 1/16	4 1/16	2 1/4	15/32	2.0	12.2	5.9
538	60	5.9	4 1/4	4 1/4	2 3/8	31/64	2.1	14.7	6.7
534	65	6.4	4 7/16	4 7/16	2 13/32	1/2	2.2	17.0	7.4
532	70	6.9	4 5/8	4 5/8	2 7/16	33/64	2.2	20.0	8.4
529	75	7.4	4 13/16	4 13/16	2 15/32	17/32	2.3	23.0	9.1
530	80	7.8	5	5	2 1/2	35/64	2.4	26.7	10.1
531	85	8.3	5 3/16	5 3/16	2 9/16	9/16	2.5	30.5	11.2
535	90	8.8	5 3/8	5 3/8	2 5/8	9/16	2.6	34.4	12.3
550	95	9.3	5 9/16	5 9/16	2 11/16	9/16	2.7	38.6	13.3
536	100	9.8	5 3/4	5 3/4	2 3/4	9/16	2.8	43.4	14.7
539	150	14.7	6	6	4 1/4	1	3.0	69.3	23.1

All sections from 40 lbs. to 100 lbs. both inclusive are Am. Soc. C. E. Standard.
For detail dimensions of Section No. 539, see page 17.

RADII OF GYRATION FOR TWO ANGLES PLACED BACK TO BACK.

ANGLES WITH EQUAL LEGS.



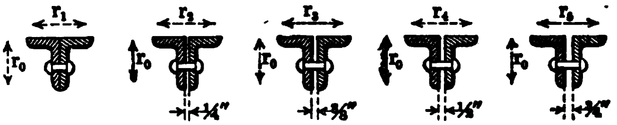
Radii of gyration correspond to directions indicated by arrowheads.

Section Number.	Dimensions.	Thickness.	Area of Two Angles.	Radii of Gyration.					
	Inches.	Inch.	Sq. Ins.	r_0	r_1	r_2	r_3	r_4	r_5
A11	1½ x 1½	3/16	1.06	0.46	0.64	0.73	0.78	0.83	0.94
"	"	⅜	1.96	0.44	0.67	0.77	0.82	0.88	0.99
*A40	1¾ x 1¾	3/16	1.24	0.54	0.74	0.83	0.88	0.93	1.03
"	"	7/16	2.68	0.51	0.78	0.88	0.93	0.98	1.09
A15	2 x 2	3/16	1.44	0.62	0.84	0.93	0.98	1.03	1.13
"	"	5/16	2.30	0.60	0.86	0.95	1.00	1.05	1.16
"	"	7/16	3.12	0.59	0.88	0.98	1.03	1.08	1.19
*A41	2¼ x 2¼	3/16	1.62	0.70	0.94	1.03	1.08	1.12	1.22
"	"	⅜	3.10	0.67	0.97	1.06	1.11	1.16	1.27
A17	2½ x 2½	¼	2.38	0.77	1.05	1.14	1.19	1.24	1.34
"	"	⅜	3.46	0.75	1.07	1.16	1.21	1.26	1.36
"	"	½	4.50	0.74	1.09	1.19	1.24	1.29	1.39
*A43	2¾ x 2¾	3/16	2.00	0.86	1.14	1.23	1.28	1.32	1.42
"	"	5/16	3.24	0.84	1.16	1.25	1.30	1.35	1.45
"	"	7/16	4.44	0.83	1.18	1.28	1.32	1.37	1.47
A19	3 x 3	¼	2.88	0.93	1.26	1.34	1.39	1.43	1.53
"	"	7/16	4.86	0.91	1.28	1.37	1.42	1.47	1.57
"	"	⅝	6.72	0.88	1.32	1.41	1.46	1.51	1.61
A21	3½ x 3½	⅜	4.96	1.07	1.48	1.56	1.61	1.66	1.75
"	"	⅝	7.96	1.04	1.52	1.61	1.66	1.71	1.81
"	"	13/16	10.06	1.02	1.55	1.65	1.70	1.75	1.85
A23	4 x 4	5/16	4.80	1.24	1.67	1.76	1.80	1.85	1.94
"	"	9/16	8.36	1.21	1.71	1.80	1.85	1.89	1.99
"	"	13/16	11.68	1.18	1.75	1.85	1.89	1.94	2.04
*A47	5 x 5	⅜	7.22	1.56	2.09	2.17	2.22	2.26	2.35
"	"	½	9.50	1.54	2.10	2.19	2.24	2.28	2.38
"	"	⅝	11.72	1.52	2.12	2.21	2.26	2.30	2.40
A27	6 x 6	7/16	10.12	1.87	2.50	2.58	2.63	2.67	2.76
"	"	⅝	14.22	1.84	2.53	2.62	2.66	2.71	2.80
"	"	⅞	19.46	1.81	2.57	2.66	2.70	2.75	2.85
A35	8 x 8	½	15.50	2.51	3.32	3.41	3.45	3.49	3.58
"	"	⅝	19.22	2.49	3.34	3.43	3.47	3.51	3.60
"	"	¾	22.88	2.47	3.36	3.44	3.49	3.53	3.62
"	"	⅞	26.46	2.45	3.38	3.46	3.51	3.55	3.64
"	"	1	30.00	2.44	3.40	3.48	3.53	3.57	3.67
"	"	1⅛	33.46	2.42	3.42	3.51	3.55	3.60	3.69

Angles marked * are special sections.

RADII OF GYRATION FOR TWO ANGLES PLACED BACK TO BACK.

ANGLES WITH UNEQUAL LEGS.

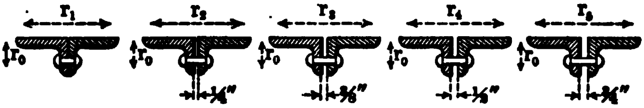


Radii of gyration correspond to directions indicated by arrowheads.

Section Number.	Dimensions.	Thickness	Area of Two Angles.	Radii of Gyration.					
	Inches.	Inch.	Sq. Ins.	r_0	r_1	r_2	r_3	r_4	r_5
A91	2½ x 2	3/16	1.62	0.79	0.79	0.88	0.93	0.97	1.07
"	"	⅜	3.10	0.77	0.82	0.91	0.96	1.01	1.12
"	"	½	4.00	0.75	0.84	0.94	0.99	1.04	1.15
*A129	3 x 2	3/16	1.80	0.97	0.75	0.83	0.88	0.93	1.03
"	"	5/16	2.94	0.95	0.76	0.85	0.90	0.95	1.05
"	"	7/16	4.00	0.93	0.79	0.88	0.93	0.98	1.09
A93	3 x 2½	¼	2.62	0.95	1.00	1.09	1.13	1.18	1.28
"	"	⅜	3.84	0.93	1.02	1.11	1.16	1.21	1.31
"	"	9/16	5.56	0.91	1.05	1.15	1.20	1.25	1.35
A95	3½ x 2½	¼	2.88	1.12	0.96	1.04	1.09	1.13	1.23
"	"	½	5.50	1.09	1.00	1.09	1.14	1.19	1.29
"	"	11/16	7.30	1.06	1.03	1.13	1.18	1.23	1.33
A97	3½ x 3	5/16	3.86	1.10	1.21	1.30	1.35	1.39	1.49
"	"	9/16	6.68	1.07	1.25	1.34	1.39	1.44	1.54
"	"	13/16	9.24	1.04	1.30	1.40	1.45	1.50	1.60
A99	4 x 3	5/16	4.18	1.27	1.17	1.25	1.30	1.34	1.44
"	"	9/16	7.24	1.24	1.21	1.30	1.34	1.39	1.49
"	"	13/16	10.06	1.21	1.25	1.35	1.40	1.45	1.55
*A131	4 x 3½	5/16	4.50	1.26	1.42	1.50	1.55	1.59	1.69
"	"	½	7.00	1.23	1.44	1.53	1.58	1.63	1.73
"	"	⅝	8.60	1.22	1.46	1.55	1.60	1.65	1.75
A101	5 x 3	5/16	4.80	1.61	1.09	1.17	1.22	1.26	1.36
"	"	9/16	8.36	1.58	1.13	1.22	1.26	1.31	1.41
"	"	13/16	11.68	1.55	1.17	1.27	1.32	1.37	1.47
A103	5 x 3½	⅜	6.10	1.60	1.34	1.42	1.46	1.51	1.60
"	"	⅝	9.84	1.56	1.37	1.46	1.51	1.56	1.66
"	"	⅞	13.34	1.53	1.42	1.51	1.56	1.61	1.71
*A135	5 x 4	⅜	6.46	1.59	1.58	1.66	1.71	1.75	1.85
"	"	½	8.50	1.57	1.60	1.68	1.73	1.78	1.87
"	"	⅝	10.46	1.55	1.62	1.71	1.75	1.80	1.90
A105	6 x 3½	⅜	6.84	1.94	1.26	1.34	1.39	1.43	1.53
"	"	⅝	11.10	1.90	1.30	1.39	1.43	1.48	1.58
"	"	⅞	15.10	1.87	1.34	1.44	1.49	1.53	1.64
A107	6 x 4	⅜	7.22	1.93	1.50	1.58	1.62	1.67	1.76
"	"	⅝	11.72	1.90	1.53	1.62	1.67	1.71	1.81
"	"	⅞	15.96	1.86	1.58	1.67	1.71	1.76	1.86
*A109	7 x 3½	7/16	8.80	2.26	1.16	1.29	1.33	1.38	1.47
"	"	½	10.00	2.25	1.22	1.30	1.35	1.39	1.48
"	"	⅝	12.34	2.24	1.24	1.33	1.37	1.42	1.51
"	"	13/16	15.74	2.21	1.27	1.36	1.41	1.46	1.56
"	"	1	19.00	2.19	1.31	1.40	1.45	1.50	1.60

Angles marked * are special sections.

RADII OF GYRATION FOR TWO ANGLES PLACED BACK TO BACK.
ANGLES WITH UNEQUAL LEGS.

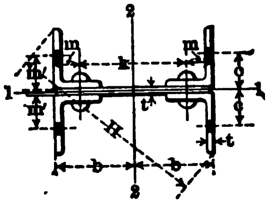


Radii of gyration correspond to directions indicated by arrowheads.

Section Number.	Dimensions.	Thickness.	Area of Two Angles.	Radii of Gyration.					
	Inches.	Inch.	Sq. Ins.	r_0	r_1	r_2	r_3	r_4	r_5
A91	2½ x 2	3/16	1.62	0.60	1.10	1.19	1.24	1.29	1.39
“	“	3/8	3.10	0.58	1.13	1.23	1.28	1.33	1.43
“	“	1/2	4.00	0.56	1.15	1.25	1.30	1.35	1.46
*A129	3 x 2	3/16	1.80	0.58	1.37	1.46	1.51	1.56	1.66
“	“	5/16	2.94	0.57	1.39	1.48	1.53	1.58	1.68
“	“	7/16	4.00	0.55	1.41	1.51	1.56	1.61	1.71
A93	3 x 2½	1/4	2.62	0.75	1.31	1.40	1.45	1.50	1.60
“	“	3/8	3.84	0.74	1.33	1.42	1.47	1.52	1.63
“	“	9/16	5.56	0.72	1.37	1.46	1.51	1.56	1.66
A95	3½ x 2½	1/4	2.88	0.74	1.58	1.67	1.72	1.76	1.86
“	“	1/2	5.50	0.70	1.62	1.72	1.77	1.81	1.92
“	“	11/16	7.30	0.69	1.66	1.75	1.80	1.86	1.96
A97	3½ x 3	5/16	3.86	0.90	1.52	1.61	1.66	1.71	1.80
“	“	9/16	6.68	0.87	1.57	1.66	1.71	1.76	1.86
“	“	13/16	9.24	0.85	1.61	1.71	1.76	1.81	1.91
A99	4 x 3	5/16	4.18	0.89	1.79	1.88	1.93	1.97	2.07
“	“	9/16	7.24	0.86	1.83	1.93	1.97	2.02	2.12
“	“	13/16	10.06	0.83	1.88	1.97	2.02	2.08	2.18
*A131	4 x 3½	5/16	4.50	1.07	1.73	1.81	1.86	1.91	2.00
“	“	1/2	7.00	1.04	1.76	1.85	1.89	1.94	2.04
“	“	5/8	8.60	1.02	1.78	1.87	1.92	1.97	2.07
A101	5 x 3	5/16	4.80	0.85	2.33	2.42	2.47	2.52	2.61
“	“	9/16	8.36	0.82	2.37	2.47	2.52	2.57	2.67
“	“	13/16	11.68	0.80	2.42	2.52	2.57	2.62	2.72
A103	5 x 3½	3/8	6.10	1.02	2.27	2.36	2.41	2.45	2.55
“	“	5/8	9.84	0.99	2.31	2.40	2.45	2.50	2.60
“	“	7/8	13.34	0.96	2.36	2.45	2.50	2.55	2.65
*A135	5 x 4	3/8	6.46	1.20	2.20	2.29	2.34	2.38	2.48
“	“	1/2	8.50	1.18	2.22	2.31	2.36	2.41	2.50
“	“	5/8	10.46	1.17	2.24	2.33	2.38	2.43	2.53
A105	6 x 3½	3/8	6.84	0.99	2.81	2.90	2.95	3.00	3.09
“	“	5/8	11.10	0.96	2.86	2.95	3.00	3.05	3.15
“	“	7/8	15.10	0.93	2.90	3.00	3.05	3.10	3.20
A107	6 x 4	3/8	7.22	1.17	2.74	2.83	2.87	2.92	3.02
“	“	5/8	11.72	1.13	2.78	2.87	2.92	2.97	3.06
“	“	7/8	15.96	1.11	2.82	2.92	2.97	3.02	3.12
*A109	7 x 3½	7/16	8.80	0.95	3.37	3.47	3.52	3.56	3.66
“	“	1/2	10.00	0.94	3.39	3.48	3.53	3.58	3.67
“	“	5/8	12.34	0.93	3.40	3.50	3.55	3.60	3.70
“	“	13/16	15.74	0.91	3.45	3.54	3.59	3.64	3.74
“	“	1	19.00	0.89	3.48	3.58	3.63	3.68	3.78

Angles marked * are special sections.

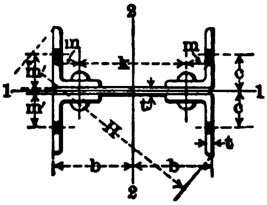
DIMENSIONS FOR PLATE AND ANGLE COLUMNS.



Size of Angles.	Size of Plates.	Weight of Column.	Area of Column Section.	b	c	m	m'	k	H
Inches.	Inches.	Lbs. per Ft.	Sq. Ins.	Inches.	Inches.	Inches.	Inches.	Inches.	Inches.
3 x 2½ x ¼ " " ⅝	6 x ¼ " ⅝	23.1 54.4	6.74 15.95	3⅛ "	1⅞ 2 1/16	1⅜ "	1¾ "	3½ "	8 11/16 9⅛
3 x 2½ x ¼ " " ⅝	8 x ¼ " ⅝	24.8 58.6	7.24 17.20	4⅛ "	1⅞ 2 1/16	1⅜ "	1¾ "	5½ "	10⅜ 10 13/16
3 x 2½ x ¼ " " ⅝	10 x ¼ " ⅝	26.5 62.9	7.74 18.45	5⅛ "	1⅞ 2 1/16	1⅜ "	1¾ "	7½ "	12 12 7/16
3 x 2½ x ¼ " " ⅝	12 x ¼ " ⅝	28.2 67.1	8.24 19.70	6⅛ "	1⅞ 2 1/16	1⅜ "	1¾ "	9½ "	13¼ 13 11/16
3½ x 2½ x ¼ " " ¾	7 x ¼ " ¾	25.6 71.5	7.51 21.01	3⅝ "	2⅜ 2⅝	1⅜ "	2¼ "	4½ "	10¼ 10⅝
3½ x 2½ x ¼ " " ¾	8 x ¼ " ¾	26.4 74.0	7.76 21.76	4⅛ "	2⅜ 2⅝	1⅜ "	2¼ "	5½ "	11 11 7/16
3½ x 2½ x ¼ " " ¾	10 x ¼ " ¾	28.1 79.1	8.26 23.26	5⅛ "	2⅜ 2⅝	1⅜ "	2¼ "	7½ "	12 7/16 12⅞
3½ x 2½ x ¼ " " ¾	12 x ¼ " ¾	29.8 84.2	8.76 24.76	6⅛ "	2⅜ 2⅝	1⅜ "	2¼ "	9½ "	14¼ 14⅝
4 x 3 x 5/16 " " ⅞	8 x 5/16 " ⅞	37.3 97.0	10.86 28.44	4⅛ "	2 1/16 2 11/16	1¾ "	2¼ "	4¾ "	11 11/16 12⅛
4 x 3 x 5/16 " " ⅞	10 x 5/16 " ⅞	39.4 108.0	11.49 30.19	5⅛ "	2 1/16 2 11/16	1¾ "	2¼ "	6¾ "	13 3/16 13 9/16
4 x 3 x 5/16 " " ⅞	12 x 5/16 " ⅞	41.6 108.9	12.11 31.94	6⅛ "	2 1/16 2 11/16	1¾ "	2¼ "	8¾ "	14 11/16 15⅛
4 x 3 x 5/16 " " ⅞	14 x 5/16 " ⅞	43.7 114.9	12.74 33.69	7⅛ "	2 1/16 2 11/16	1¾ "	2¼ "	10¾ "	16¼ 16 11/16

Dimensions m' and c may be varied to suit requirements.

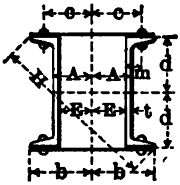
DIMENSIONS FOR PLATE AND ANGLE COLUMNS.



Size of Angles.	Size of Plates.	Weight of Column.	Area of Column Section.	b	c	m	m'	k	H
Inches.	Inches.	Lbs. per Ft.	Sq. Ins.	Inches.	Inches.	Inches.	Inches.	Inches.	Inches.
5 x 3½ x 5/16 " " 15/16	10 x 5/16 " 15/16	45.4 128.7	13.37 37.74	5⅛ "	2 5/16 2¾	2¼ "	2¼ "	5¾ "	14 9/16 15
5 x 3½ x 5/16 " " 15/16	12 x 5/16 " 15/16	47.6 135.1	13.99 39.61	6⅛ "	2 5/16 2¾	2¼ "	2¼ "	7¾ "	16 16 7/16
5 x 3½ x 5/16 " " 15/16	14 x 5/16 " 15/16	49.7 141.5	14.62 41.49	7⅛ "	2 5/16 2¾	2¼ "	2¼ "	9¾ "	17 9/16 17 15/16
5 x 3½ x 5/16 " " 15/16	16 x 5/16 " 15/16	51.8 147.8	15.24 43.36	8⅛ "	2 5/16 2¾	2¼ "	2¼ "	11¾ "	19¼ 19 9/16
6 x 3½ x ⅜ " " 1	12 x ⅜ " 1	62.1 156.4	18.18 46.00	6⅛ "	2 5/16 2¾	2¼ "	2¼ "	7¾ "	17 7/16 17⅞
6 x 3½ x ⅜ " " 1	14 x ⅜ " 1	64.7 163.2	18.93 48.00	7⅛ "	2 5/16 2¾	2¼ "	2¼ "	9¾ "	18⅞ 19 5/16
6 x 3½ x ⅜ " " 1	16 x ⅜ " 1	67.2 170.0	19.68 50.00	8⅛ "	2 5/16 2¾	2¼ "	2¼ "	11¾ "	20 7/16 20 13/16
6 x 3½ x ⅜ " " 1	18 x ⅜ " 1	69.8 176.8	20.43 52.00	9⅛ "	2 5/16 2¾	2¼ "	2¼ "	13¾ "	22 1/16 22 7/16
7 x 3½ x 7/16 " " 1	14 x 7/16 " 1	80.8 176.8	23.73 52.00	7⅛ "	2⅛ 2¾	2¼ "	2¼ "	9¾ "	20 5/16 20 11/16
7 x 3½ x 7/16 " " 1	16 x 7/16 " 1	83.8 183.6	24.60 54.00	8⅛ "	2⅛ 2¾	2¼ "	2¼ "	11¾ "	21¾ 22⅛
7 x 3½ x 7/16 " " 1	18 x 7/16 " 1	86.8 190.4	25.48 56.00	9⅛ "	2⅛ 2¾	2¼ "	2¼ "	13¾ "	23¼ 23⅝
7 x 3½ x 7/16 " " 1	20 x 7/16 " 1	89.8 197.2	26.35 58.00	10⅛ "	2⅛ 2¾	2¼ "	2¼ "	15¾ "	24⅞ 25 3/16

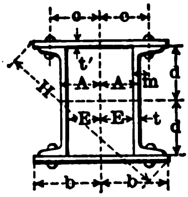
Dimensions m' and c may be varied to suit requirements.

DIMENSIONS FOR LATTICED CHANNEL COLUMNS.



Depth of Channel and Section Number.	Weight per Foot.	t	b	d	H	e	E	A	m
	Pounds.	Inch.	Inches.	Inches.	Inches.	Inches.	Inches.	Inches.	Inches.
6" C17	8.00	.20	3¼	3	9¹⁄₁₆	2⅝	1¹³⁄₁₆	2	1³⁄₁₆
	10.50	.32	"	"	"	"	1¹¹⁄₁₆	"	1⁵⁄₁₆
	13.00	.44	"	"	"	"	1⁹⁄₁₆	"	1⁷⁄₁₆
	15.50	.56	"	"	"	"	1⁷⁄₁₆	"	1⁹⁄₁₆
7" C21	9.75	.21	4¼	3½	11	3⅝	2³⁄₁₆	2⅜	1³⁄₁₆
	12.25	.32	"	"	"	"	2¹⁄₁₆	"	1⁵⁄₁₆
	14.75	.42	"	"	"	"	1¹⁵⁄₁₆	"	1⁷⁄₁₆
	17.25	.53	"	"	"	"	1⅞	"	1½
	19.75	.63	"	"	"	"	1¾	"	1⅝
8" C25	11.25	.22	4¹¹⁄₁₆	4	12½	3¾	2½	2¾	1¼
	13.75	.31	"	"	"	"	2⁷⁄₁₆	"	1⁵⁄₁₆
	16.25	.40	"	"	"	"	2⅜	"	1⅜
	18.75	.49	"	"	"	"	2¼	"	1½
	21.25	.58	"	"	"	"	2³⁄₁₆	"	1⁹⁄₁₆
9" C29	13.25	.23	5¹⁄₁₆	4½	13¾	4⅛	2¾	3	1⅜
	15.00	.29	"	"	"	"	2¹¹⁄₁₆	"	1⁷⁄₁₆
	20.00	.45	"	"	"	"	2⁹⁄₁₆	"	1⁹⁄₁₆
	25.00	.61	"	"	"	"	2⅜	"	1¾
10" C33	15.00	.24	5¾	5	15¼	4⅝	3⅛	3⅜	1½
	20.00	.38	"	"	"	"	3	"	1⅝
	25.00	.53	"	"	"	"	2⅞	"	1¾
	30.00	.68	"	"	"	"	2¹¹⁄₁₆	"	1¹⁵⁄₁₆
	35.00	.82	"	"	"	"	2⁹⁄₁₆	"	2¹⁄₁₆
12" C41	20.50	.28	6¹¹⁄₁₆	6	18⅛	5⅝	3⅞	4⅛	1¾
	25.00	.39	"	"	"	"	3¾	"	1⅞
	30.00	.51	"	"	"	"	3⅝	"	2
	35.00	.64	"	"	"	"	3½	"	2⅛
	40.00	.76	"	"	"	"	3⅜	"	2¼
15" C53	33.00	.40	8⅛	7½	22⅛	6⅝	4¾	5⅛	1⅞
	35.00	.43	"	"	"	"	4¹¹⁄₁₆	"	1¹⁵⁄₁₆
	40.00	.52	"	"	"	"	4⅝	"	2
	45.00	.62	"	"	"	"	4½	"	2⅛
	50.00	.72	"	"	"	"	4⁷⁄₁₆	"	2¼
	55.00	.82	"	"	"	"	4⁵⁄₁₆	"	2⁵⁄₁₆

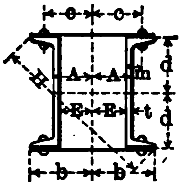
DIMENSIONS FOR PLATE AND CHANNEL COLUMNS.



SERIES B.

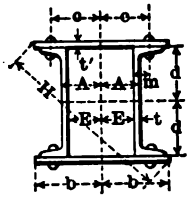
Depth of Channel and Section No.	Weight per Foot.	Size of Plates.		t	b	d	H	c	E	A	m
		Width.	Thickness. t'								
	Pounds.	Inches.	Inch.	Inch.	Inches.	Inches.	Inches.	Inches.	Inches.	Inches.	Inches.
10" C33	15.0	15	¼	.24	7½	5¼	18 1/16	6	4½	4¾	1½
	"	"	⅝	"	"	5⅝	18¾	"	"	"	"
	20.0	"	¼	.38	"	5¼	18 1/16	"	4⅜	"	1⅝
	"	"	⅝	"	"	5⅝	18¾	"	"	"	"
	25.0	"	¼	.53	"	5¼	18 1/16	"	4¼	"	1¾
	"	"	⅝	"	"	5⅝	18¾	"	"	"	"
	30.0	"	¼	.68	"	5¼	18 1/16	"	4 1/16	"	1 15/16
	"	"	⅝	"	"	5⅝	18¾	"	"	"	"
	35.0	"	¼	.82	"	5¼	18 1/16	"	3 15/16	"	2 1/16
	"	"	⅝	"	"	5⅝	18¾	"	"	"	"
12" C41	20.5	16	¼	.28	8	6¼	20 1/16	6⅝	4⅞	5⅛	1¾
	"	"	⅝	"	"	6⅝	20¾	"	"	"	"
	25.0	"	¼	.39	"	6¼	20 1/16	"	4¾	"	1⅞
	"	"	⅝	"	"	6⅝	20¾	"	"	"	"
	30.0	"	¼	.51	"	6¼	20 1/16	"	4⅝	"	2
	"	"	⅝	"	"	6⅝	20¾	"	"	"	"
	35.0	"	¼	.64	"	6¼	20 1/16	"	4½	"	2⅛
	"	"	⅝	"	"	6⅝	20¾	"	"	"	"
	40.0	"	¼	.76	"	6¼	20 1/16	"	4⅜	"	2¼
	"	"	⅝	"	"	6⅝	20¾	"	"	"	"
15" C53	33.0	20	⅜	.40	10	7⅞	25 1/16	8¼	6⅜	6¾	1⅞
	"	"	¾	"	"	8¼	25 11/16	"	"	"	"
	35.0	"	⅜	.43	"	7⅞	25 1/16	"	6 5/16	"	1 15/16
	"	"	¾	"	"	8¼	25 11/16	"	"	"	"
	40.0	"	⅜	.52	"	7⅞	25 1/16	"	6¼	"	2
	"	"	¾	"	"	8¼	25 11/16	"	"	"	"
	45.0	"	⅜	.62	"	7⅞	25 1/16	"	6⅛	"	2⅛
	"	"	¾	"	"	8¼	25 11/16	"	"	"	"
	50.0	"	⅜	.72	"	7⅞	25 1/16	"	6 1/16	"	2¼
	"	"	¾	"	"	8¼	25 11/16	"	"	"	"
	55.0	"	⅜	.82	"	7⅞	25 1/16	"	5 15/16	"	2 5/16
	"	"	¾	"	"	8¼	25 11/16	"	"	"	"

DIMENSIONS FOR LATTICED CHANNEL COLUMNS.



Depth of Channel and Section Number.	Weight per Foot.	t	b	d	E	c	E	A	m
	Pounds.	Inch.	Inches.	Inches.	Inches.	Inches.	Inches.	Inches.	Inches.
6″ C17	8.00	.20	3¼	3	9¹⁄₁₆	2⅞	1¹³⁄₁₆	2	1¹⁄₁₆
	10.50	.32	“	“	“	“	1¹¹⁄₁₆	“	1³⁄₁₆
	13.00	.44	“	“	“	“	1⁹⁄₁₆	“	1⁵⁄₁₆
	15.50	.56	“	“	“	“	1⁷⁄₁₆	“	1⁷⁄₁₆
7″ C21	9.75	.21	4¼	3½	11	3⅜	2³⁄₁₆	2⅜	1³⁄₁₆
	12.25	.32	“	“	“	“	2¹⁄₁₆	“	1⁵⁄₁₆
	14.75	.42	“	“	“	“	1¹⁵⁄₁₆	“	1⁷⁄₁₆
	17.25	.53	“	“	“	“	1⅞	“	1½
	19.75	.63	“	“	“	“	1¾	“	1⅝
8″ C25	11.25	.22	4¹¹⁄₁₆	4	12⅝	3¾	2½	2¾	1¼
	13.75	.31	“	“	“	“	2⁷⁄₁₆	“	1⁵⁄₁₆
	16.25	.40	“	“	“	“	2⅜	“	1⅜
	18.75	.49	“	“	“	“	2¼	“	1½
	21.25	.58	“	“	“	“	2³⁄₁₆	“	1⁹⁄₁₆
9″ C29	13.25	.23	5¹⁄₁₆	4½	13¾	4⅛	2¾	3	1⅜
	15.00	.29	“	“	“	“	2¹¹⁄₁₆	“	1⁷⁄₁₆
	20.00	.45	“	“	“	“	2⁹⁄₁₆	“	1⁹⁄₁₆
	25.00	.61	“	“	“	“	2⅜	“	1¾
10″ C33	15.00	.24	5¾	5	15¼	4⅝	3⅛	3⅜	1½
	20.00	.38	“	“	“	“	3	“	1⅝
	25.00	.53	“	“	“	“	2⅞	“	1¾
	30.00	.68	“	“	“	“	2¹¹⁄₁₆	“	1¹⁵⁄₁₆
	35.00	.82	“	“	“	“	2⁹⁄₁₆	“	2¹⁄₁₆
12″ C41	20.50	.28	6¹¹⁄₁₆	6	18⅛	5⅝	3⅞	4⅛	1¾
	25.00	.39	“	“	“	“	3¾	“	1⅞
	30.00	.51	“	“	“	“	3⅝	“	2
	35.00	.64	“	“	“	“	3½	“	2⅛
	40.00	.76	“	“	“	“	3⅜	“	2¼
15″ C53	33.00	.40	8⅛	7½	22⅛	6⅝	4¾	5⅛	1⅞
	35.00	.43	“	“	“	“	4¹¹⁄₁₆	“	1¹⁵⁄₁₆
	40.00	.52	“	“	“	“	4⅝	“	2
	45.00	.62	“	“	“	“	4½	“	2⅛
	50.00	.72	“	“	“	“	4⁷⁄₁₆	“	2¼
	55.00	.82	“	“	“	“	4⁵⁄₁₆	“	2⁵⁄₁₆

DIMENSIONS FOR PLATE AND CHANNEL COLUMNS.



SERIES B.

Depth of Channel and Section No.	Weight per Foot.	Size of Plates.		t	b	d	H	c	E	A	m
		Width.	Thickness. t'								
	Pounds.	Inches.	Inch.	Inch.	Inches.	Inches.	Inches.	Inches.	Inches.	Inches.	Inches.
10" C33	15.0	15	¼	.24	7½	5¼	18 1/16	6	4½	4¾	1½
	"	"	⅜	"	"	5⅜	18¼	"	"	"	"
	20.0	"	¼	.38	"	5¼	18 1/16	"	4⅜	"	1⅝
	"	"	⅜	"	"	5⅜	18¼	"	"	"	"
	25.0	"	¼	.53	"	5¼	18 1/16	"	4¼	"	1¾
	"	"	⅜	"	"	5⅜	18¼	"	"	"	"
	30.0	"	¼	.68	"	5¼	18 1/16	"	4 1/16	"	1 11/16
	"	"	⅜	"	"	5⅜	18¼	"	"	"	"
	35.0	"	¼	.82	"	5¼	18 1/16	"	3 11/16	"	2 1/16
	"	"	⅜	"	"	5⅜	18¼	"	"	"	"
12" C41	20.5	16	¼	.28	8	6¼	20 1/16	6⅝	4⅞	5⅛	1¾
	"	"	⅜	"	"	6⅜	20¼	"	"	"	"
	25.0	"	¼	.39	"	6¼	20 1/16	"	4¾	"	1⅞
	"	"	⅜	"	"	6⅜	20¼	"	"	"	"
	30.0	"	¼	.51	"	6¼	20 1/16	"	4⅝	"	2
	"	"	⅜	"	"	6⅜	20¼	"	"	"	"
	35.0	"	¼	.64	"	6¼	20 1/16	"	4½	"	2⅛
	"	"	⅜	"	"	6⅜	20¼	"	"	"	"
	40.0	"	¼	.76	"	6¼	20 1/16	"	4⅜	"	2¼
	"	"	⅜	"	"	6⅜	20¼	"	"	"	"
15" C58	33.0	20	⅜	.40	10	7⅞	25 1/16	8¼	6⅜	6¾	1⅞
	"	"	¾	"	"	8¼	25 11/16	"	"	"	"
	35.0	"	⅜	.43	"	7⅞	25 1/16	"	6 1/16	"	1 11/16
	"	"	¾	"	"	8¼	25 11/16	"	"	"	"
	40.0	"	⅜	.52	"	7⅞	25 1/16	"	6¼	"	2
	"	"	¾	"	"	8¼	25 11/16	"	"	"	"
	45.0	"	⅜	.62	"	7⅞	25 1/16	"	6⅛	"	2⅛
	"	"	¾	"	"	8¼	25 11/16	"	"	"	"
	50.0	"	⅜	.72	"	7⅞	25 1/16	"	6 1/16	"	2¼
	"	"	¾	"	"	8¼	25 11/16	"	"	"	"
	55.0	"	⅜	.82	"	7⅞	25 1/16	"	5 11/16	"	2 1/16
	"	"	¾	"	"	8¼	25 11/16	"	"	"	"

SHEARING VALUE OF RIVETS AND BEARING VALUE OF RIVETED PLATES.

All Dimensions in Inches.

Shearing Value = Area of Rivet × Allowable Shearing Stress per Square Inch.

Diameter of Rivet.	Area in Square Inches.	Unit Stress = 6 000 lbs.		Bearing Value for Different			
		Single Shear.	Double Shear.	$\frac{1}{4}$	$\frac{5}{16}$	$\frac{3}{8}$	$\frac{7}{16}$
⅜	.1105	663	1325	1125	1406	1688	
½	.1964	1178	2356	1500	1875	2250	2625
⅝	.3068	1841	3682	1875	2344	2813	3281
¾	.4418	2651	5301	2250	2813	3375	3938
⅞	.6013	3608	7216	2625	3281	3938	4594
1	.7854	4712	9425	3000	3750	4500	5250

Diameter of Rivet.	Area in Square Inches.	Unit Stress = 6 750 lbs.		Bearing Value for Different			
		Single Shear.	Double Shear.	$\frac{1}{4}$	$\frac{5}{16}$	$\frac{3}{8}$	$\frac{7}{16}$
⅜	.1105	746	1491	1266	1582	1898	
½	.1964	1325	2651	1688	2109	2531	2953
⅝	.3068	2071	4142	2109	2637	3164	3691
¾	.4418	2982	5964	2531	3164	3797	4430
⅞	.6013	4059	8118	2953	3691	4430	5168
1	.7854	5301	10603	3375	4219	5063	5906

Diameter of Rivet.	Area in Square Inches.	Unit Stress = 7 500 lbs.		Bearing Value for Different			
		Single Shear.	Double Shear.	$\frac{1}{4}$	$\frac{5}{16}$	$\frac{3}{8}$	$\frac{7}{16}$
⅜	.1105	828	1657	1406	1758	2109	
½	.1964	1473	2945	1875	2344	2813	3281
⅝	.3068	2301	4602	2344	2930	3516	4102
¾	.4418	3313	6627	2813	3516	4219	4922
⅞	.6013	4510	9020	3281	4102	4922	5742
1	.7854	5891	11781	3750	4688	5625	6563

Diameter of Rivet.	Area in Square Inches.	Unit Stress = 10 000 lbs.		Bearing Value for Different			
		Single Shear.	Double Shear.	$\frac{1}{4}$	$\frac{5}{16}$	$\frac{3}{8}$	$\frac{7}{16}$
⅜	.1105	1105	2209	1875	2344	2813	
½	.1964	1964	3927	2500	3125	3750	4375
⅝	.3068	3068	6136	3125	3906	4688	5469
¾	.4418	4418	8836	3750	4688	5625	6563
⅞	.6013	6013	12026	4375	5469	6563	7656
1	.7854	7854	15708	5000	6250	7500	8750

In the above tables the bearing values between the lower and upper zigzag black lines are greater than single and less than double shear for the corresponding dimensions, so that in case of single shear, the single shearing value governs, and in case of double shear, the bearing value governs the design.

SHEARING VALUE OF RIVETS AND BEARING VALUE OF RIVETED PLATES.

All Dimensions in Inches.

Bearing Value = Diameter of Rivet × Thickness of Plate × Allowable Bearing Stress per Square Inch.

Thicknesses of Plate in Inches at 12 000 Pounds per Square Inch.

$\frac{1}{2}$	$\frac{9}{16}$	$\frac{5}{8}$	$\frac{11}{16}$	$\frac{3}{4}$	$\frac{13}{16}$	$\frac{7}{8}$	$\frac{15}{16}$	1
3000								
3750	4219	4688						
4500	5063	5625	6188	6750				
5250	5906	6563	7219	7875	8531	9188	9844	
6000	6750	7500	8250	9000	9750	10500	11250	12000

Thicknesses of Plate in Inches at 13 500 Pounds per Square Inch.

$\frac{1}{2}$	$\frac{9}{16}$	$\frac{5}{8}$	$\frac{11}{16}$	$\frac{3}{4}$	$\frac{13}{16}$	$\frac{7}{8}$	$\frac{15}{16}$	1
3375								
4219	4746	5273						
5063	5695	6328	6961	7594				
5906	6645	7383	8121	8859	9598	10336	11074	
6750	7594	8438	9281	10125	10969	11813	12656	13500

Thicknesses of Plate in Inches at 15 000 Pounds per Square Inch.

$\frac{1}{2}$	$\frac{9}{16}$	$\frac{5}{8}$	$\frac{11}{16}$	$\frac{3}{4}$	$\frac{13}{16}$	$\frac{7}{8}$	$\frac{15}{16}$	1
3750								
4688	5273	5859						
5625	6328	7031	7734	8438				
6563	7383	8203	9023	9844	10664	11484	12305	
7500	8438	9375	10313	11250	12188	13125	14063	15000

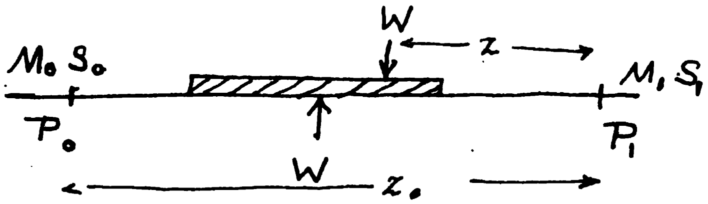
Thicknesses of Plate in Inches at 20 000 Pounds per Square Inch.

$\frac{1}{2}$	$\frac{9}{16}$	$\frac{5}{8}$	$\frac{11}{16}$	$\frac{3}{4}$	$\frac{13}{16}$	$\frac{7}{8}$	$\frac{15}{16}$	1
5000								
6250	7031	7813						
7500	8438	9375	10313	11250				
8750	9844	10938	12031	13125	14219	15313	16406	
10000	11250	12500	13750	15000	16250	17500	18750	20000

The bearing values above and to the right of the upper zigzag black lines are greater than double shear for the corresponding dimensions, so that in these cases the shearing values govern the design.

The bearing values below and to the left of the lower zigzag black lines are less than single shear, so that in these cases the bearing values govern the design.

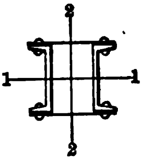
General Formula.



$$S_1 = S_0 + \Sigma W$$

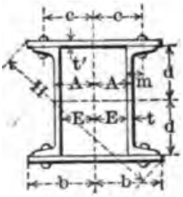
$$M_1 = M_0 + S_0 z_0 \pm \Sigma Wx$$

PROPERTIES OF LATTICED CHANNEL COLUMNS.



Depth of Channel and Section Number.	Weight per Foot.	Axis 1-1.		Axis 2-2.	
		Moment of Inertia.	Section Modulus.	Moment of Inertia.	Section Modulus.
	Pounds.	Inches.^{4}	Inches.^{3}	Inches.^{4}	Inches.^{3}
6" C17	8.00	26.0	8.7	27.0	7.3
	10.50	30.2	10.1	31.1	8.4
	13.00	34.6	11.5	35.2	9.5
	15.50	39.0	13.0	38.7	10.4
7" C21	9.75	42.2	12.1	44.0	10.3
	12.25	48.4	13.8	50.5	11.9
	14.75	54.4	15.5	56.4	13.3
	17.25	60.4	17.3	61.4	14.4
	19.75	66.4	19.0	66.5	15.6
8" C25	11.25	64.6	16.2	67.5	14.0
	13.75	72.0	18.0	75.8	15.8
	16.25	79.8	20.0	84.5	17.6
	18.75	87.7	21.9	92.3	19.3
	21.25	95.6	23.9	99.7	20.8
9" C29	13.25	94.6	21.0	92.4	17.8
	15.00	101.8	22.6	100.0	19.2
	20.00	121.6	27.0	120.1	23.1
	25.00	141.4	31.4	139.1	26.8
10" C33	15.00	133.8	26.8	131.7	23.0
	20.00	157.4	31.5	158.5	27.6
	25.00	182.0	36.4	183.8	32.0
	30.00	206.4	41.3	205.4	35.8
	35.00	231.0	46.2	226.0	39.4
12" C41	20.50	256.2	42.7	256.9	37.9
	25.00	288.0	48.0	295.6	43.6
	30.00	323.2	53.9	335.8	49.5
	35.00	358.6	59.8	370.5	54.6
	40.00	393.8	65.6	405.7	59.8
15" C53	33.00	625.2	83.4	618.7	76.1
	35.00	639.8	85.3	636.1	78.3
	40.00	695.0	92.7	700.8	86.3
	45.00	750.2	100.0	763.0	93.9
	50.00	805.4	107.4	819.5	100.9
	55.00	860.4	114.7	874.3	107.6

DIMENSIONS FOR PLATE AND CHANNEL COLUMNS.



SERIES B.

Depth of Channel and Section No.	Weight per Foot.	Size of Plates.		t	b	d	H	c	E	A	m
		Width.	Thickness t'								
	Pounds.	Inches.	Inch.	Inch.	Inches.	Inches.	Inches.	Inches.	Inches.	Inches.	Inches.
6″ C17	8.0	9	¼	.20	4½	3¼	11⅛	3⅝	2 5/16	2½	1 1/16
	“	“	⅝	“	“	3⅝	11 9/16	“	“	“	“
	10.5	“	¼	.32	“	3¼	11⅛	“	2 3/16	“	1 3/16
	“	“	⅝	“	“	3⅝	11 9/16	“	“	“	“
	13.0	“	¼	.44	“	3¼	11⅛	“	2 1/16	“	1 5/16
	“	“	⅝	“	“	3⅝	11 9/16	“	“	“	“
	15.5	“	¼	.56	“	3¼	11⅛	“	1 15/16	“	1 7/16
	“	“	⅝	“	“	3⅝	11 9/16	“	“	“	“
7″ C21	9.75	11	¼	.21	5½	3¾	13 5/16	4¼	3 1/16	3¼	1 3/16
	“	“	⅝	“	“	4⅛	13¾	“	“	“	“
	12.25	“	¼	.32	“	3¾	13 5/16	“	2 15/16	“	1 5/16
	“	“	⅝	“	“	4⅛	13¾	“	“	“	“
	14.75	“	¼	.42	“	3¾	13 5/16	“	2 13/16	“	1 7/16
	“	“	⅝	“	“	4⅛	13¾	“	“	“	“
	17.25	“	¼	.53	“	3¾	13 5/16	“	2¾	“	1½
	“	“	⅝	“	“	4⅛	13¾	“	“	“	“
	19.75	“	¼	.63	“	3¾	13 5/16	“	2⅝	“	1⅝
	“	“	⅝	“	“	4⅛	13¾	“	“	“	“
8″ C25	11.25	12	¼	.22	6	4¼	14 11/16	4⅝	3 7/16	3⅝	1¼
	“	“	⅝	“	“	4⅝	15⅛	“	“	“	“
	13.75	“	¼	.31	“	4¼	14 11/16	“	3 5/16	“	1 5/16
	“	“	⅝	“	“	4⅝	15⅛	“	“	“	“
	16.25	“	¼	.40	“	4¼	14 11/16	“	3¼	“	1⅜
	“	“	⅝	“	“	4⅝	15⅛	“	“	“	“
	18.75	“	¼	.49	“	4¼	14 11/16	“	3⅛	“	1½
	“	“	⅝	“	“	4⅝	15⅛	“	“	“	“
	21.25	“	¼	.58	“	4¼	14 11/16	“	3 1/16	“	1 9/16
	“	“	⅝	“	“	4⅝	15⅛	“	“	“	“
9″ C29	13.25	13	¼	.23	6½	4¾	16⅛	5⅛	3¾	4	1⅜
	“	“	⅝	“	“	5⅛	16 9/16	“	“	“	“
	15.00	“	¼	.29	“	4¾	16⅛	“	3 11/16	“	1 7/16
	“	“	⅝	“	“	5⅛	16 9/16	“	“	“	“
	20.00	“	¼	.45	“	4¾	16⅛	“	3 9/16	“	1 9/16
	“	“	⅝	“	“	5⅛	16 9/16	“	“	“	“
	25.00	“	¼	.61	“	4¾	16⅛	“	3⅜	“	1¾
	“	“	⅝	“	“	5⅛	16 9/16	“	“	“	“

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